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STOCHASTIC MODELING OF FLOWER POLLINATION AND SEED FERTILIZATION WITH EXPONENTIALLY DISTRIBUTED LIFE SPANS OF POLLEN AND OVULE

Abstract. The study considers the development of Stochastic Model for plant reproduction systems by describing the processes of pollen's spread and ovule's fertilization among self-pollinated flowering plants. It intends to assess the rates of fertilization of the ovule with a successful interaction with pollen by studying the stochastic processes like spread and transitions of pollen to stigma, passage of pollen through pollen tube, interaction of pollen grain with ovule after logging on to stigma,, etc. The life spans of the pollen grain and the ovule are assumed to be random variables and they follow truncated exponential distribution. This model will measure the chance of successful mating of male and female gametes. The joint probability distribution for fertilization due to the successful interaction of both pollen (male gamete) and ovule (female gamete) is obtained through bi-variate truncated exponential distribution. Mathematical formulae for different statistical measures are derived. Model behavior is analyzed through numerical illustrations.

Mathematics Subject Classification 2020. Primary: 62E115; Secondary: 62P10, 93A30

1. Introduction

Flowers are the reproductive sites in plants referred to angiosperms. They have more varied structures of organisms. Because of this variation, plant's sexual diversity has a long and venerable history in biology. A filament is a long stalk that connects to and holds up the anther. Production of fully-grown pollen (stamen) will be in a sac-like structure in anther located at the tip of the filament. Stamen is a male portion of a plant that produces sperm and housed within pollen. Pistil is the female reproductive part of a flower. It consists of stigma, style, and ovary. The tip of the pistil is the stigma and is sticky so as it can collect and withhold pollen on its upper place. This slender, neck-like portion of the pistil provides a pathway for sperm to the ovary. The ovary is located at the base of the pistil and houses the ovules. The processes of transitions of pollen grains from anther to stigma consist of several acts. The times of full pledged pollen and ovule for the reproduction processes are random and highly influenced by several uncertainty factors. Understanding the dynamics of pollen transitions and the successful conversion of ovule as seed after fertilization is feasible with proper formulation of mathematical modeling. The numbers of pollen grains in anther for reproduction, the number of ovules in the stack, the timings of transitions of pollen and ovules,

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2000 *Mathematics Subject Classification.*

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etc. are the random variables for study. Assessment of success rate of the pollination and fertilization through mathematical models will explore the real insights of plant reproduction system.

1.1. Reported Research. Darwin explained the plant reproductive biology, and Fisher provided the population genetic principles for understanding the evolution of plant mating systems. Flowering plants have an unrivaled diversity of sexual methods [3]. Change in reproductive structures of plants is the basis for this classification[1]. Many flowering plants rely either on external pollinators, or on self-pollination, or on both for reproduction. The models for plant mating system based on population dynamics include an allele effect. Pollen and seed vectors influence the genetic structure of plant populations. When the wind is the dispersing agent, pollen and seed movement are relatively straightforward and described by leptokurtic distributions, with most of the pollen grains or seeds transported to short distances and tails extending long distances away from the source. The evidenced literature revealed the human interest in the mechanism of pollination and fertilization in plants is that of a relief carving from the palace of Syrian King Ashuirnasir-pal II, 883859 BC. It states that a winged man with a male inflorescence from a palm tree distributing pollen over a female inflorescence[8].

It was observed that pollen tubes germinating on a stigma. The pollen tube carried the sperm cells to the ovule where the egg resided. The enclosure of the egg is inside the ovule within the ovary. Fertilization is a fusion of egg and sperm nuclei in the flowering plants (angiosperms)[5]. Double fertilization was recognized as well, a feature unique to flowering plants. A major review has been carried out for covering the molecular and genetic basis of pollen/pistil interaction. The pollination and fertilization processes in flowering plants will be happened when pollen grains germinate to form pollen tubes that transport male gametes (sperm cells) to the egg cell in the embryo sac during sexual reproduction. Pollen tube cells elongate on active extracellular matrix in the style [10]. Several models addressed the plant population dynamics, pollinator for aging and the selection of self-fertilization [6]. The consequences of the features for plant population dynamics and mating system evolution and evolution of self-fertilization are vital in Meta populations. The selection of mating system modifies genes within and among populations was studied through pollination model. Levels of selection play a key role in the evolution of the mating system. Selection among populations could maintain out crossing through higher extinction rates of selfing groups [7].

The flower fertility and factors influencing seed production in winter oilseed rape are studied with a probabilistic model. It has investigated the variability in the yields of ovules per pod, seeds per pod and pod per axis in relation to the variables namely flower and inflorescence position and time of pod appearance. Probability distribution of the number of pollen grains per stigma is deduced thorough this model. The number of ovules per ovary and seeds per pod linked with the flower fertility were also simulated with the derived distribution [9]. A phenotypic model for the evaluation of ovule number per flower incorporated the aerodynamics of pollen capture. Studies established fixed resource pool for provisioning of flowers, ovules, and seeds [2]. The plant pollination and dispersal linked with ecology and environment. Plants are stationary and depend on external agencies to help them reproduce and disperse their seeds. They depend on animal pollinators and seed dispersers, although in specific ecosystems some plants can use wind or water for such transport. In order to attract animal vectors, plants use various food rewards including pollen, nectar, seeds, and fruits. In terms of species numbers, insects are the major pollinators, and vertebrates are the major seed dispersers [4].

1.2. Research Gap and the Motivation of the Study. The reported research studies on pollination and fertilization have given much emphasis on biological processing and addressing the flower pathology related to interception of pollen and ovule. They have discussed much on the sperm cell journey towards the ovule by creating a pollen tube by the pollen grain once it logon the stigma. The happening of pollinations is more described with different pollination mediums. The focus is more on describing the biological processes for pollination and seed fertilization. It is evidenced that there is very thin mentioning about usage of mathematical modeling for assessing the successful pollination among plants. There is no visible attempt on mathematical formulation for understanding the spread mechanism of pollen grains. The issues on different stochastic transition stages of pollen grain in the process of seed fertilization, mating mechanism of male and female plant gametes, etc. are still needs the attention of the researchers of mathematical biology.

The processes of pollination are influenced mostly by random factors. Success of the processes of both pollination and fertilization are based on the accuracy in mating of male and female gametes in the ovule. Proper modeling of structures will address the fertilization well when the randomness in transitions from one stage to another stage is competing for successful mating with their counterparts. For proper understanding of seed fertilization mechanism, we need to understand the process of pollen grain journey as reproductive male gamete. The happenings of pollen grain dispersion from stamen to stigma, the survival span of the pollen grain on the stigma, the survival time of the pollen grain in pollen tube, the ability of successful joining of sperm cell to the egg cell in ovule, etc. are very vital for mathematical modeling. In order to get the rate of pollen grain mating with ovule so as it leads to successful seeds formation, we need to model the heterogeneous transitions. This sort of context motivated us to develop stochastic models for measuring the success rate of seed fertilization. This model will explore the indicators on success rate of seed fertilizations through development of suitable probability distributions. The study aims (i) to develop stochastic model for fertility successes and seed formations; (ii) to study different random processes like spread of pollen grain, release of ovule, interactions of pollen grain with ovule; (iii) to deduce the joint probability distribution of combined processes ovule release and pollen contact with ovule; (iv) to derive mathematical formulae for moments; (v) to study mathematical properties of the formulated model; and (vi) to conduct the sensitivity analysis and verification of model validity.

2. Stochastic Model

This section deals with the development of stochastic model of pollen pollination and ovule fertilization after thorough alignment of different biological issues with mathematical assumptions. The individual and joint processes of pollination and fertilization will explain with suitable probabilistic models.

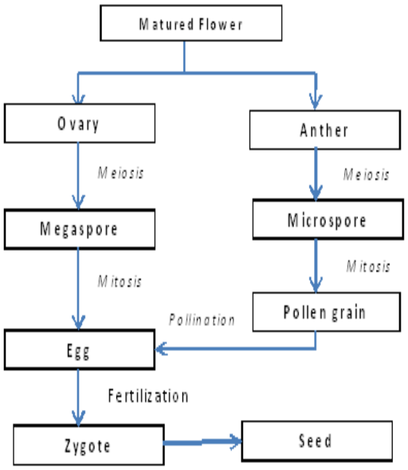

FIGURE 1. Schematic diagram of Pollination and Fertilization

2.1. Pollination and Fertilization Processes. Pollen grain is a full pledged male gamete of the reproduction process. It will be released from anther at any time in between t_{11} and t_{12}. The logging of pollen grain on the stigma (the female reproductive part is on the tip of the pistil) is random and influenced by chance factors. Therefore, the event of reaching the pollen to the stigma is uncertain. The processes of Meiosis, Mitosis to the pollen grains will make the pollen as a full pledged male gamete for reproduction activity. The waiting time of pollen from its logon to stigma and up to joining the egg in ovule is having the length of a units of time. It is a partial time and sub set of total life span of the pollen grain. It is studied with a random variable on X-axis. Ovule in the ovary will also be prepared as full pledged female gamete for reproduction process after Meiosis and Mitosis. It will bear the egg in ovule and the seed vector of ovules in Ovary. The time interval of release of ovule will be in between t_{21} and t_{22}. The actual time length of release of egg in ovule is b units of time. Every released pollen grain may or may not reach to the stigma. However, if it reaches to stigma, it will stay there in the pistil up to 'a' units of time for successful mating with ovule. Therefore the time from logging of pollen grains on stigma and it will be in the reproduction process either for successful mating with ovule or it may discard from the reproduction activity. Successful fertilization will be happened if the release time of egg in ovule will coincide with the time of logging of the pollen grain on stigma. If these two events of happening are non-overlapping, then there may be a failure of fertilization.

2.2. Notation and Terminology.

X	The random variable representing the total life span of the pollen grain (male gamete)
t_{11}	The initial time of release of prepared pollen grain (which is suitable for reproduction process).
t_{12}	End time of the life span of pollen grains.

a	The survival time of the pollen grain during the period from logon to the stigma to either its death or its successful mating with female gametes.
Y	The random variable representing the time span of opening of stigma (prepared for fertilization)situated on the style of flower and up to release of ovule in the ovary (a female gamete).
t_{21}	The initial time that the process of prepared (full pledged) stigma to hold the pollen grain for further reproduction activity.
t_{22}	The end time that the process of prepared (full pledged) stigma to make reproduction activity due to stopping of fertilization.
b	The life span/ access time of the ovule during which it can participate in the successful reproduction process.
Z	Random variable (index) by taking values 1 and 0.
$Z = 0$	release of the pollen grain during $(t_{12} - t_{11}) - a$ and release of ovule during$(t_{22} - t_{21}) - b$ failure of fertility.
$Z = 1$	successful fertility(complementary period with $Z = 0$).
μ	= location parameter for pollen's life expectancy distribution
θ	= location parameter for ovule's life expectancy distribution

2.3. Probability Distribution. Let Z be a random variable representing the intersection time of the joint event. It is equivalent to happening of both the events of reaching pollen grain to stigma and release of egg in the ovule in ovary. Z can be expressed as an indicator variable such that, Z=1 when the mating of male gametophyte (pollen grain) and female gametophyte (egg) become successful fertilization and Z=0, when the mating of male gametophyte and female gametophyte become failure of fertilization.

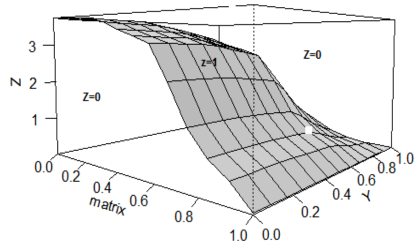

FIGURE 2. Schematic diagram

2.4. Mathematical formulae. It is known that the life spans of both the pollen grain (after releasing from the anther) and the ovule (after releasing from ovary) are finite as they will get expired after a specific time. Usually the life span of pollen is more than that of ovule. Further, the chances of the survival of both male gamete (pollen) and female gamete (ovule) are significantly more in the beginning and their survival probabilities will be decreasing gradually and become very small and tend to zero after its life time. Hence, it is appropriate to consider that the chance of life times of the reproductive gametes follows truncated exponential distributions with parameters $\mu and \theta$ respectively. The pollen's stay time in the ovary (after making a pollen tube from the stigma) is considered to be 'a' and the ovule's stay time in the ovary (between its being released from the stack of ovary up to either successful mating or getting death without mating with pollen) is considered to be 'b'. Though the values are considered to be variables, usually they are of constant lengths of time. Hence, 'a' and 'b' are the time epochs form their arrival to the ovary and up to the successful mating with their counterparts.

The Joint Probability Distribution of Pollen Grain and Ovule:

$$P(Z=i)=\begin{cases} P\left[\{(t_{11}+a)\leq X\leq t_{12}\}\cap\{(t_{21}+b)\leq Y\leq t_{22}\}\right] & ;\ i=0\\ 1-P\left[\{(t_{11}+a)\leq X\leq t_{12}\}\cap\{(t_{21}+b)\leq Y\leq t_{22}\}\right] & ;\ i=1\end{cases}$$

$$P(Z=i)=\begin{cases} \left[e^{-t_{12}\theta}-e^{-t_{11}\theta-a\theta}\right]\left[e^{-t_{22}\mu}-e^{-t_{21}\mu-b\mu}\right]\left[(1-e^{-a\theta})(1-e^{-b\mu})\right]^{-1} & ;\ i=0\\ \left[(1-e^{-a\theta})(1-e^{-b\mu})\right]-\left[e^{-t_{12}\theta}-e^{-t_{11}\theta-a\theta}\right]\left[e^{-t_{22}\mu}-e^{-t_{21}\mu-b\mu}\right]\left[(1-e^{-a\theta})(1-e^{-b\mu})\right]^{-1} & ;\ i=1\end{cases} \tag{2.1}$$

Average Successful Fertilization is as the following

$$mean=\left[1-e^{-a\theta}-e^{-b\mu}+e^{-a\theta-b\mu}-e^{-t_{12}\theta-t_{22}\mu}+e^{-t_{12}\theta-t_{21}\mu-b\mu}+e^{-t_{11}\theta-a\theta-t_{22}\mu}-e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right] \left[1-e^{-a\theta}-e^{-b\mu}+e^{-a\theta-b\mu}\right]^{-1} \tag{2.2}$$

Variance of the mating of male and female gametophyte is given below

$$variance=\left[1-e^{-a\theta}-e^{-b\mu}+e^{-a\theta-b\mu}-e^{-t_{12}\theta-t_{22}\mu}+e^{-t_{12}\theta-t_{21}\mu-b\mu}+e^{-t_{11}\theta-a\theta-t_{22}\mu}-e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right] \left[1-e^{-a\theta}-e^{-b\mu}+e^{-a\theta-b\mu}\right]^{-2}\left[e^{-t_{12}\theta-t_{22}\mu}-e^{-t_{12}\theta-t_{21}\mu-b\mu}-e^{-t_{11}\theta-a\theta-t_{22}\mu}+e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right] \tag{2.3}$$

The third and fourth central moments are given below

$$\begin{aligned}\mu_3=&\left[1-e^{-a\theta}-e^{-b\mu}+e^{-a\theta-b\mu}-e^{-t_{12}\theta-t_{22}\mu}+e^{-t_{12}\theta-t_{21}\mu-b\mu}+e^{-t_{11}\theta-a\theta-t_{22}\mu}-e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]\\ &\left[1-e^{-a\theta}-e^{-b\mu}+e^{-(a\theta+b\mu)}\right]^{-3}\left[1-e^{-a\theta}-e^{-b\mu}+e^{-(a\theta+b\mu)}\right]^{2}-3\left[1-e^{-a\theta}-e^{-b\mu}+e^{-(a\theta+b\mu)}\right]\\ &\left[1-e^{-a\theta}-e^{-b\mu}+e^{-a\theta-b\mu}-e^{-t_{12}\theta-t_{22}\mu}+e^{-t_{12}\theta-t_{21}\mu-b\mu}+e^{-t_{11}\theta-a\theta-t_{22}\mu}-e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]\\ &+2\left[1-e^{-a\theta}-e^{-b\mu}+e^{-a\theta-b\mu}-e^{-t_{12}\theta-t_{22}\mu}+e^{-t_{12}\theta-t_{21}\mu-b\mu}+e^{-t_{11}\theta-a\theta-t_{22}\mu}-e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]^{2}\end{aligned} \tag{2.4}$$

$$\begin{aligned}\mu_4=&\left[1-e^{-a\theta}-e^{-b\mu}+e^{-a\theta-b\mu}-e^{-t_{12}\theta-t_{22}\mu}+e^{-t_{12}\theta-t_{21}\mu-b\mu}+e^{-t_{11}\theta-a\theta-t_{22}\mu}-e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]\\ &\left[1-e^{-a\theta}-e^{-b\mu}+e^{-(a\theta+b\mu)}\right]^{-4}\left[1-e^{-a\theta}-e^{-b\mu}+e^{-(a\theta+b\mu)}\right]^{3}-4\left[1-e^{-a\theta}-e^{-b\mu}+e^{-(a\theta+b\mu)}\right]^{2}\\ &\left[1-e^{-a\theta}-e^{-b\mu}+e^{-a\theta-b\mu}-e^{-t_{12}\theta-t_{22}\mu}+e^{-t_{12}\theta-t_{21}\mu-b\mu}+e^{-t_{11}\theta-a\theta-t_{22}\mu}-e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]\\ &+6\left[1-e^{-a\theta}-e^{-b\mu}+e^{-(a\theta+b\mu)}\right]\\ &\left[1-e^{-a\theta}-e^{-b\mu}+e^{-a\theta-b\mu}-e^{-t_{12}\theta-t_{22}\mu}+e^{-t_{12}\theta-t_{21}\mu-b\mu}+e^{-t_{11}\theta-a\theta-t_{22}\mu}-e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]^{2}\\ &-3\left[1-e^{-a\theta}-e^{-b\mu}+e^{-a\theta-b\mu}-e^{-t_{12}\theta-t_{22}\mu}+e^{-t_{12}\theta-t_{21}\mu-b\mu}+e^{-t_{11}\theta-a\theta-t_{22}\mu}-e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]^{3}\end{aligned} \tag{2.5}$$

The shape and peakedness measures of the successful mating of male and female gametophyte is given below

$$\begin{aligned}
\beta_1 = {} & \left[1 - e^{-a\theta} - e^{-b\mu} + e^{-(a\theta+b\mu)}\right]^4 - 3\left[1 - e^{-a\theta} - e^{-b\mu} + e^{-(a\theta+b\mu)}\right]^2 \\
& \left[1 - e^{-a\theta} - e^{-b\mu} + e^{-a\theta-b\mu} - e^{-t_{12}\theta-t_{22}\mu} + e^{-t_{12}\theta-t_{21}\mu-b\mu} + e^{-t_{11}\theta-a\theta-t_{22}\mu} - e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]^{-1} \\
& \left[1 - e^{-a\theta} - e^{-b\mu} + e^{-a\theta-b\mu} - e^{-t_{12}\theta-t_{22}\mu} + e^{-t_{12}\theta-t_{21}\mu-b\mu} + e^{-t_{11}\theta-a\theta-t_{22}\mu} - e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]^{2} \\
& \left[e^{-t_{12}\theta-t_{22}\mu} + e^{-t_{12}\theta-t_{21}\mu-b\mu} + e^{-t_{11}\theta-a\theta-t_{22}\mu} - e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]^{-1} \\
& + 2\left[1 - e^{-a\theta} - e^{-b\mu} + e^{-a\theta-b\mu} - e^{-t_{12}\theta-t_{22}\mu} + e^{-t_{12}\theta-t_{21}\mu-b\mu} + e^{-t_{11}\theta-a\theta-t_{22}\mu} - e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]^{4}
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
\beta_2 = {} & \left[1 - e^{-a\theta} - e^{-b\mu} + e^{-(a\theta+b\mu)}\right]^3 - 4\left[1 - e^{-a\theta} - e^{-b\mu} + e^{-(a\theta+b\mu)}\right] \\
& \left[1 - e^{-a\theta} - e^{-b\mu} + e^{-a\theta-b\mu} - e^{-t_{12}\theta-t_{22}\mu} + e^{-t_{12}\theta-t_{21}\mu-b\mu} + e^{-t_{11}\theta-a\theta-t_{22}\mu} - e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]^{-1} \\
& \left[e^{-t_{12}\theta-t_{22}\mu} + e^{-t_{12}\theta-t_{21}\mu-b\mu} + e^{-t_{11}\theta-a\theta-t_{22}\mu} - e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]^{-1} \\
& \left[1 - e^{-a\theta} - e^{-b\mu} + e^{-a\theta-b\mu} - e^{-t_{12}\theta-t_{22}\mu} + e^{-t_{12}\theta-t_{21}\mu-b\mu} + e^{-t_{11}\theta-a\theta-t_{22}\mu} - e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right] \\
& + 6\left[1 - e^{-a\theta} - e^{-b\mu} + e^{-(a\theta+b\mu)}\right] \\
& \left[1 - e^{-a\theta} - e^{-b\mu} + e^{-a\theta-b\mu} - e^{-t_{12}\theta-t_{22}\mu} + e^{-t_{12}\theta-t_{21}\mu-b\mu} + e^{-t_{11}\theta-a\theta-t_{22}\mu} - e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]^{2} \\
& - 3\left[1 - e^{-a\theta} - e^{-b\mu} + e^{-a\theta-b\mu} - e^{-t_{12}\theta-t_{22}\mu} + e^{-t_{12}\theta-t_{21}\mu-b\mu} + e^{-t_{11}\theta-a\theta-t_{22}\mu} - e^{-t_{11}\theta-a\theta-t_{21}\mu-b\mu}\right]^{3}
\end{aligned} \tag{2.7}$$

Lemma 2.1. *If the reproductive time of the pollen grain is equal to the reproductive time of ovule after reaching the stigma is equal to a, i.e.* $a = b = a$; $\theta = \mu = \theta$; $t_{11} \neq t_{21}$ *and* $t_{12} \neq t_{22}$, *then mean and variance is given below:*

$$\begin{aligned}
P(Z=0) &= \left[e^{-t_{12}\theta} - e^{-t_{11}\theta-a\theta}\right]\left[e^{-t_{22}\theta} - e^{-t_{21}\theta-a\theta}\right]\left[(1-e^{-a\theta})\right]^{-2} \\
P(Z=1) &= 1 - \left[(e^{-t_{12}\theta} - e^{-t_{11}\theta-a\theta})(e^{-t_{22}\theta} - e^{-t_{21}\theta-a\theta})\right]\left[(1-e^{-a\theta})\right]^{-2} \\
mean &= \left[1 - 2e^{-a\theta} + e^{-2a\theta} - e^{-2a\theta-t_{11}\theta-t_{21}\theta} + e^{-a\theta-t_{12}\theta-t_{21}\theta} + e^{-a\theta-t_{11}\theta-t_{22}\theta} - e^{-t_{12}\theta-t_{22}\theta}\right]\left[1 - 2e^{-a\theta} + e^{-2a\theta}\right]^{-1} \\
Variance &= \left[1 - 2e^{-a\theta} + e^{-2a\theta} - e^{-2a\theta-t_{11}\theta-t_{21}\theta} + e^{-a\theta-t_{12}\theta-t_{21}\theta} + e^{-a\theta-t_{11}\theta-t_{22}\theta} - e^{-t_{12}\theta-t_{22}\theta}\right]\left[-1 + e^{a\theta}\right]^{-4} \\
& \quad \left[e^{-2a\theta-t_{11}\theta-t_{21}\theta} - e^{-a\theta-t_{12}\theta-t_{21}\theta} - e^{-a\theta-t_{11}\theta-t_{22}\theta} + e^{-t_{12}\theta-t_{22}\theta}\right]
\end{aligned}$$

Lemma 2.2. *If both reproductive time and life time of the pollen grain and ovule is equal* $a = b = a; \theta = \mu = \theta; t_{11} = t_{21} = t_{11}; t_{12} = t_{22} = t_{12}$ *then*

$$\begin{aligned}
P(Z=0) &= \left[e^{-t_{12}\theta} - e^{-t_{11}\theta-a\theta}\right]^2\left[(1-e^{-a\theta})\right]^{-2} \\
P(Z=1) &= 1 - \left[e^{-t_{12}\theta} - e^{-t_{11}\theta-a\theta}\right]^2\left[(1-e^{-a\theta})\right]^{-2} \\
&= (1-e^{-a\theta})^2 - \left[e^{-t_{12}\theta} - e^{-t_{11}\theta-a\theta}\right]^2\left[(1-e^{-a\theta})\right]^{-2} \\
mean &= \left[1 - 2e^{-a\theta} + e^{-2a\theta} - e^{-2t_{12}\theta} + 2e^{-t_{12}\theta-t_{11}\theta-a\theta} - e^{-2t_{11}\theta-a\theta}\right]\left[1 - 2e^{-a\theta} + e^{-2a\theta}\right]^{-1}
\end{aligned}$$

$$\begin{aligned} variance = & \left[1 - 2e^{-a\theta} + e^{-2a\theta} - e^{-2t_{12}\theta} + 2e^{-t_{12}\theta - t_{11}\theta - a\theta} - e^{-2t_{11}\theta - a\theta}\right] \left[1 - 2e^{-2a\theta} + e^{-2a\theta}\right]^{-2} \\ & \left[2e^{-a\theta} - 2e^{-2a\theta} + e^{-2t_{12}\theta} - 2e^{-t_{12}\theta - t_{11}\theta - a\theta} + e^{-2t_{11}\theta - a\theta}\right] \end{aligned}$$

2.5. Generating Functions. Moment Generating Function is given below where

$$c = \frac{-\mu}{t} + t_{12}; \qquad d = \frac{-\mu}{t} - t_{11} + a$$

$$M_x(t) = \frac{e^{-\mu\theta}}{(1 - e^{-a\theta})(1 - e^{-b\mu})} \left[\sum_{n=1}^{\infty} \frac{(t(t_{22}) - \theta)^n}{nn!} (c - d)^n - \frac{(t(t_{21} + b) - \theta)^n}{nn!} (c + d)^n\right] \tag{2.8}$$

Characteristic Function is given below

$$\phi_x(t) = \frac{e^{-\mu\theta}}{(1 - e^{-a\theta})(1 - e^{-b\mu})} \left[\sum_{n=1}^{\infty} \frac{(it(t_{22}) - \theta)^n}{nn!} (c - d)^n - \frac{(it(t_{21} + b) - \theta)^n}{nn!} (c + d)^n\right] \tag{2.9}$$

Probability Generating Function is given below

$$P(s) = \frac{e^{-\mu\theta}}{(1 - e^{-a\theta})(1 - e^{-b\mu})} \left[\sum_{n=1}^{\infty} \frac{(s(t_{22}) - \theta)^n}{nn!} (c - d)^n - \frac{(s(t_{21} + b) - \theta)^n}{nn!} (c + d)^n\right] \tag{2.10}$$

3. Results and Discussion

In order to understand the model behavior in Layman point of view, we have simulated numerical data sets and observed the patterns of the probabilities by plotting graphs for changing value and fixed value of 'a' and 'b' are given below.

a	b	θ	μ	t_{11}	t_{12}	t_{21}	t_{22}	mean	variance	β_1	β_2
1	7	0.25	0.2	4	10	15	25	0.1651	0.1378	3.2559	4.2559
2	7	0.25	0.2	4	10	15	25	0.2952	0.2081	0.8063	1.8063
3	7	0.25	0.2	4	10	15	25	0.3966	0.2393	0.1789	1.1789
4	7	0.25	0.2	4	10	15	25	0.4755	0.2494	0.0096	1.0096
5	7	0.25	0.2	4	10	15	25	0.5370	0.2486	0.0220	1.0220
1	0.5	0.25	0.2	4	10	15	25	0.0101	0.0100	96.2802	97.2802
2	0.5	0.25	0.2	4	10	15	25	0.0289	0.0281	31.6376	32.6376
3	0.5	0.25	0.2	4	10	15	25	0.0436	0.0417	20.0058	21.0058
4	0.5	0.25	0.2	4	10	15	25	0.0550	0.0519	15.2501	16.2501
5	0.5	0.25	0.2	4	10	15	25	0.0639	0.0598	12.7275	13.7275
8	1	0.25	0.2	4	10	15	25	0.1550	0.1310	3.6333	4.6333
8	2	0.25	0.2	4	10	15	25	0.2837	0.2032	0.9205	1.9205
8	3	0.25	0.2	4	10	15	25	0.3891	0.2377	0.2070	1.2070
8	4	0.25	0.2	4	10	15	25	0.4754	0.2494	0.0097	1.0097
8	5	0.25	0.2	4	10	15	25	0.5460	0.2479	0.0341	1.0341
0.9	1	0.25	0.2	4	10	15	25	0.0265	0.0258	34.7234	35.7234
0.9	2	0.25	0.2	4	10	15	25	0.0586	0.0552	14.1269	15.1269
0.9	3	0.25	0.2	4	10	15	25	0.0849	0.0777	8.8769	9.8769
0.9	4	0.25	0.2	4	10	15	25	0.1064	0.0950	6.5211	7.5211
0.9	5	0.25	0.2	4	10	15	25	0.1240	0.1086	5.2086	6.2086

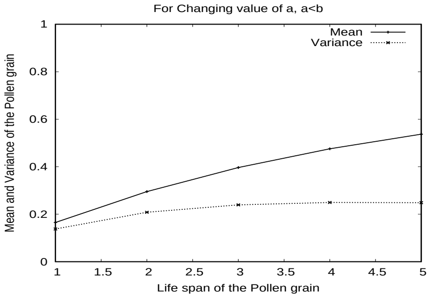

FIGURE 3. Changing value of a,a<b

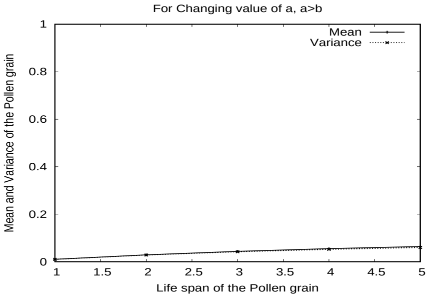

FIGURE 4. Changing value of a,a>b

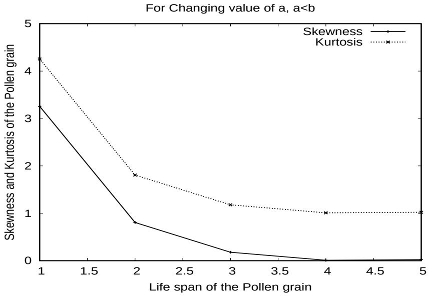

FIGURE 5. Changing value of a,a<b

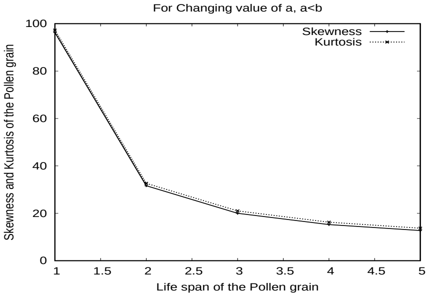

FIGURE 6. Changing value of a,a>b

Figure 3, 4, 5 and 6 represent the diagrammatic display of success rate of fertilization revealing the average, variance, skewness and kurtosis measures. The behavior of the said measures are observed with the changing value of a, for two cases namely (i) the reproductive time of pollen grain a is less than reproductive time of ovule b, (ii) the case where pollens reproductive time a is greater than ovules reproductive time b. Figure 3 and 4 reveals that average success rate is an increasing function of the life span of the pollen grain and the variance of success rate is a decreasing function of the life span of the pollen grain. This observation may interpret that the increased longevity of pollen grain will have the positive influence on the success rate of fertilization. It implies that the more time of stay of pollen in the pistil will have more likely to get succeeded in fertilization. Similarly, the display of variance measure reveals that the more life span of the pollen grain makes more consistency in the success of the fertility. Figure 5 and 6 displayed the information on shaping and peakedness behaviour of fertility success. It is observed that the right tail of the skewness curve is stretched more, interpret that average success rate of fertilization (mean) is more than the median success rate of fertilization due to pollens influence. Figure 7, 8, 9 and 10 exhibiting the similar pattern display with respect to the duration of active reproductive time of Ovule. We observed the behaviour of the model for changing values of b for two cases namely (i) the reproductive time of ovule is less than pollens reproductive time; (ii) the reproductive time of ovule is greater than pollens reproductivity time. Figure 7 and 8 denote the mating time of the pollen grain and ovule is increased when the effective reproductive time of the ovary is increased. The variance of the matting time is decreased in the reproductive time of the ovary. It conveys that the increased reproductive time of the ovule gives consistent increment in fertility success rate.

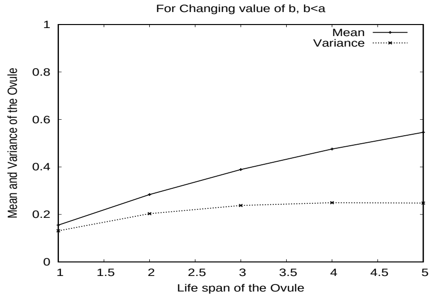

FIGURE 7. Changing value of b,b<a

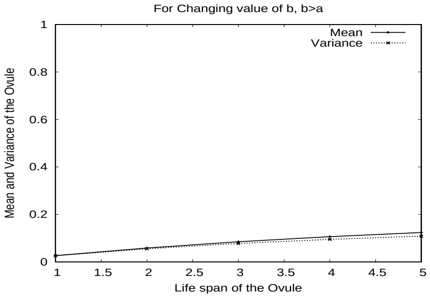

FIGURE 8. Changing value of b,b>a

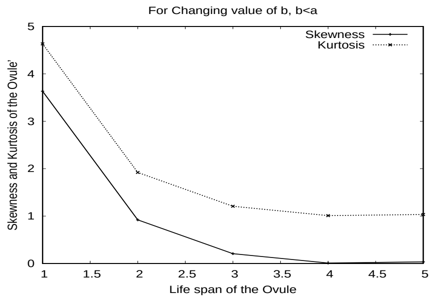

FIGURE 9. Changing value of b,b<a

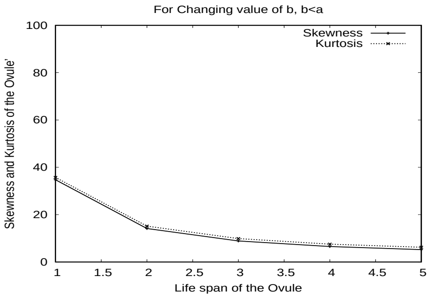

FIGURE 10. Changing value of b,b>a

Figures 9 and 10 compares the measures of Skewness and Kurtosis for successful fertility rate with respect to the effective reproductive time of the ovule. It is noticed that there is positive skewness. Hence we may interpret the happening as the average success rate of fertilization is more than the median success rate due to the influence of the active time of ovule.

4. Conclusion

This study addressed the concern of developing a stochastic model to explore the joint probability distribution for fertilization due to successful interaction of both pollen and ovules. The model has explored the joint probability density function of the fertilization by considering the life span of both pollen and ovule are random variables and following Truncated Exponential distributions. The indicator variable Z is defined with the assumption of Z=1 when there is an intersection/ overlapping of the events of pollen mating with egg in ovule; and Z=0 when the events of egg release and sperm cell interaction are non-overlapping. We derived the probability distribution of Z which is the core contribution of this study. This study explored different statistical measures and explained the patterns for successful pollination and fertilization with the active participation and non-successful interaction of pollen and ovule. The sensitivity analyses with numerical illustrations have explained the model behavior on more detailed way. The location and scaling parameters of the joint probability distribution are arrived with statistical measures are also explained with Truncated Exponential distribution. This study has explored and analyzed the mean, variance, shaping and kurtosis measures. All these measures suggested that with respect to pollen grain, the average success rate of fertilization is increases when the life span of pollen grain increases. Similarly with respect to ovule, the average success rate of fertilization increases when the life span of the ovule increases. It is observed that there is a positive relationship between the life spans of both pollen and ovule on the success rate of fertilization.

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Department of Statistics, Pondicherry University, Puducherry - 605 014
E-mail address: `drtrpadi@gmail.com`

Department of Statistics, Pondicherry University, Puducherry - 605 014
E-mail address: `abarna.0791@gmail.com`

Department of Statistics, Pondicherry University, Puducherry - 605 014
E-mail address: `mohammed.hisham19@gmail.com`