Global and Stochastic Analysis
Vol. 12 No. 2 (March, 2025)

Received: 05th January 2025 *Revised: 21st January 2025* *Accepted: 25th January 2025*

A NOTE ON COMPLEMENTS IN A LATTICE

KUNCHAM S.P., TAPATEE S.*, SRIDHARA K. B., KEDUKODI B.S., AND HARIKRISHNAN P.K.*

Abstract. The concept of modular lattices naturally exhibit the links between the theoretical aspects of discrete structures and corresponding applications. In this paper, we consider the complements in a modular lattice with finite Goldie dimension (in short, FGD). We prove several properties and characterizations that involve θ-complement, θ-closed and relative θ-complemented, weak θ-complemented, etc in a lattice. We provide the necessary illustrations to justify the notions and generalizations in this paper.

1. Introduction

The notion of 'essential submodule' of a module over a ring is an analogy to the concept of 'dense subspace' in a topological space [2]. A submodule L is essential in a module M in case $K \cap L \neq (0)$, for each non-zero submodule K of M. However, as we know, a lattice need not contain a zero element, and so essentiality concept in a lattice with respect to an arbitrary element was introduced in [19]. Nevertheless, the concept of module over a ring is well interpreted in terms of the lattice structure of its submodules. Grzeszczuk and Puczylowski [9] established the idea of Goldie dimension from the module theory, to the modular lattices. They defined an essential element in a lattice with the least element 0. The theory has become significant and later many developments found in Calugareanu [6] wherein several ideas from modules over rings were generalized to the lattice theory. Goldie [10] introduced the concept of the Goldie dimension of modules over rings, and proved a characterization for a module to have finite Goldie dimension. Bhavanari [3] obtained several equivalent conditions in terms of descending chain conditions on essential submodules. There are good connections between semiprime ideals and uniform ideals of module over rings. Tapatee et.al [20, 21] studied relative essential ideals and relative complements and in [23], the authors studied the partial order aspects of modules over generalized rings. We refer to [19, 22] for the developments in modular lattices. The notion complement plays an important role in modules, specifically, as in [3, 21], to establish the dimension of a quotient submodule and the dimension of sum of two submodules. Analogously, in a lattice with 0, the notion pseudo-complement has been defined in [6], and some recent developments can be seen in [7]. Saki and Kiani [18] studied the properties of complements and pseudo-complements of finite modular lattices of subracks, and obtained some

---

2000 *Mathematics Subject Classification.* 16Y30.

Key words and phrases. complement; essential element; uniform element.

* Corresponding author.

equivalent conditions. Rao and Beyene [16] have explored on irreducible elements in almost semilattices. The semi-complement undirected graph of lattice modules have been studied by Phadatare et.al [15].
In this paper, we consider a lattice $(L, \wedge, \vee)$ with the smallest element 0 and whenever necessary we assume 1 to be the greatest element in L. For $x, y \in L$ and $x \leq y$, the interval between x and y is denoted by $[x, y] = \{a \in L \mid x \leq a \leq y\}$, is a sublattice of L. If $a \neq 1$ in L, then a is called proper. In a bounded lattice an element a is called an atom (respectively, dual atom) if there is no $x \in L$ such that $0<x<a$ (respectively, $a<x<1$).
In this paper, we deal with the modular lattices and define θ-complement and weak θ-complement which generalize both the notions pseudo-complement and complement in L. We prove several properties as generalizations of results in [6, 13], wherein the lattice is upper continuous.

For comprehensive literature in lattice theory, we refer to [8].

The following definitions are from [1, 6].

A subset $\mathcal{D}$ of a poset is called upper directed, if each finite subset of $\mathcal{D}$ has an upper bound in $\mathcal{D}$. A complete lattice L is called upper continuous if $a \wedge (\bigvee \mathcal{D}) = \bigvee_{d \in \mathcal{D}} (a \wedge d)$ holds for every $a \in L$ and every upper directed subset $\mathcal{D} \subseteq L$. L is called modular if for any $x, y, z \in L$, $x \leq z$ implies $(x \vee y) \wedge z = x \vee (y \wedge z)$. If $y \in L$ is maximal with respect to the property $x \wedge y = 0$, then y is called a pseudo-complement of x in L. L is pseudo-complemented if for every $x \in L$, there exists a pseudo-complement in L, and is relative pseudo-complemented, if each sublattice of L is pseudo-complemented. In a lattice with 0 and 1, $y \in L$ is called a complement of $x \in L$ if $x \wedge y = 0$ and $x \vee y = 1$.

2. θ-complement

Throughout, let $\theta \in L$ be an arbitrary but fixed element, where L is a modular lattice.

Definition 2.1. [19]

(1) $\theta \neq a \in L$ is θ-essential if $a \wedge x \neq \theta$ for every $\theta \neq x \in L$, we denote it as $a \leq_\theta^e L$. The set of all θ-essential elements in L is denoted by $E_\theta(L)$.
(2) Let $x \leq y \in L$. Then x is θ-essential in y if $x \leq_\theta^e [\theta, y]$. In other words, $x \wedge k \neq \theta$ for every $k \in (\theta, y]$, denoted by $x \leq_\theta^e y$. In this case, we call y as θ-essential extension of x.
(3) $a \in [x, y]$ is said to be θ-essential, if $\theta \in [x, y]$ and $a \wedge b \neq \theta$ for every $\theta \neq b \in [x, y]$.

Evidently, if $\theta = 0$, then the notion of 'θ-essential' coincides with the notion 'essential'.

Example 2.2. Let L be the lattice given in Fig. 1. Now $x \not\leq^e L$, for every $0 \neq x \in L$, whereas $x \leq_{\theta=b}^e L$, for every $\theta \neq x \in L$.

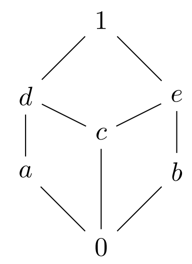

FIGURE 1

Example 2.3. [8] Consider the free distributive lattice L, on three generators given in Fig. 2. Then we have the following.

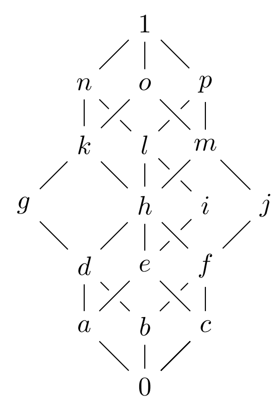

FIGURE 2

(1) $e \leq^e_{\theta=b} p$, whereas $e \nleq^e p$, since $e \wedge b = 0$ and $b \neq 0$.
(2) $i \leq^e_{\theta=b} p$, whereas $i \nleq^e p$, since $i \wedge b = 0$ and $b \neq 0$.

Definition 2.4. [6] A function $\phi : L_1 \to L_2$ between two lattices is called a lattice homomorphism if $\phi(x \vee y) = \phi(x) \vee \phi(y)$ and $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$, for all $x, y \in L_1$.

Theorem 2.5. [6] If $s, t \in L$, then $[t, (s \vee t)]$ and $[(s \wedge t), s]$ are isomorphic.

Lemma 2.6. Let $f : L_1 \to L_2$ be an isomorphism and $a \leq^e_\theta L_1$ implies $f(a) \leq^e_{f(\theta)} L_2$.

Proof. Let $a \leq^e_\theta L_1$. Let $b \in L_2$ such that $f(a) \wedge b = f(\theta)$. Then $a \wedge f^{-1}(b) = \theta$. Since $a \leq^e_\theta L_1$ and $f^{-1}(b) \in L$, we get $f^{-1}(b) = \theta$. Therefore, $b = f(\theta)$. This shows that $f(a) \leq^e_{f(\theta)} L_2$. □

Unlike in case of module over rings, essentiality need not be closed under homomorphic images. Indeed, in a lattice, the image of a θ-essential element under a

lattice homomorphism need not be θ-essential.
Consider the following example.

Example 2.7. Let L_1 and L_2 be two lattices given in Fig. 3. Let $f : L_1 \to L_2$ be a lattice homomorphism defined by $f(x) = x$, for $x \in \{0, a, b\}$ and $f(c) = 0$. Clearly, for $\theta = 0$, $a \leq_\theta^e L_1$, but $f(a) = a \not\leq_{f(\theta)=0}^e L_2$.

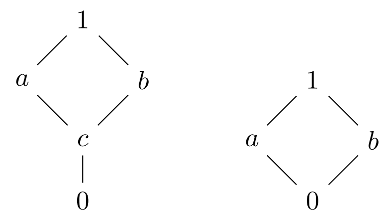

FIGURE 3. L_1 and L_2

Notation 2.8. If $a \leq_\theta^e c$, and $b \leq c$, then $a \leq_\theta^e b$.

Definition 2.9. [19] $S = \{a_i \mid i \in I, \text{ where } I \text{ is finite}\} \subseteq L \setminus \{\theta\}$, is said to be θ-$\vee$-independent if $a_i \wedge \big(\bigvee_{j \neq i} a_j\big) = \theta$, for every $i \in I$.

Definition 2.10. For any $a, b \in L$, an element a is θ-closed in b, if a has no proper θ-essential extension in b, we denote it by $a \leq_\theta^{cl} b$.

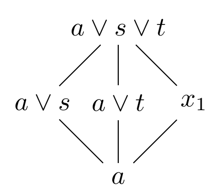

FIGURE 4

Proposition 2.11. For $\theta < a < b < c$ in L, if a is θ-closed in b and b is θ-closed in c, then a is θ-closed in c.

Proof. Suppose $a \leq_\theta^{cl} b$, $b \leq_\theta^{cl} c$, and $a \not\leq_\theta^{cl} c$. Then there exists $x \in L$ such that $a < x \leq_\theta^{cl} c$ and $a \leq_\theta^e x$. Now $x \wedge b \leq x$, implies $a \leq_\theta^e x \wedge b$. Since $a \leq_\theta^e x \wedge b \leq b$ and $a \leq_\theta^{cl} b$, we get $a = x \wedge b \ \cdots\ (1)$.
If $x \leq b$, then $a = x$, a contradiction. So, $x \not\leq b$, and so $b < b \vee x \leq c$. Since, $b \leq_\theta^{cl} c$, it follows that $b \not\leq_\theta^{cl} b \vee x$, thus there exists s such that $\theta < s \leq b \vee x$ and $s \wedge b = \theta$. Now, $a \leq_\theta^e x$, $s \wedge x \leq x$ and $a \wedge (s \wedge x) \leq a \wedge s < b \wedge s = \theta$, implies $s \wedge x = \theta \ \cdots\ (2)$.

If $a = (s \wedge x) \wedge b$, then

$$\begin{aligned} s &= s \wedge (b \vee x) \\ &= [s \wedge (s \wedge x)] \wedge (b \vee x) \\ &= s \wedge [((s \wedge x) \wedge b) \vee x], \text{ by modular law, } x \leq s \vee x \\ &= s \wedge (a \vee x) \\ &= s \wedge x \\ &= \theta, \end{aligned}$$

a contradiction. Hence, $a < (s \vee x) \wedge b \leq b$. Since $a \leq_\theta^{cl} b$, we get $a \not\leq_\theta^{cl} (s \vee x) \wedge b$, thus there exists t such that $\theta < t \leq (s \vee x) \wedge b$ and $a \wedge t = \theta \ \cdots \ (3)$.
Then from (1) it follows that $x \wedge t = x \wedge (b \wedge t) = a \wedge t = \theta$. Thus, if $s \wedge (x \wedge t) = \theta$, then the (s, t, x) is θ-$\vee$-independent, and thus $t \wedge (x \vee s) = \theta$, a contradiction, since $\theta < t \leq x \wedge s$. This shows that $s' = s \wedge (x \vee t) \neq \theta$.
Moreover, from (3) it follows,

$$\begin{aligned} x \vee s' &= x \vee (s \wedge (x \vee t)) \\ &= (x \vee s) \wedge (x \vee t), \text{ by modular law, } x \leq x \vee t \\ &= x \vee t \\ &\geq t \end{aligned}$$

So, we may replace s by s' without changing the validity of (3).
Therefore, we may assume that $\theta < s \leq x \vee t$ and $s \wedge b = \theta \ \cdots \ (4)$.
Now from (2) and (4), $x \vee s = x \vee t$. Also since $a \wedge t \leq b \leq b$ and $s \wedge b = \theta$, we have $s \wedge (a \wedge t) = \theta$. Since $a \wedge t = \theta$, and by modular law, $\{a, s, t\}$ is θ-$\vee$-independent. Moreover, since $a \leq x$ and by (2), $x \wedge (a \vee s) = a \vee (x \wedge s) = a \vee \theta = a$. Similarly, by using the equality $x \wedge t = \theta$, yields $x \wedge (a \vee t) = a$. Let $x_1 = x \wedge (a \vee s \vee t)$.
Next we claim that the elements $a \vee s$, $a \vee t$ and x_1 are the atoms of a lattice shown in Fig. 4, with bottom a and top $a \vee s \vee t$. Clearly, by modular law and the fact that $\{a, s, t\}$ is θ-$\vee$-independent, we get $(a \vee s) \wedge (a \vee t) = a \vee [s \wedge (a \vee t)] = a \vee \theta = a$. Again by modular law, since $a \leq x_1 \leq x$,

$$\begin{aligned} x_1 \wedge (a \vee s) &= a \vee (x_1 \wedge s) \\ &\leq a \vee (x \wedge s) \\ &= a \vee (x \wedge t) \\ &= x \wedge (a \vee t) \\ &= a. \end{aligned}$$

Now $x_1 \wedge (a \vee s) = a$, since $a \leq a \vee s$ and $a \leq x_1$. Clearly $x_1 \leq (a \vee s) \vee (a \vee t) = a \vee s \vee t$. Since $x \vee s = x \vee t \geq a \vee s \vee t$, we get $x_1 \vee (a \vee s) = (x \vee a \vee s) \wedge (a \vee s \vee t) = a \vee s \vee t$. Similarly, $x_1 \vee (a \vee t) = a \vee s \vee t$. From $a \wedge t = \theta$ and $t > \theta$, it follows that $a < a \vee t$, thus $a < x_1$. Now let $x_0 = x \wedge (s \vee t)$. Since $a \leq x$ and by modularity, we get

$$\begin{aligned} x_0 \vee a &= [x \wedge (s \vee t)] \vee a \\ &= x \wedge (a \vee s \vee t) \\ &= x_1 \\ &\geq a > \theta. \end{aligned}$$

Thus $x_0 \vee a > \theta$, hence $x_0 > \theta$.
Whereas,

$$\begin{aligned} a \wedge x_0 &= a \wedge x \wedge (s \vee t) \\ &= a \wedge (s \vee t), \text{ as } a \leq x \\ &= \theta, \end{aligned}$$

a contradiction to the assumption that $a \leq_\theta^e x$.

□

The converse of the Proposition 2.11 not necessarily true.

Example 2.12. Let L be the lattice given in Fig. 2. Then for $\theta = 0$, $d \leq_\theta^{cl} l$, $d \leq_\theta^{cl} h$, but $h \not\leq_\theta^{cl} l$, as $h \wedge x \neq 0$, for all $x \in [0, l]$.

The following definition is a generalization of pseudo-complement defined in [6].

Definition 2.13. $c \in L$ is called a θ-complement of b in L if c is maximal with respect to $b \wedge c = \theta$. Further, L is θ-complemented if every $x \in L$ has at least one θ-complement.

Example 2.14. Let $L = (D_{30}, \leq)$, the elements are positive divisors of 30, given in the Fig. 5. Write $x \leq y \Leftrightarrow x$ *divides* y, $x \vee y = \text{l.c.m}\{x, y\}$ and $x \wedge y = \text{g.c.d}\{x, y\}$. Then, d is a $(\theta = b)$-complement of f, but d is not a pseudo-complement of f,

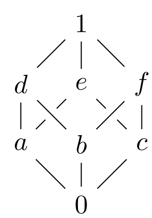

FIGURE 5

since $d \wedge f = b \neq 0$.

Definition 2.15. L is called relative θ-complemented if for every $x \in L$, $[\theta, x]$ is θ-complemented. Further, $x \in L$ is called a weak θ-complement if there exists $x' \in L$ such that $x \wedge x' = \theta$ and $x \vee x' = 1$. L is called weak θ-complemented if every $x \in L$ has at least one weak θ-complement in L.

Example 2.16. Let L be the non-modular lattice given in Fig. 6, of all subgroups of the group D_8, the dihedral group of order 8. Then, f is a $(\theta = c)$-complement of g, but f is not a pseudo-complement of g, since $f \wedge g = c \neq 0$.

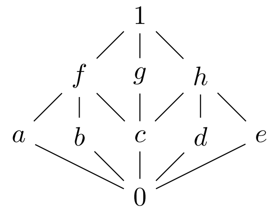

FIGURE 6

Example 2.17. Consider the Lattice L_1 given in Fig. 3. Here, for $(\theta = c)$, L_1 is θ-complemented, relative θ-complemented and weak θ-complemented. But L_1 is not pseudo-complemented or complemented, since $x \wedge y \neq 0$, for any $0 \neq x, y \in L_1$.

Lemma 2.18. Let $\theta \leq a \leq p \leq b$ be elements in L with 1, and r a weak θ-complement of p in L. Then $q = (a \vee r) \wedge b = a \vee (r \wedge b)$, a weak θ-complement of p in $[a, b]$.

Proof. Since r is weak θ-complement of p in L, we have $r \wedge p = \theta$ and $r \vee p = 1$. Now,

$$\begin{aligned} p \vee q &= p \vee a \vee (r \wedge b) \\ &= p \vee (r \wedge b) \\ &= (p \vee r) \wedge b, \text{ since } p \leq b \text{ and by modular law} \\ &= 1 \wedge b \\ &= b \end{aligned}$$

and

$$\begin{aligned} p \wedge q &= q \wedge p \\ &= (a \vee r) \wedge b \wedge p \\ &= (a \vee r) \wedge p \\ &= a \vee (r \wedge p), \text{ since } a \leq p \text{ and by modular law} \\ &= a \vee \theta \\ &= a. \end{aligned}$$

Therefore, q is a weak θ-complement of p in $[a, b]$. □

Corollary 2.19. A weak θ-complemented lattice is relative θ-complemented.

Proof. Follows from Lemma 2.18. □

Definition 2.20. L is called θ-inductive, if every sublattice $[x, y]$ of L satisfies the condition that: for any chain $\{b_i\}_{i \in I}$ in L and for any $a \in [x, y]$ with $a \wedge b_i = \theta$, for all $i \in I$, imply $a \wedge (\bigvee_{i \in I} b_i) = \theta$. If $\theta = 0$, θ-inductive coincides with the inductive defined in [6], and if L is upper continuous, then it is θ-inductive.

Notation 2.21. Every inductive lattice is θ-inductive. In a lattice of finite length, both inductive and θ-inductive exists. However, we may have an infinite lattice, which is θ-inductive but not inductive. Consider the infinite lattice $L = (\mathbb{Z}, \leq)$. Since the lattice has no least element, it is not inductive, whereas, L is $(\theta = 1)$-inductive.

For any θ-inductive lattice L and $\theta \leq a \in L$, the set $S = \{x \in L : a \wedge x = \theta,\ b \leq x\}$ has a maximal element by Zorn's lemma, which will be a θ-complement of a in L. Precisely, we state the following Lemma.

Lemma 2.22. Let L be θ-inductive lattice. Then every $\theta \leq a \in L$ has a θ-complement in L.

Corollary 2.23. If L is upper continuous, then every $\theta < a \in L$, has a θ-complement in L.

Proof. Follows from Lemma 2.22. □

Lemma 2.24. Let $1 \in L$, and $a, b \in L$. Then b is a θ-complement of a if and only if $a \wedge b = \theta$ and $a \vee b \leq^e_\theta [b, 1]$.

Proof. Let b be a θ-complement of a in L. Clearly, $b \wedge a = \theta$, and for any $d \in L$, $b < d$ implies $d \wedge a \neq \theta$. In particular, $d \in [b, 1]$, $d \neq b$ implies $d \neq \theta$. Then by modular law and since $d \wedge a \leq d \nleq b$, we have $\theta \leq b < b \vee (d \wedge a) = (a \vee b) \wedge d$. Hence $(a \vee b) \wedge d \neq \theta$, shows that $a \vee b \leq^e_\theta [b, 1]$. Conversely, suppose that $a \wedge b = \theta$ and $a \vee b \leq^e_\theta [b, 1]$. Then, for every $d \in [b, 1]$, $d \neq b$, we have $b < (a \vee b) \wedge d = b \vee (a \wedge d)$. That is, $a \wedge b = \theta$, and for every $b < d$, we have $a \wedge d \nleq b$. This implies $a \wedge d \neq \theta$. □

Corollary 2.25. Let $1 \in L$ and b be a θ-complement of $a \in L$, then $a \wedge b = \theta$ and $a \vee b \leq^e_\theta L$.

Proof. Let b be a θ-complement of a in L. Then by Lemma 2.24, we have $a \wedge b = \theta$ and $a \vee b \leq^e_\theta [b, 1]$. Then clearly, $\theta \leq b$. To show, $a \vee b \leq^e_\theta L$, take $d \in L$.
Case (i): If $\theta \neq d \leq b$, then $(a \vee b) \wedge d = d \neq \theta$. Therefore, $a \vee b \leq^e_\theta L$.
Case (ii): If $d \nleq b$, then clearly $d \nleq \theta$. Now $b \leq b \vee d \in [b, 1]$, and by Lemma 2.24 and by modular law, we have $\theta \leq b \neq (a \vee b) \wedge (b \vee d) = ((a \vee b) \wedge d) \vee b$, shows that $a \vee b \leq^e_\theta L$. □

Lemma 2.26. Let L be upper continuous and b be a θ-complement of a in L. If $c \in L$ is maximal such that $a \leq c$, $b \wedge c = \theta$, then c is maximal with respect to $a \leq^e_\theta c$.

Proof. Let $\mathcal{K} = \{y \in L : a \leq y, b \wedge y = \theta\}$. Since $a \in \mathcal{K}$, $\mathcal{K} \neq \emptyset$. By Zorn's lemma, $\mathcal{K}$ has a maximal element, say c. To show $a \leq^e_\theta c$, let $x \in [\theta, c]$ such that

$a \wedge x = \theta$. Now,

$$\begin{aligned} a \wedge (b \vee x) &= (a \wedge c) \wedge (b \vee x) \\ &= a \wedge (c \wedge (b \vee x)) \\ &= a \wedge ((c \wedge b) \vee x), \text{ since } x \leq c, \text{ by modularity} \\ &= a \wedge (\theta \vee x) \\ &= a \wedge x \\ &= \theta. \end{aligned}$$

Since b is θ-complement of a, we have $b \vee x = b$ implies $x \leq b$. So, $x = x \wedge b \leq c \wedge b = \theta$, implies $x \leq \theta$. Therefore, $x = \theta$. For the maximality, let $a \leq c'$, and $a \leq_\theta^e c'$ such that $c \leq c'$. Then by hypothesis, we have $b \wedge c' \neq \theta$. Therefore, $a \wedge (b \wedge c') \neq \theta$. But $a \wedge (b \wedge c') = (a \wedge b) \wedge c' = \theta \wedge c' = \theta$, a contradiction. □

Lemma 2.27. Let $\theta < a \in L$, where L is upper continuous. Then a is θ-closed if and only if a is a θ-complement.

Proof. Suppose a is a θ-complement of b in L. In a contrary, assume that $a \leq_\theta^e c$, for some $c \in L$. Since $a \leq c$, by maximality of a, we have $b \wedge c \neq \theta$. Moreover, since $a \leq_\theta^e c$ and $b \wedge c \in [\theta, c]$, we have $a \wedge (b \wedge c) \neq \theta$, whereas, $a \wedge (b \wedge c) = (a \wedge b) \wedge c = \theta \wedge c = \theta$, a contradiction. Conversely, since L is upper continuous, and $\theta < a \in L$, by Corollary 2.23, we have a has a θ-complement, say b'. That is, b' is maximal such that $b' \wedge a = \theta$. Now to show, a is θ-complement of b', let c be maximal with respect to $a \leq c$ and $b' \wedge c = \theta$. Then by Lemma 2.26, c is maximal with respect to $a \leq_\theta^e c$. But since a is θ-closed in L, we get $a = c$. Therefore, a is θ-complement of b'. □

Proposition 2.28. Let L be θ-complemented. For any $b, c \in L$, if $b \wedge c = \theta$, $b \vee c \leq_\theta^e L$, and c is θ-essentially closed, then c is a θ-complement of b.

Proof. Let $b \wedge c = \theta$, $b \vee c \leq_\theta^e L$ and c is θ-essentially closed. In view of Lemma 2.24, it is enough to show $b \vee c \leq_\theta^e [c, 1]$. In a contrary, suppose that $(b \vee c) \wedge d = \theta$, for $\theta \neq d \in [c, 1]$. Now, $(b \vee c) \wedge d = \theta \leq c$ and $c \leq (b \vee c) \wedge d$. Therefore, $(b \vee c) \wedge d = c$. Since c is θ-closed, there exists $x \in L$ such that $\theta < x < d$, and $c \wedge x = \theta$. Then,

$$\begin{aligned} \theta &= c \wedge x \\ &= ((b \vee c) \wedge d) \wedge x \\ &= (b \vee c) \wedge (d \wedge x) \\ &= (b \vee c) \wedge x, \end{aligned}$$

a contradiction to $(b \vee c) \leq_\theta^e L$. □

Notation 2.29. If $a \wedge b = \theta$ and $(a \vee b) \wedge c = \theta$, then $a \wedge (b \vee c) = \theta$.

Proof. Let $a \wedge b = \theta$ and $(a \vee b) \wedge c = \theta$. Then,

$$\begin{aligned} a \wedge (b \vee c) &\leq (a \vee b) \wedge (b \vee c) \\ &= ((a \vee b) \wedge c) \vee b, \text{ by modular law} \\ &= \theta \vee b \\ &= b. \end{aligned}$$

Hence $a \wedge (b \vee c) \leq a \wedge b = \theta$. Also, $\theta \leq a$, $\theta \leq b \leq (b \vee c)$, imply $\theta \leq a \wedge (b \vee c)$. Therefore, $a \wedge (b \vee c) = \theta$. □

Proposition 2.30. Let c, b be θ-complements of b, a respectively in L such that $a \leq c$. Then

(1) b is a θ-complement of c in L; and $b \vee c \leq^e_\theta [b, 1]$;
(2) $a \leq^e_\theta c$.

Proof. (1) Suppose b is maximal with respect to $b \wedge a = \theta$. Let $d \in L$ and $\theta \leq b < d$ such that $c \wedge d = \theta$. Then $a \wedge d \leq c \wedge d = \theta$. Also, since $\theta \leq a \wedge d$, we get $a \wedge d = \theta$, a contradiction to the maximality of b. Thus, b is θ-complement of c in L. Now, by Lemma 2.24, we get $b \vee c \leq^e_\theta [b, 1]$.

(2) To show, $a \leq^e_\theta c$, let $a \wedge d = \theta$, where $d \in [\theta, c]$. Now, $(a \vee d) \wedge b \leq c \wedge b = \theta$. Also, $\theta \leq a \leq (a \vee d)$, $\theta \leq b$, implies $\theta \leq (a \vee d) \wedge b$. Therefore, $(a \vee d) \wedge b = \theta$. Then by Note 2.29, we have $a \wedge (d \vee b) = \theta$. Now, by maximality of b, we get $d \vee b = b$. Therefore, $d \leq b$ and $d \leq b \wedge c = \theta$, shows that $d = \theta$.

□

Proposition 2.31. Let b be a θ-complement of a in L. If $\theta < c \leq^e_\theta L$, then $b \vee c \leq^e_\theta [b, 1]$.

Proof. Let $d \in [b, 1]$ such that $(b \vee c) \wedge d = \theta$. Then clearly, $(b \vee c) \wedge d = \theta \leq b$, and $b \leq (b \vee c) \wedge d$, implies $(b \vee c) \wedge d = b$. Now, by modular law $b = (b \vee c) \wedge d = b \vee (c \wedge d)$, and so $c \wedge d \leq b$. Then, $a \wedge (c \wedge d) \leq a \wedge b = \theta$. Also, $\theta \leq a$, $\theta \leq c \leq (c \wedge d)$ implies $c \wedge (a \wedge d) = a \wedge (c \wedge d) = \theta$. Since, $c \leq^e_\theta L$, we get $a \wedge d = \theta$. Then by the maximality of b, we get $d = b = \theta$, and shows $b \vee c \leq^e_\theta [b, 1]$. □

Lemma 2.32. Let $1 \in L$, $b < a$ in L and $a \leq^e_\theta [b, 1]$. Then $a \wedge c \leq^e_\theta [b \wedge c, c]$, for all $c \in L$.

Proof. Suppose $(a \wedge c) \wedge x = \theta$, where $x \in [b \wedge c, c]$. Now, taking the join with b on both side we get $[a \wedge (c \wedge x)] \vee b = \theta \vee b = \theta$, since $\theta \in [b, 1]$. By modular law, since $b < a$, we have $a \wedge [(c \wedge x) \vee b] = \theta$. Since $a \leq^e_\theta [b, 1]$ and $(c \wedge x) \vee b \in [b, 1]$, we get $(c \wedge x) \vee b = \theta$. Since $x \leq c$, $x \vee b = \theta$, and hence $x \leq \theta$. Also, $\theta \leq x$. Therefore, $x = \theta$, as desired. □

Lemma 2.33. [19] Let $\theta < b < a$ be in L. Then, $a \leq^e_\theta L$ and $b \leq^e_\theta [\theta, a]$ if and only if $b \leq^e_\theta L$.

Notation 2.34. Let x, y be elements of L. If $x \vee y \leq^e_\theta L$, then $x \vee y \in [\theta, 1]$.

Lemma 2.35. Let $1 \in L$. If L is θ-complemented, then for every $a \in L$, $[\theta, a]$ is also θ-complemented.

Proof. Suppose L is θ-complemented. Let $x \in [\theta, a] \subseteq L$. By Corollary 2.25, there exists $y \in L$ such that $x \wedge y = \theta$ and $x \vee y \leq_\theta^e L$. Then by Note 2.34, we have $x \vee y \leq_\theta^e [\theta, 1]$. Now, $a \wedge (x \wedge y) = a \wedge \theta = \theta$, implies $x \wedge (y \wedge a) = \theta$. Then, by Lemma 2.32, $(x \vee y) \wedge a \leq_\theta^e [(\theta \wedge a), 1 \wedge a] = [\theta, a]$. Since $x \leq a$, by modular law we get $x \vee (y \wedge a) \leq_\theta^e [\theta, a]$. Therefore, $[\theta, a]$ is complemented. □

Proposition 2.36. If $[\theta, a]$ is θ-complemented in L, for some $a \leq_\theta^e L$, then L is also θ-complemented.

Proof. Let $[\theta, a]$ be θ-complemented. For $x \in L$, $x \wedge a \in [\theta, a]$ has a θ-complement in $[\theta, a]$, say y. Then by Corollary 2.25, we have $y \wedge (x \wedge a) = \theta$, and $y \vee (x \wedge a) \leq_\theta^e [\theta, a]$. By Lemma 2.33, $y \vee (x \wedge a) \leq_\theta^e L$. Now to show $y \vee x \leq_\theta^e L$, let $z \in L$ such that $(y \vee x) \wedge b = \theta$. Then, $[(y \vee x) \wedge a] \wedge b \leq (y \vee x) \wedge b = \theta$. Also, since $\theta \leq b$, $\theta \leq [y \vee (x \wedge a)]$, implies $\theta \leq [y \vee (x \wedge a)] \wedge b$. Hence, $[y \vee (x \wedge a)] \wedge b = \theta$. Since $y \vee (x \wedge a) \leq_\theta^e L$, we get $b = \theta$. Thus $y \vee x \leq_\theta^e L$, shows that y is θ-complement of x in L. □

Proposition 2.37. Let L is θ-complemented and $1 \in L$. If a is a θ-complement in L, then $[a, 1]$ is also θ-complemented.

Proof. Suppose a is a θ-complement of b in L. Then, by Lemma 2.24, we have $a \wedge b = \theta$, and $a \vee b \leq_\theta^e [a, 1]$, and by Lemma 2.35, $[\theta, b]$ is also θ-complemented. Now, $[a, a \vee b] \cong [a \wedge b, b] = [\theta, b]$, is θ-complemented. That is, $[a, a \vee b]$ is θ-complemented. Therefore by Proposition 2.36, $[a, 1]$ is θ-complemented. □

Theorem 2.38. If b is a θ-complement of a in L and $\theta < c \leq_\theta^e L$, then b is a θ-complement of $a \wedge c$ in L.

Proof. Let b be a θ-complement of a in L. Then $a \wedge b = \theta$ and $a \vee b \leq_\theta^e L$. Clearly, $(a \wedge c) \wedge b = \theta$. Now let $d = a \vee b$ and $u = (a \wedge c) \vee b$. To show $u \leq_\theta^e L$, let $u \wedge y = \theta$, for some $y \in L$. Let $x = y \wedge d$. Now, $u \wedge x = u \wedge (y \wedge d) = (u \wedge y) \wedge d = \theta \wedge d = \theta$. Since $b \leq u$ by modular law, $u \wedge (b \vee x) = b \vee (u \wedge x) b \vee \theta = b$, and

$$\begin{aligned} \theta &= a \wedge b \\ &= a \wedge [u \wedge (b \vee x)] \\ &= a \wedge [(a \wedge c) \vee b] \wedge (b \vee x). \end{aligned}$$

Now since $a \wedge c \leq a$ and by modularity we get $\theta = [(a \wedge c) \vee (a \wedge b)] \wedge (b \vee x) = (a \wedge c) \wedge (b \vee x)$. Since $c \leq_\theta^e L$, we get $a \wedge (b \vee x) = \theta$. Then,

$$\begin{aligned} b &= b \vee \theta \\ &= b \vee [a \wedge (b \vee x)] \\ &= (b \vee x) \wedge (b \vee a), \text{ by modular law, } b \leq b \vee x \\ &= b \vee x, \end{aligned}$$

which implies $x \leq b$. Now $\theta = u \wedge x \geq b \wedge x = x$. Also, $\theta \leq x$, implies $x = \theta$. Since $d \leq_\theta^e L$, we get $y = \theta$. Thus, b is θ-complement of $a \wedge c$ in L.

□

Conclusions

We have defined the module theoretical concepts such as θ-complement, θ-closed and relative θ-complemented in a lattice. In a modular lattice, we have proved characterizations involving θ-complements with necessary illustrations. The results can be extended to study the dual aspects like supplements, superfluous and radicals etc. in a lattice. Possibly, one can study the concepts in hyperlattices, as the authors explored several hyperstructural aspects of lattices in [17, 14].

Acknowledgment

All the authors thank the referees for their careful reading and suggestions. A part of this research work is carried out during the collaborative visit of author[2] at Manipal Institute of Technology, Manipal under the award number INSA/SP/VSP-56/2023-24/. The author[2] acknowledges MIT Bengaluru, Manipal Academy of Higher Education, Manipal, and the authors[1,3,4] acknowledge Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, India for their kind encouragement. The author Harikrishnan P.K. acknowledges SERB, Govt. of India for the Teachers Associateship for Research Excellence (TARE) fellowship TAR/2022/000219.

References

[1] Alizade R, Toksoy S (2009) Cofinitely weak supplemented lattices. Indian J. Pure Appl. Math. **40**:337-346.

[2] Anderson FW, Fuller KR (1992) Rings and Categories of Modules Graduate Texts in Mathematics (Springer-Verlag New York).

[3] Bhavanari S (1990) On modules with finite Goldie Dimension. J. Ramanujan Math. Soc. **5**(1):61-75.

[4] Bhavanari S, Kuncham SP (2003) An Isomorphism Theorem on Directed Hypercubes of dimension n. Indian J. pure appl. Math. **34**(10):1453-1457.

[5] Bhavanari S, Kuncham SP (2014) Discrete Mathematics and Graph theory (Prentice Hall learning Ltd. Second edition).

[6] Calugareanu G (2003) Lattice concepts of module theory 22 (Springer Science & Business Media).

[7] Chajda I, Länger H (2021) Filters and congruences in sectionally pseudocomplemented lattices and posets. Soft Computing. **25**:8827–8837.

[8] Gratzer G (2011) Lattice theory: foundation Springer Science & Business Media.

[9] Grzeszczuk P, Puczyłowski ER (1984) On Goldie and dual Goldie dimensions. J. Pure Appl. Algebra. **31**(1-3):47-54.

[10] Goldie AW (1972) The structure of Noetherian rings In Lectures on Rings and Modules. (Springer), Berlin, Heidelberg, 246:213-321.

[11] Lopez AF, Rus EG (1998) Algebraic lattices and nonassociative structures. Proceedings of the American Mathematical Society. 3211-3221.

[12] Nimbhorkar SK, Shroff RC (2017) Goldie extending elements in modular lattices. Math. Bohem. **142**(2):163-180.

[13] Nimbhorkar S, Deshmukh V (2020) The essential element graph of a lattice. Asian-Eur. J. Math. **13**(01):2050023 (9 pages). https://doi.org/10.1142/S1793557120500230

[14] Pallavi P, Kuncham SP, Vadiraja GRB, Harikrishnan PK (2022) Computation of prime hyperideals in meet hyperlattices, Bull. Comput. Appl. Math. **10**(01):33-58.

[15] Phadatare N, Vilas K, Sachin B (2019) Semi-complement graph of lattice module. Soft Computing **23**:3973-3978.

[16] Rao GN, Beyene TG (2019) Relative Annihilators and Filters in Almost Semilattice. Southeast Asian Bull. Math. **43**(4):553–576.

[17] Rasouli S, Davvaz B (2010) Lattices derived from hyperlattices. Comm. Algebra **38**(8):2720-2737.

[18] Saki A, Kiani D (2021) Complemented lattices of subracks. Journal of Algebraic Combinatorics, **53**(2): 455-468.

[19] Tapatee S, Kedukodi BS, Shum KP, Harikrishnan PK, Kuncham SP (2021) On essential elements in a lattice and Goldie analogue theorem. Asian-Eur. J. Math. **15**(05):2250091. https://doi.org/10.1142/S1793557122500917

[20] Tapatee S, Davvaz B, Harikrishnan PK, Kedukodi BS, Kuncham SP (2021) Relative Essential Ideals in N-groups. Tamkang J. Math. **54**(01):69-82. https://doi.org/10.5556/j.tkjm.54.2023.4136

[21] Tapatee S, Kedukodi BS, Harikrishnan PK, Kuncham SP (2021) On the finite Goldie dimension of sum of two ideals of an R-group, Discuss. Math. Gen. Algebra Appl. **43**(02):177-187. https://doi.org/10.7151/dmgaa.1419

[22] Tapatee S, Harikrishnan PK, Kedukodi BS, Kuncham SP (2022) Graph with respect to superfluous elements in a lattice. Miskolc Math. Notes **23**(02):929-945. DOI:10.18514/MMN.2022.3620

[23] Tapatee S, Meyer JH, Harikrishnan PK, Kedukodi BS, Kuncham SP (2022) Partial Order in Matrix Nearrings. Bull. Iranian Math. Soc. **48**(06):3195-3209. https://doi.org/10.1007/s41980-022-00689-w

Kuncham S.P.: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education,576104, India

Email address: `syamprasad.k@manipal.edu`

Tapatee S.: Department of Mathematics, Manipal Institute of Technology Bengaluru, Manipal Academy of Higher Education,India

Email address: `sahoo.tapatee@manipal.edu`

Sridhara K. B.: Department of Mathematics, Manipal Institute of Technology Bengaluru, Manipal Academy of Higher Education,India

Email address: `sridhara.mitblr2024@learner.manipal.edu`

Kedukodi B.S.: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education,576104, India

Email address: `babushrisrinivas.k@manipal.edu`

Harikrishnan P.K. : Department of Mathematics, Manipal Institute of Technology Manipal, Manipal Academy of Higher Education,576104, India

Email address: `pk.harikrishnan@manipal.edu`