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AGGREGATION OPERATORS ON MULTIPLE SETS AND ITS APPLICATION IN DECISION-MAKING PROBLEMS

SANJITHA R AND BAIJU THANKACHAN*

Abstract. The ability of fuzzy set theory to offer a mathematical framework for the integration of arbitrary categories represented by membership functions is one of its most alluring aspects. But complexity arises when the number of criteria to be considered increases. In this paper, we propose a new concept called 'ordered weighted aggregation operators on multiple sets', by which the difficulty in considering the choices is minimized. This paper begins with the introduction of aggregation operators and ordered weighted aggregation operators in multiple sets. Along with certain key findings, a theoretical discussion based on them has also been made. Considering all the theories discussed, an application in personnel selection has been depicted in detail. The idea of dual aggregation operators on multiple sets has also been briefly introduced.

1. Introduction

Multiple set [1] is a broadened version of the fuzzy set theory that can handle the uncertainty of an element together with its multiplicity. It was Zadeh [2] who originally presented the idea of fuzzy sets, a significant area in mathematics to deal with uncertainty. While developing the concept of fuzzy sets, the notion of partial membership is considered as a crucial point and the degree to which an element belongs to one set is indicated by its membership value. However, an element's multiplicity is not taken into account while tackling uncertainty, and in 2016, Shijina et al.first put up the idea of multiple sets in the light of both multiplicity and uncertainty.

Aggregation operators are mathematical tools that aggregate a set of objects into a single one. Fuzzy aggregation operators are much explored by many researchers [6, 7, 8]. It has a wide range of practical applications in decision-making, artificial neural networks, fuzzy system modeling, etc [3, 4, 5]. In many disciplines, the issue of combining criteria functions to create overall decision functions is quite significant. The link between the various criteria is a key element in determining the structure of such aggregation functions.

The ordered weighted averaging operator (OWA) [9, 10, 11]is a fuzzy aggregation operator that aggregates the fuzzy sets in such a manner that the subjected

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fuzzy sets after aggregation will provide a fuzzy set whose membership function lies in between *min* and *max* aggregation operators. Yager introduced the OWA operator in the year 1987 as a result of his attempt to obtain a more generalized operator which accommodates almost all criteria in a decision-making problem. Furthermore, several authors [12, 13, 14] have investigated various applications of multi-criteria decision-making (MCDM) problems. Now, the same idea can be presented in multiple sets that can resolve the complexity of the decision-making process. The purpose of this research work is to devise a decision-making strategy for problems that involve multiple information sources, which can include both crisp and fuzzy data. The ordered weighted aggregation operator on multiple sets introduced in this work is a new concept defined as an extended form of the general aggregation operator. Additionally, a real-world example has been provided here to illustrate how simple it is to use while taking MCDM into consideration. This work also presents the dual aggregation operator on multiple sets using the dual operator for fuzzy sets, which has been the focus of numerous previous investigations.

The paper is presented in the following manner. Section 2 contains the survey of related work. In Section 3, we introduce the concept of OWA operators on multiple sets and an application in the area of personnel selection is demonstrated. An introduction to the concept of dual aggregation operator on multiple sets has also been included in the same section. Section 4 provides summary of the results.

2. Preliminaries

Definition 2.1. [6, 15] "An *aggregation operation* on m fuzzy sets ($m \geq 2$) is a function $H : [0,1]^m \to [0,1]$ which satisfies the following axioms

(1) $H(0,0,\dots,0) = 0$ and $H(1,1,\dots,1) = 1$ (Boundary condition).
(2) For any pair $(a_1, a_2, \dots, a_m)$ and $(b_1, b_2, \dots, b_m)$ of m-tuples such that $a_i, b_i \in [0,1]$, if $a_i \leq b_i$ for all $i \in N_m$ then $H(a_1, a_2, \dots, a_m) \leq H(b_1, b_2, \dots, b_m)$. i.e., H is monotonic increasing in all its arguments.
(3) H is continuous.
(4) $H(a_1, a_2, \dots, a_m) = H(a_{p(1)}, a_{p(2)}, \dots, a_{p(m)})$ for any permutation p on N_m.
(5) H is idempotent function that is , $H(a,a,a,\dots,a) = a$ for all $a \in [0,1]$."

Definition 2.2. [1] Let X be a non-empty crisp set called the universal set and $A_1, A_2, \dots, A_n$ be n distinct fuzzy sets on X corresponding to distinct attributes associated with each element in X. For each $i \in N_n; A_i^1(x), A_i^2(x), \dots, A_i^k(x)$ are membership values of the fuzzy set A_i for k identical copies of the element $x \in X$ in decreasing order. Then a multiple set $\mathbf{A}$ of order (n,k) over X is an object of the form $\{(x, \mathbf{A}(x)); x \in X\}$ where for each $x \in X$ its membership value is an $n \times k$ matrix in $\mathbf{M}$(collection of all matrices of order n $\times$ k with entries from $[0,1]$) given by,

$$\mathbf{A(x)} = \begin{bmatrix} A_1^1(x) & A_1^2(x) & \cdots & A_1^k(x) \\ A_2^1(x) & A_2^2(x) & \cdots & A_2^k(x) \\ \cdots & \cdots & \cdots & \cdots \\ A_n^1(x) & A_n^2(x) & \cdots & A_n^k(x) \end{bmatrix}$$

Definition 2.3. [1] Let $\mathbf{A}, \mathbf{B} \in \mathbf{MS}_{(n,k)}(X)$ where $\mathbf{MS}_{(n,k)}(X)$ denotes the collection of all multiple sets of order (n,k), then

(1) **Subset**:$\mathbf{A} \subseteq \mathbf{B}$ if and only if $\mathbf{A}(x) \leq \mathbf{B}(x) \forall x \in X$.
(2) **Union**: The union of $\mathbf{A}$ and $\mathbf{B}$ is a multiple set in $\mathbf{MS}_{(n,k)}(X)$ given by by $(\mathbf{A} \cup \mathbf{B})(x) = \mathbf{A}(x) \vee \mathbf{B}(x)$
(3) **Complement**: Complement of $\mathbf{A} \in \mathbf{MS}_{(n,k)}(X)$ is a multiple set $\overline{\mathbf{A}}$ whose membership matrix is an $n \times k$ matrix $\overline{\mathbf{A}}(x) = [\overline{A}_i^j(x)]_{n \times k}$ where $\overline{A}_i^j(x) = 1 - A_i^{k-j+1} \forall i \in N_n$, $j \in N_k$, $x \in X$.

Let $\mathbf{M^m}$ *denote the cartesian product* $\mathbf{M} \times \mathbf{M} \times \ldots \times \mathbf{M}$ $(m \ \ times)$ where $\mathbf{M} = \mathbf{M}_{n \times k}([0,1])$ be the collection of all matrices of order n $\times$ k with entries from $[0,1]$.

Definition 2.4. [16] Let H_{ij} be fuzzy aggregation operators for every $i \in N_n$ and $j \in N_k$. Define a function $\mathbf{H} : \mathbf{M^m} \longrightarrow \mathbf{M}$ as follows:
Matrices $N_1 = [(n_1)_{ij}]_{n \times k}, N_2 = [(n_2)_{ij}]_{n \times k}, \ldots, N_m = [(n_m)_{ij}]_{n \times k}$ in $\mathbf{M}$ are mapped to a matrix $\mathbf{P}$=$[p_{ij}]_{n \times k}$ such that $p_{ij} = H_{ij}((n_1)_{ij}, (n_2)_{ij}, \ldots, (n_m)_{ij})$ for every $i \in N_n$ and $j \in N_k$. Then $\mathbf{H}$ is called an aggregation operator induced by the fuzzy aggregation operators H_{ij} for every $i \in N_n$ and $j \in N_k$. It is represented as $\mathbf{H} = [H_{ij}]_{n \times k}$.

Definition 2.5. [10] A mapping $F : I^n \rightarrow I$ ($I = [0,1]$) is called an OWA of dimension n if there is a weighted vector $W = [W_1, W_2, \ldots, W_n]^t$ associated with F such that

(1) $W_i \in (0,1)$
(2) $\sum_i W_i = 1$

where $F(a_1, a_2, \ldots, a_n) = W_1 b_1 + W_2 b_2 + \cdots + W_n b_n$ and b_i is the i^{th} largest element in the collection $a_1, a_2, \ldots, a_n$.

3. Aggregation operators on Multiple sets.

Fuzzy sets have undergone several expansions and generalizations in the literature, including Atanassov's intuitionistic fuzzy sets [17], type 2 fuzzy sets [18], Torra's hesitant fuzzy sets [19], fuzzy multisets [20],multi-fuzzy sets [21] etc. The multiple sets [1] described by Shijina et al. is a general variant of fuzzy sets that may be thought of as a more straightforward tool for handling an object's multiplicity as well as its uncertainty. While dealing with many of the practical problems, the aforementioned theories exhibit increasingly complex behavior. Because of such, it takes longer and is more difficult. Multiple sets main benefits include ease of operation and a straightforward way to describe the issue. Each uncertain aspect of an item is represented by a different fuzzy membership function, and values are assigned to each function in line with the object's multiplicity. The fundamental benefit of multiple sets is the capacity to compress all the data into a single matrix-like structure. As a result, a multiple set assigns each object a matrix, where each row represents a distinct fuzzy membership function in accordance with each attribute of the item.

Aggregation operators on multiple sets have been defined as a more inclusive version of fuzzy aggregation operators. The fuzzy aggregation operator operates on fuzzy membership values in the membership matrix to produce the membership matrix of the aggregated multiple set. With the introduction of m-bag, the way of defining aggregation operators in multiple sets has changed, and the discussion is now carried further just by treating the operators as value functional.

Definition 3.1. An m-bag defined on X is a collection of multiple sets over X with repetition allowed. For example, let $\widehat{B} =< \mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \mathbf{A_2} >$, then $\widehat{B}$ is a m-bag and $|\widehat{B}|= 4$.

Definition 3.2. Consider two m-bags $\widehat{A}$ and $\widehat{B}$, then their sum is defined as $\widehat{A} \oplus \widehat{B} = \{\mathbf{A_i} | \mathbf{A_i} \in \widehat{A} \ or \ \mathbf{A_i} \in \widehat{B}\}$.

Definition 3.3. U^X denotes the collection of all m-bags defined over X. The m-bag operator $F : U^X \longrightarrow \mathbf{MS}_{(n,k)}(X)$ satisfying the following conditions

(1) $F(\mathbf{A}) = \mathbf{A}$
(2) $F(\widehat{A} \oplus \widehat{B}) = F(< F(\widehat{A}), F(\widehat{B}) >)$ is called a value functional.

Definition 3.4. Consider two m-bags $\widehat{A}$ and $\widehat{B}$, then $\widehat{A} \geq \widehat{B}$ if $\mathbf{A_i}(\mathbf{x}) \geq \mathbf{B_i}(\mathbf{x}), \ \forall \mathbf{x} \in \mathbf{X}$.

Definition 3.5. The m-bag mapping $H : U^X \longrightarrow MS_{(n,k)}(X)$ is an aggregation operator defined as

$$H < \mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_m} > (\mathbf{x}) = \mathbf{P}(\mathbf{x})$$

where

$$p_{ij} = H_{ij}[(a_1)_{ij}, (a_2)_{ij}, \ldots, (a_m)_{ij}]$$

and H_{ij} denotes the fuzzy aggregation operator. Hence the aggregation operator H is said to be induced by the fuzzy aggregation operator H_{ij}.

Definition 3.6. Consider the aggregation operator $H : U^X \longrightarrow MS_{(n,k)}(X)$, then the dual aggregation operator of H can be defined as $\overline{H} : U^X \longrightarrow \mathbf{MS}_{(n,k)}(X)$, where $\overline{H}(\mathbf{P}) = (\mathbf{I} - H(\bar{\mathbf{P}}))^*$ and $\bar{\mathbf{P}} = \mathbf{I} - \mathbf{P}^*$. $\mathbf{I}$ is the multiple set with a membership matrix having all entries 1 and $\mathbf{P}^*$ is a matrix of order (n, k) with entries given by

$$\mathbf{P}^*(x) = \begin{bmatrix} (P)_1^k(x) & (P)_1^{k-1}(x) & \cdots & (P)_1^1(x) \\ (P)_2^k(x) & (P)_2^{k-1}(x) & \cdots & (P)_2^1(x) \\ \cdots & \cdots & \cdots & \cdots \\ (P)_n^k(x) & (P)_n^{k-1}(x) & \cdots & (P)_n^1(x) \end{bmatrix}$$

Theorem 3.7. *The dual of the identity operator i is i.*

Proof.

$$\begin{aligned} \bar{i}(\mathbf{P}) &= (\mathbf{I} - i(\bar{\mathbf{P}}))^* \\ &= (\mathbf{I} - (\mathbf{I} - \mathbf{P}^*))^* \\ &= (\mathbf{P}^*)^* \\ &= \mathbf{P} \\ &= i(\mathbf{P}) \end{aligned}$$

Theorem 3.8. *The dual of the null operator o is o.*

Proof.

$$\begin{aligned}\bar{o}(\mathbf{P}) &= (\mathbf{I} - o(\bar{\mathbf{P}}))^* \\ &= (\mathbf{I} - (\mathbf{O}))^* \\ &= (\mathbf{I})^* \\ &= \mathbf{O} \\ &= o(\mathbf{P})\end{aligned}$$

Definition 3.9. An aggregation operator $H : U^X \longrightarrow MS_{(n,k)}(X)$ is said to be self dual if

$$\overline{(H < \overline{\mathbf{A_1}}, \overline{\mathbf{A_2}}, \ldots, \overline{\mathbf{A_m}} >)} = H < \mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_m} >$$

.

Definition 3.10. Let H and K be two aggregation operators defined over X such that H is induced by the fuzzy aggregation operator H_{ij} and K is induced by the fuzzy aggregation operator K_{ij}, then H is said to be stronger than K if

$$H < \mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_m} > (\mathbf{x}) \geq \mathbf{K} < \mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_m} > (\mathbf{x}), \quad \forall \mathbf{x} \in \mathbf{X}$$

Example 3.11.

$$max < \mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_m} > \ \geq \quad min \ < \mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_m} >$$

Definition 3.12. Consider the aggregation operator $H : U^X \longrightarrow MS_{(n,k)}(X)$ defined by

$$H < \mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_m} > (\mathbf{x}) = \mathbf{P}(\mathbf{x})$$

where

$$p_{ij} = H_{ij}[(a_1)_{ij}, (a_2)_{ij}, \ldots, (a_m)_{ij}]$$

and H_{ij} denotes the corresponding fuzzy aggregation operator. Let W be a weighted matrix of order (n, k) given by

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1k} \\ w_{21} & w_{22} & \cdots & w_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ w_{n1} & w_{n2} & \cdots & w_{nk} \end{bmatrix}$$

where $\sum_{i,j} w_{ij} = 1$.

Then the weighted aggregation operator $H^{'}$ with respect to the weighted matrix W is defined as

$$\begin{aligned}&H^{'} < \mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_m} > (\mathbf{x}) = \mathbf{P'}(\mathbf{x}) \\ &p'_{ij} = H'_{ij}[(a_1)_{ij}, (a_2)_{ij}, \ldots, (a_m)_{ij}] \\ &= H_{ij}[w_{ij}(a_1)_{ij}, w_{ij}(a_2)_{ij}, \ldots, w_{ij}(a_m)_{ij}]\end{aligned}$$

Example 3.13. $\mathbf{M_1}, \mathbf{M_2}$ be two multiple sets of order $(2,2)$ defined over $X = \{x_1, x_2\}$ whose membership matrix is given by

$$\mathbf{M_1}(\mathbf{x_1}) = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.1 \end{bmatrix}$$

$$\mathbf{M_1}(\mathbf{x_2}) = \begin{bmatrix} 0.5 & 0.3 \\ 0.4 & 0.2 \end{bmatrix}$$

$$\mathbf{M_2}(\mathbf{x_1}) = \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.1 \end{bmatrix}$$

$$\mathbf{M_2}(\mathbf{x_2}) = \begin{bmatrix} 0.4 & 0.3 \\ 0.4 & 0.2 \end{bmatrix}$$

Let H be the average aggregation operator

$$H(<\mathbf{M_1}, \mathbf{M_2}>)(\mathbf{x_1}) = \begin{bmatrix} 0.55 & 0.35 \\ 0.2 & 0.1 \end{bmatrix}$$

$$H(<\mathbf{M_1}, \mathbf{M_2}>)(\mathbf{x_2}) = \begin{bmatrix} 0.45 & 0.3 \\ 0.4 & 0.2 \end{bmatrix}$$

Now take $W = \begin{bmatrix} 0.3 & 0.2 \\ 0.4 & 0.1 \end{bmatrix}$ as a weighted matrix. Then the weighted aggregation operator $H^{'}$ with respect to the matrix W is given by

$$H^{'}(<\mathbf{M_1}, \mathbf{M_2}>)(\mathbf{x_1}) = \begin{bmatrix} (0.6 \times 0.3 + 0.5 \times 0.3)/2 & (0.4 \times 0.2 + 0.3 \times 0.2)/2 \\ (0.2 \times 0.4 + 0.2 \times 0.4)/2 & (0.1 \times 0.1 + 0.1 \times 0.1)/2 \end{bmatrix}$$

$$H^{'}(<\mathbf{M_1}, \mathbf{M_2}>)(\mathbf{x_1}) = \begin{bmatrix} 0.165 & 0.07 \\ 0.08 & 0.01 \end{bmatrix}$$

$$H^{'}(<\mathbf{M_1}, \mathbf{M_2}>)(\mathbf{x_2}) = \begin{bmatrix} (0.5 \times 0.3 + 0.4 \times 0.3)/2 & (0.3 \times 0.2 + 0.3 \times 0.2)/2 \\ (0.4 \times 0.4 + 0.4 \times 0.4)/2 & (0.2 \times 0.1 + 0.2 \times 0.1)/2 \end{bmatrix}$$

$$H^{'}(<\mathbf{M_1}, \mathbf{M_2}>)(\mathbf{x_2}) = \begin{bmatrix} 0.135 & 0.06 \\ 0.16 & 0.02 \end{bmatrix}$$

Definition 3.14. Let W be a weighted matrix corresponding to the aggregation operator $H : U^X \longrightarrow \mathbf{MS}_{(n,k)}(X)$, then $W^{\star}_{ij}$ denotes the weighted matrix with 1 in the $(i,j)^{th}$ entry and 0 otherwise.

Theorem 3.15. *M be a multiple set of order (n,k) defined over X then for any fixed $i, 1 \leq i \leq n$*

$$< H, W^{\star}_{ij} > (M) \;\; \leq \;\; < H, W^{\star}_{ij'} > (M) \;\; \textit{where} \;\; 1 \leq j^{'} \leq j \leq k$$

Proof.

$$H_{\star} =< H, W^{\star}_{ij} > (M)(x) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & H_{ij}(a_{ij}) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Now,

$$H^{\star} =< H, W^{\star}_{ij'} > (M)(x) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & H_{ij}(a_{ij'}) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

From the monotonicity of the fuzzy aggregation operator H_{ij}, we have

$$H_{ij}(a_{ij}) \leq H_{ij}(a_{ij'})$$
$$\Longrightarrow < H, W_{ij} > (M) \;\; \leq \; < H, W_{ij'} > (M)$$

Example 3.16. Let $X = \{x_1, x_2\}$ and $\mathbf{A_1}, \mathbf{A_2}$ be two multiple sets of order (2,2) whose membership matrix is given as follows

$$\mathbf{A_1}(\mathbf{x_1}) = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.2 \end{bmatrix}$$

$$\mathbf{A_1}(\mathbf{x_2}) = \begin{bmatrix} 0.1 & 0.1 \\ 0.3 & 0.2 \end{bmatrix}$$

$$\mathbf{A_2}(\mathbf{x_1}) = \begin{bmatrix} 0.5 & 0.4 \\ 0.3 & 0.3 \end{bmatrix}$$

$$\mathbf{A_2}(\mathbf{x_2}) = \begin{bmatrix} 0.4 & 0.3 \\ 0.2 & 0.2 \end{bmatrix}$$

Let H be the average aggregation operator defined over $< \mathbf{A_1}, \mathbf{A_2} >$ for the weighted operator $W = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,we have

$$H^{'}(< \mathbf{A_1}, \mathbf{A_2} >)(\mathbf{x_1}) = \begin{bmatrix} 0 & 0.25 \\ 0 & 0 \end{bmatrix}$$

$$H^{'}(< \mathbf{A_1}, \mathbf{A_2} >)(\mathbf{x_2}) = \begin{bmatrix} 0 & 0.2 \\ 0 & 0 \end{bmatrix}$$

The resultant $H^{'} < \mathbf{A_1}, \mathbf{A_2} >$ provides a membership matrix whose membership values are not in decreasing order and hence cannot be considered as a multiple set. To overcome this issue we define an ordered weighted aggregation operator for multiple sets.

Definition 3.17. Let H be an aggregation operator and

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1k} \\ w_{21} & w_{22} & \cdots & w_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ w_{n1} & w_{n2} & \cdots & w_{nk} \end{bmatrix}$$

be the associated weighted matrix. The ordered weighted matrix of W is given by

$$W' = \begin{bmatrix} w'_{11} & w'_{12} & \cdots & w'_{1k} \\ w'_{21} & w'_{22} & \cdots & w'_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ w'_{n1} & w'_{n2} & \cdots & w'_{nk} \end{bmatrix}$$

where $\{w_{i1}, w_{i2}, \ldots, w_{ik}\} = \{w'_{i1}, w'_{i2}, \ldots, w'_{ik}\}$ but $w'_{i1} \geq w'_{i2} \geq \ldots \geq w'_{ik}$ for $i = 1, 2, \ldots, n$.

Then the ordered weighted aggregation operator $H^{'}$ with respect to the ordered weighted matrix W' is defined as,

$$H^{'} < \mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_m} > (\mathbf{x}) = \mathbf{P}'(\mathbf{x})$$

where

$$\begin{aligned} p'_{ij} &= H'_{ij}[(a_1)_{ij}, (a_2)_{ij}, \ldots, (a_m)_{ij}] \\ &= H_{ij}[w'_{ij}(a_1)_{ij}, w'_{ij}(a_2)_{ij}, \ldots, w'_{ij}(a_m)_{ij}] \end{aligned}$$

Theorem 3.18. *M_1 and M_2 be two multiple sets of order (n, k) defined over X such that $M_1 \geq M_2$ and H be an ordered weighted aggregation operator with the corresponding ordered weighted matrix W, then*

$$H < M_1 > \; \geq \; H < M_2 >$$

Proof: For any $x \in X$, we have $M_1(x) \geq M_2(x) \forall x \in X$

$$H < M_1 > (x) = \begin{bmatrix} H_{11}(w_{11}a_{11}) & H_{12}(w_{12}a_{12}) & \cdots & H_{1k}(w_{1k}a_{1k}) \\ H_{21}(w_{21}a_{21}) & H_{22}(w_{22}a_{22}) & \cdots & H_{2k}(w_{2k}a_{2k}) \\ \cdots & \cdots & \cdots & \cdots \\ H_{n1}(w_{n1}a_{n1}) & H_{n2}(w_{n2}a_{n2}) & \cdots & H_{nk}(w_{nk}a_{nk}) \end{bmatrix}$$

$$H < M_2 > (x) = \begin{bmatrix} H_{11}(w_{11}a^{'}_{11}) & H_{12}(w_{12}a^{'}_{12}) & \cdots & H_{1k}(w_{1k}a^{'}_{1k}) \\ H_{21}(w_{21}a^{'}_{21}) & H_{22}(w_{22}a^{'}_{22}) & \cdots & H_{2k}(w_{2k}a^{'}_{2k}) \\ \cdots & \cdots & \cdots & \cdots \\ H_{n1}(w_{n1}a^{'}_{n1}) & H_{n2}(w_{n2}a^{'}_{n2}) & \cdots & H_{nk}(w_{nk}a^{'}_{nk}) \end{bmatrix}$$

Now

$$\begin{gathered} a_{ij} > a^{'}_{ij} \\ w_{ij}a_{ij} > w_{ij}a^{'}_{ij} \\ H(w_{ij}a_{ij}) > H(w_{ij}a^{'}_{ij}) \\ H < M_1 > (x) \;\; \geq \;\; H < M_2 > (x) \forall x \in X \\ \Longrightarrow H < M_1 > \; \geq \; H < M_2 > \end{gathered}$$

Theorem 3.19. *Let W be a weighted matrix and W' be the corresponding ordered weighted matrix. Then for any aggregation operator H defined over an m-bag $\widehat{B} =< \mathbf{P_1}, \mathbf{P_2}, \ldots, \mathbf{P_m} >$ of order (n, k), we have*

$$H(W\widehat{B}) \leq H(W'\widehat{B})$$

3.1. Application of Ordered Weighted Aggregation Operators in Decision-Making Process. There are numerous instances in real life where selecting an option from the options offered evolves into a laborious effort. The selection procedure is challenging as it has to deal with all the traits and factors. The situation typically occurs during "personnel selection." The process of identifying people who have the skills necessary to carry out a specific task in the best way is known as personnel selection. It establishes the level of personnel input and plays a crucial part in managing human resources. Organizations are being pushed by rising global market competition to place more emphasis on the hiring process.

The fuzzy set theory seems to be a crucial tool for creating a framework for making decisions that take into account the imprecise assessments made during the hiring process. Mathematically, this issue can be expressed more simply by using multiple sets. For each entity, a membership matrix that corresponds to the problem's criteria can be created. By choosing the weighted aggregation operator it will be much easier to aggregate the factors and sum them up. Here is a real-world example that demonstrates the use of ordered weighted aggregation operators (OWA) and multiple sets in the personnel selection process.

Here, we present a method for solving the personnel selection problem that makes use of multiple sets and ordered weighted aggregation operators. The resulting multiple set with the membership matrix is the result of combining the expert-weighted valuations for multi- criteria of each entity.

Algorithm

Let $X = \{C_1, C_2, \ldots, C_n\}$ be the universe of discourse under the evaluation of $\{P_1, P_2, \ldots, P_n\}$ based on the attributes $\{A_1, A_2, \ldots, A_k\}$.

Step 1: Define multiple sets $\{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n\}$ of order (k, s) over the set X. Here s denotes the number of levels of evaluation made by P_i's over C_i's.

$$\mathbf{P_i}(\mathbf{C_i}) = \begin{bmatrix} (a_i)_1^1 & (a_i)_1^2 & \cdots & (a_i)_1^s \\ (a_i)_2^1 & (a_i)_2^2 & \cdots & (a_i)_2^s \\ \cdots & \cdots & \cdots & \cdots \\ (a_i)_k^1 & (a_i)_k^2 & \cdots & (a_i)_k^s \end{bmatrix}$$

The membership matrix $\mathbf{P_i}(\mathbf{C_i})$ indicates the report given by the P_i to each C_i and each row of the membership matrix indicates the score granted in the decreasing order at different levels for the criterion involved .

Step 2: Construct an ordered weighted matrix

$$W = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1k} \\ w_{21} & w_{22} & \cdots & w_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ w_{n1} & w_{n2} & \cdots & w_{nk} \end{bmatrix}$$

from the weightage of the attributes given by the examiners in such a way that $\sum_{i,j} w_{ij} = 1$.

Step 3: Choose an aggregation operator H considering the weighted matrix W and define the ordered weighted aggregation operator $H' =< H, W >$. The ordered weighted aggregation operator $H^{'}$ with respect to the ordered weighted matrix W is given by

$$H^{'} < \mathbf{P_1}, \mathbf{P_2}, \ldots, \mathbf{P_n} > (\mathbf{C_i}) = \mathbf{P}'(\mathbf{C_i})$$

where

$$p'_{ij} = H'_{ij}[(a_1)_i^j, (a_2)_i^j, \ldots, (a_n)_i^j]$$

$$= H_{ij}[w_{ij}(a_1)_i^j, w_{ij}(a_2)_i^j, \ldots, w_{ij}(a_n)_i^j]$$

Step 4: Aggregate the multiple sets $\{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n\}$ with the ordered weighted aggregation operator obtained in step 3. The resultant multiple set will provide the required selection.

Example 3.20. A multinational company is interviewing for the post of HR manager, the interview panel consists of two experts and it is conducted on two levels based on communication skills, general awareness, and critical reasoning. Three candidates were selected for the final interview. One of the three will be selected finally based on the evaluation conducted considering the information given below. Each candidate's performance has been assessed by experts 1 and 2 on two different levels, and the results are summarised in Tables 1 and 2.

Candidate name	Communication skill	General Awareness	Critical Reasoning
C_1	$(0.7, 0.4)$	$(0.8, 0.6)$	$(0.8, 0.5)$
C_2	$(0.8, 0.5)$	$0.9, 0.6)$	$(0.6, 0.5)$
C_3	$(07, 0.5)$	$(0.8, 0.6)$	$(0.8, 0.5)$

Table 1:*Evaluation of Expert-1*

Candidate name	Communication skill	General Awareness	Critical Reasoning
C_1	$(0.8, 0.4)$	$(0.7, 0.7)$	$(0.9, 0.5)$
C_2	$(0.9, 0.5)$	$(0.9, 0.7)$	$(0.8, 0.5)$
C_3	$(07, 0.4)$	$(0.9, 0.6)$	$(0.7, 0.4)$

Table 2:*Evaluation of Expert-2*

The expert committee has given weightage to the skills by their personal choice given in table-3.

Skills	Expert 1- Score	Expert 2- Score
Communication skill	3	2
General Awareness	1	1
Critical Reasoning	1	2

Table 3: *Professional skill evaluation*

Now, we model the aforementioned personnel selection problem using the algorithm provided and hence obtain a solution for it.

• The multiple set $\mathbf{P} = \{(C_1, \mathbf{P}(C_1)), (C_2, \mathbf{P}(C_2)), (C_3, \mathbf{P}(C_3))\}$ represents the performance of the candidates evaluated by expert-1 in two levels with respect to above mentioned criteria where $\mathbf{P}(C_1), \mathbf{P}(C_2)$ *and* $\mathbf{P}(C_3)$ represent the corresponding membership matrix given as follows,

$$\mathbf{P(C_1)} = \begin{bmatrix} 0.7 & 0.4 \\ 0.8 & 0.6 \\ 0.8 & 0.5 \end{bmatrix}$$

$$\mathbf{P(C_2)} = \begin{bmatrix} 0.8 & 0.5 \\ 0.9 & 0.6 \\ 0.6 & 0.5 \end{bmatrix}$$

$$\mathbf{P(C_3)} = \begin{bmatrix} 0.7 & 0.5 \\ 0.8 & 0.6 \\ 0.8 & 0.5 \end{bmatrix}$$

• The multiple set $\mathbf{Q} = \{(C_1, \mathbf{Q}(C_1)), (C_2, \mathbf{Q}(C_2)), (C_3, \mathbf{Q}(C_3))\}$ represents the performance of the candidates evaluated by expert-2 in two levels where $\mathbf{Q}(C_1), \mathbf{Q}(C_2)$ *and* $\mathbf{Q}(C_3)$ represent the corresponding membership matrix given as follows,

$$\mathbf{Q(C_1)} = \begin{bmatrix} 0.8 & 0.4 \\ 0.7 & 0.7 \\ 0.9 & 0.5 \end{bmatrix}$$

$$\mathbf{Q(C_2)} = \begin{bmatrix} 0.9 & 0.5 \\ 0.9 & 0.7 \\ 0.8 & 0.5 \end{bmatrix}$$

$$\mathbf{Q(C_3)} = \begin{bmatrix} 0.7 & 0.4 \\ 0.9 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}$$

•Now the weighted matrix W is constructed using the weightage of each skill given by the experts.

$$W = \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}$$

and $\sum_{i,j} w_{ij} = 1$. Here the first row of the matrix represents the weightage given by the two experts for communication skills, the second row indicates the weightage for general awareness, and the third row for critical reasoning.

• By considering the average aggregation operator H together with the weighted matrix W, we obtain the weighted average aggregation operator H'.The overall performance of each candidate is calculated below

$$H^{'}(<\mathbf{P},\mathbf{Q}>)(\mathbf{C_1}) = \begin{bmatrix}(0.7\times 0.3+0.8\times 0.3)/2 & (0.4\times 0.2+0.4\times 0.2)/2\\(0.8\times 0.1+0.7\times 0.1)/2 & (0.6\times 0.1+0.7\times 0.1)/2\\(0.8\times 0.2+0.9\times 0.2)/2 & (0.5\times 0.1+0.5\times 0.1)/2\end{bmatrix}$$

$$H^{'}(<\mathbf{P},\mathbf{Q}>)(\mathbf{C_1}) = \begin{bmatrix}0.33 & 0.08\\0.075 & 0.065\\0.17 & 0.05\end{bmatrix}$$

$$H^{'}(<\mathbf{P},\mathbf{Q}>)(\mathbf{C_2}) = \begin{bmatrix}(0.8\times 0.3+0.9\times 0.3)/2 & (0.5\times 0.2+0.5\times 0.2)/2\\(0.9\times 0.1+0.9\times 0.1)/2 & (0.6\times 0.1+0.7\times 0.1)/2\\(0.6\times 0.2+0.8\times 0.2)/2 & (0.5\times 0.1+0.5\times 0.1)/2\end{bmatrix}$$

$$H^{'}(<\mathbf{P},\mathbf{Q}>)(\mathbf{C_2}) = \begin{bmatrix}0.255 & 0.1\\0.09 & 0.065\\0.140 & 0.05\end{bmatrix}$$

$$H^{'}(<\mathbf{P},\mathbf{Q}>)(\mathbf{C_3}) = \begin{bmatrix}(0.7\times 0.3+0.7\times 0.3)/2 & (0.5\times 0.2+0.4\times 0.2)/2\\(0.8\times 0.1+0.9\times 0.1)/2 & (0.6\times 0.1+0.6\times 0.1)/2\\(0.8\times 0.2+0.7\times 0.2)/2 & (0.5\times 0.1+0.4\times 0.1)/2\end{bmatrix}$$

$$H^{'}(<\mathbf{P},\mathbf{Q}>)(\mathbf{C_3}) = \begin{bmatrix}0.21 & 0.09\\0.085 & 0.06\\0.15 & 0.045\end{bmatrix}$$

On comparing the final performance matrices $H^{'}(<\mathbf{P},\mathbf{Q}>)(\mathbf{C_i})$, it can easily be verified that the performance level of candidate C_1 is better than the other two candidates.

4. Conclusion

Fuzzy sets and fuzzy decision-making procedures can successfully handle the imprecision and vagueness that are frequently present in decision-making. The structure and resolution of decision and planning problems incorporating multiple criteria are the focus of multi-criteria decision-making (MCDM) and hence numerous studies examine MCDM. Regular work is taking place in fuzzy set theory to address the growing number of options in MCDM. At each step, fuzzy aggregation operators are changed to suit the requirements. According to a generalization in the definition of aggregation operators for multiple sets, we make a new attempt to present ordered weighted aggregation operators in this study. A personnel selection problem has been used to demonstrate how ordered weighted aggregation operators and multiple sets make it simple to handle MCDM challenges. This work has also established an introduction to dual aggregation operators to multiple set theory.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Sanjitha R: Department of Mathematics, Manipal Institute of Technology, Manipal, Manipal Academy of Higher Education, Manipal-576104, Karnataka, India
Email address: sanjitharadhakrishnan97@gmail.com

Baiju Thankachan: Department of Mathematics, Manipal Institute of Technology, Manipal, Manipal Academy of Higher Education, Manipal-576104, India
Email address: baiju.t@manipal.edu *Corresponding author