TRUNCATION OF PICTURE FUZZY SOFT GRAPH

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Abstract. The purpose of this paper is to place forward to the notion of picture fuzzy soft graph (PFSG). In this paper, we introduced picture fuzzy soft graph, lower and upper truncation of picture fuzzy soft graph. Also studied truncation of subdivision picture fuzzy soft graph and truncation of strong picture fuzzy soft graph.

1. Introduction

The origin of the graph theory started with the Konigsberg bridge problem in 1735. This problem led to the concept of the Eulerian graph. Euler studied the Konigsberg bridge problem and constructed a structure that solves the problem that is referred to as an Eulerian graph. The concept of fuzzy set theory was introduced by Zadeh [9] to solve difficulties in dealing with uncertainties. Since then the theory of fuzzy sets and fuzzy logic have been examined by many researchers to solve many real life problems involving ambiguous and uncertain environment.


Cen Zuo [10] introduced picture fuzzy graph. A kram and Davvaz [1] discussed Strong intuitionistic fuzzy graphs. [2] introduced many new concepts, including soft graphs, fuzzy soft graphs and operations on fuzzy soft graphs, during this paper, we introduced picture fuzzy soft graph, lower and upper truncation of picture fuzzy soft graph. Also studied truncation of subdivision picture fuzzy soft graph and truncation of strong picture fuzzy soft graph.

2. Preliminaries

Definition 2.1. If \( Z \) is a collection of object (or element) bestowed by \( z \). Then fuzzy set [9] \( A' \) in \( Z \) is expressed as a set of ordered pair.

\[ A' = \{(Z, \lambda_{A'}(z)) : z \in Z\} \]

where, \( \lambda_{A'}(z) \) is called the membership function (or characteristic function) which maps \( Z \) to the closed interval [0, 1].

Key words and phrases. Picture fuzzy soft graph, lower truncation, upper truncation strong picture fuzzy soft graph, subdivision picture fuzzy soft graph.
Definition 2.2. Let $D$ be initial universal set, $Q$ be a set of parameters, $\varphi(D)$ be the power set of $D$ and $K \subseteq Q$. A pair $(J, K)$ is called soft set [3] over $D$ if and only if $J$ is a mapping of $K$ into the set of all subsets of the set $D$.

Definition 2.3. A pair $(J, K)$ is called fuzzy soft set [2, 6] over $D$, where $J$ is a mapping given by $J : K \rightarrow I^D$, $I^D$ denote the collection of all fuzzy subset of $D, K \subseteq Q$.

Definition 2.4. Let $A'$ be a picture fuzzy set [10, 4]. $A'$ in $Z$ defined by

$$A' = \{(z, \lambda_A(z), \delta_A(z), \varphi_A(z)) : z \in Z\}$$

where, $\lambda_A(z) \in [0, 1], \delta_A(z) \in [0, 1]$ and $\varphi_A(z) \in [0, 1]$ follow the condition $0 \leq \lambda_A(z) + \delta_A(z) + \varphi_A(z) \leq 1$. The $\lambda_A(z)$ is used to represent the positive membership degree, $\delta_A(z)$ is used to represent the neutral membership degree and $\varphi_A(z)$ is used to represent the negative membership degree of the element $z$ in the set $A'$. For each picture fuzzy set $A'$ in $Z$, the refusal membership degree is described as

$$\pi_{A'}(z) = 1 - (\lambda_A(z) + \delta_A(z) + \varphi_A(z)).$$

3. Picture Fuzzy Soft Graph

Definition 3.1. A pair $(J, K)$ is called picture fuzzy soft set over $D$, where $J$ is a mapping given by $J : K \rightarrow IP^D$, where $IP^D$ denote the collection of all picture fuzzy subset of $D, K \subseteq Q$.

Definition 3.2. Let $G'^* = (W, Y)$ be a graph, $W = \{W_1, w_2, \ldots W_n\}$ be a non-empty set, $Y \subseteq W \times W, Q$ be parameter set and $K \subseteq Q$. Also let,

i. a) $\lambda_A$ is a positive membership function defined on $W$ by $\lambda_A : K \rightarrow IP^D(W)$ ($IP^D(W)$ denote collection of all picture fuzzy subset in $W$) $k \rightarrow \lambda_A(k) = \lambda_{Ak}$ (say), $k \in K$ and $\lambda_{Ak} : W \rightarrow [0, 1], w_i \rightarrow \lambda_{Ak}(w_i)$ ($K, \lambda_A$) picture fuzzy soft vertex of positive membership function.

b) $\delta_A$ is a neutral membership function defined on $W$ by $\delta_A : K \rightarrow IP^D(W)$ ($IP^D(W)$ denote collection of all picture fuzzy subset in $W$) $k \rightarrow \delta_A(k) = \delta_{Ak}$ (say), $k \in K$ and $\delta_{Ak} : W \rightarrow [0, 1], w_i \rightarrow \delta_{Ak}(w_i)$ ($K, \delta_A$) picture fuzzy soft vertex of neutral membership function.

c) $\varphi_A$ is a negative membership function defined on $W$ by $\varphi_A : K \rightarrow IP^D(W)$ ($IP^D(W)$ denote collection of all picture fuzzy subset in $W$) $k \rightarrow \varphi_A(k) = \varphi_{Ak}$ (say), $k \in K$ and $\varphi_{Ak} : W \rightarrow [0, 1], w_i \rightarrow \varphi_{Ak}(w_i)$ ($K, \varphi_A$) picture fuzzy soft vertex of negative membership function. such that $0 \leq \lambda_{Ak}(w_i) + \delta_{Ak}(w_i) + \varphi_{Ak}(w_i) \leq 1 \forall w_i \in W, k \in K$, where $A$ is a picture fuzzy soft set on $W$.

ii. a) $\lambda_B$ is a positive membership function defined on $Y$ by $\lambda_B : K \rightarrow IP^D(W \times W)$ ($IP^D(W \times W)$ denote collection of all picture fuzzy subset in $Y$) $k \rightarrow \lambda_B(k) = \lambda_{Bk}$ (say), $k \in K$ and $\lambda_{Bk} : W \times W \rightarrow [0, 1], (w_i, w_j) \rightarrow \lambda_{Bk}(w_i, w_j)$ ($K, \lambda_B$) picture fuzzy soft edge of positive membership function.

b) $\delta_B$ is a neutral membership function defined on $Y$ by $\delta_B : K \rightarrow IP^D(W \times W)$ ($IP^D(W \times W)$ denote collection of all picture fuzzy subset in $Y$) $k \rightarrow
The fuzzy relation \( \lambda \) and \( \delta \) are defined as:

\[
\delta_B(k) = \delta_{Bk} \quad \text{for} \quad k \in K \text{ and } \delta_{Bk} : W \times W \to [0,1], (w_i, w_j) \to \delta_{Bk}(w_i, w_j)
\]

and

\[
(K, \delta_B) \text{ picture fuzzy soft edge of neutral membership function.}
\]

c) The picture fuzzy soft graph \( \varphi_B : W \times W \to [0,1], (w_i, w_j) \to \varphi_{Bk}(w_i, w_j) \) is said to be strong picture fuzzy soft graph, if

\[
\forall (W_i, W_j) \in Y, i, j = 1, 2, \ldots n \text{ and } k \in K. \text{ Then } \]

\[
G' = (W, Y, (K, \lambda_A), (K, \delta_A), (K, \varphi_A), (K, \lambda_B), (K, \delta_B), (K, \varphi_B))
\]

is said to be picture fuzzy soft graph and this denoted by \( G'_{K,W,Y} \).

**Definition 3.3.** Picture fuzzy soft graph

\[
G'_{K,W,Y} = (W, Y, (K, \lambda_A), (K, \delta_A), (K, \varphi_A), (K, \lambda_B), (K, \delta_B), (K, \varphi_B))
\]

is said to be strong picture fuzzy soft graph, if

\[
\lambda_{Bk}(w, w_j) = \min(\lambda_{Ak}(w), \lambda_{Ak}(w_j)), \quad \delta_{Bk}(w, w_j) = \min(\delta_{Ak}(w), \delta_{Ak}(w_j)), \quad \varphi_{Bk}(w, w_j) = \max(\varphi_{Ak}(w), \varphi_{Ak}(w_j)) \quad \forall (w_i, w_j) \in Y, k \in K.
\]

**Definition 3.4.** The vertices and edges of \( G'_{K,W,Y} \) are taken together as the vertex set of the subdivision picture fuzzy soft graph \( sd(G'_{K,W,Y}) \) is defined as,

\[
(\lambda_{Ak})_{sd} (w) = \begin{cases} 
\lambda_{Ak}(w) & \text{if } w \in W \\
\lambda_{Bk}(w) & \text{if } w \in Y
\end{cases}, \quad (\delta_{Ak})_{sd} (w) = \begin{cases} 
\delta_{Ak}(w) & \text{if } w \in W \\
\delta_{Bk}(w) & \text{if } w \in Y
\end{cases}
\]

\[
(\varphi_{Ak})_{sd} (w) = \begin{cases} 
\varphi_{Ak}(w) & \text{if } w \in W \\
\varphi_{Bk}(w) & \text{if } w \in Y
\end{cases}
\]

The fuzzy relation \( (\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd} \) on \( W \cup Y \) is,

\[
(\lambda_{Bk})_{sd} (w, y) = \begin{cases} 
(\lambda_{Ak})_{sd} (w) \land (\lambda_{Ak})_{sd} (y), & \text{if } w \in W, y \in Y \text{ and } w \text{ lies on } y \\
0, & \text{otherwise}
\end{cases}
\]

\[
(\delta_{Bk})_{sd} (w, y) = \begin{cases} 
(\delta_{Ak})_{sd} (w) \land (\delta_{Ak})_{sd} (y), & \text{if } w \in W, y \in Y \text{ and } w \text{ lies on } y \\
0, & \text{otherwise}
\end{cases}
\]

\[
(\varphi_{Bk})_{sd} (w, y) = \begin{cases} 
(\varphi_{Ak})_{sd} (w) \land (\varphi_{Ak})_{sd} (y), & \text{if } w \in W, y \in Y \text{ and } w \text{ lies on } y \\
0, & \text{otherwise}
\end{cases}
\]
4. Lower and Upper Truncation of Picture Fuzzy Soft Graph

Definition 4.1. Let $(\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})$ be a picture fuzzy soft subset of the set $W$. Then lower truncation $(\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)}$ and upper truncation $(\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})^{(t)}$ of $(\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})$ at level ‘$t$’, $0 < t \leq 1$ are defined by,

$$(\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)} (w) = \begin{cases} 
(\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak}) (w), & \text{if } w \in (\lambda_{Ak}^t, \delta_{Ak}^t, \varphi_{Ak}^t) \\
0, & \text{if } w \notin \lambda_{Ak}^t(\text{or})W \notin \delta_{Ak}^t(\text{or})w \notin \varphi_{Ak}^t
\end{cases}$$

$$(\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})^{(t)} (w) = \begin{cases} 
(t, & \text{if } w \in (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak}) \\
(\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak}) (w), & \text{if } w \notin \lambda_{Ak}^t(\text{or})W \notin \delta_{Ak}^t(\text{or})w \notin \varphi_{Ak}^t
\end{cases}$$

where, $\lambda_{Ak}^t \{ w \in W | \lambda_{Ak} \geq t \}$, $\delta_{Ak}^t \{ w \in W/\delta_{Ak} \geq t \}$, $\varphi_{Ak}^t \{ w \in W/\varphi_{Ak} \leq t \}$ and $k \in K$.

Definition 4.2. Let $G^{(t)}_{K,W,Y}$ be a picture fuzzy soft graph. Take $W_{(t)} = (\lambda_{Ak}^t, \delta_{Ak}^t, \varphi_{Ak}^t)$, $Y_{(t)} = (\lambda_{Bk}^t, \delta_{Bk}^t, \varphi_{Bk}^t)$. Then for $w \in W_{(t)}$, $(\lambda_{Ak})_{(t)} (w) = \lambda_{Ak}(w)$, $(\varphi_{Ak})_{(t)} (w) = \varphi_{Ak}(w)$ and for $y \in Y_{(t)}$, $(\lambda_{Bk})_{(t)} (y) = \lambda_{Bk}(y)$, $(\delta_{Bk})_{(t)} (y) = \delta_{Bk}(y)$, $(\varphi_{Bk})_{(t)} (y) = \varphi_{Bk}(y)$.

Therefore,

$$(\lambda_{Bk})_{(t)} (w) = \lambda_{Bk}(w) \leq \lambda_{Ak}(w) \land \lambda_{Ak}(w) = (\lambda_{Ak})_{(t)} (w) \land \lambda_{Ak}(w)$$

$$(\delta_{Bk})_{(t)} (w) = \delta_{Bk}(w) \leq \delta_{Ak}(w) \land \delta_{Ak}(w) = (\delta_{Ak})_{(t)} (w) \land \delta_{Ak}(w)$$

$$(\varphi_{Bk})_{(t)} (w) = \varphi_{Bk}(w) \leq \varphi_{Ak}(w) \land \varphi_{Ak}(w) = (\varphi_{Ak})_{(t)} (w) \land \varphi_{Ak}(w)$$

for all $w_i, w_j \in W_{(t)}, k \in K$.

Hence, $(G^{(t)}_{K,W,Y})_{(t)}$ is a lower truncation of picture fuzzy soft graph.

Take $W_{(t)} = W, Y_{(t)} = Y$.

By definition,

(i) either $(\lambda_{Bk})_{(t)} (w) = t$ or $(\lambda_{Bk})_{(t)} (w) = \lambda_{Bk}(w)$,

(ii) either $(\delta_{Bk})_{(t)} (w) = t$ or $(\delta_{Bk})_{(t)} (w) = \delta_{Bk}(w)$,

(iii) either $(\varphi_{Bk})_{(t)} (w) = t$ or $(\varphi_{Bk})_{(t)} (w) = \varphi_{Bk}(w)$.

(i) Suppose, $(\lambda_{Bk})_{(t)} (w) = t$ then $\lambda_{Bk}(w) \geq t$.

$$\Rightarrow t \leq \lambda_{Bk}(w) \leq \lambda_{Ak}(w) \land \lambda_{Ak}(w)$$

$$\Rightarrow \lambda_{Ak}(w) \geq t, \lambda_{Ak}(w) \geq t$$

$$\Rightarrow (\lambda_{Ak})_{(t)} (w) = t, (\lambda_{Ak})_{(t)} (w) = t$$

$$\Rightarrow (\lambda_{Ak})_{(t)} (w) \land (\lambda_{Ak})_{(t)} (w) = t$$

Therefore, $(\lambda_{Bk})_{(t)} (w) = t = (\lambda_{Ak})_{(t)} (w) \land (\lambda_{Ak})_{(t)} (w)$

Suppose, $(\lambda_{Bk})_{(t)} (w) = \lambda_{Bk}(w) = \lambda_{Bk} (w) < t$

Now, we have two cases $\lambda_{Bk}(w) < t \leq \lambda_{Ak}(w) \land \lambda_{Ak}(w)$ or $\lambda_{Bk}(w) \leq \lambda_{Ak}(w) \land \lambda_{Ak}(w) < t$.

when, $t \leq \lambda_{Ak}(w) \land \lambda_{Ak}(w)$, proceeding as above $(\lambda_{Ak})_{(t)} (w) \land (\lambda_{Ak})_{(t)} (w) = t$. 

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Therefore, \((\lambda_{Bk}^{(t)}(w_i,w_j)) = \lambda_{Bk}(w_i,w_j) < t = (\lambda_{Ak}^{(t)}(w_i) \wedge (\lambda_{Ak}^{(t)}(w_j))\) when, \(\lambda_{Ak}(w_i) \wedge \lambda_{Ak}(w_j) < t\), we have

\[\Rightarrow \lambda_{Ak}(w_i) < t, \lambda_{Ak}(w_j) \geq t \lor \lambda_{Ak}(w_i) \geq t, \lambda_{Ak}(w_j) < t \lor \lambda_{Ak}(w_i) < t \lambda_{Ak}(w_j) < t.\]

\[\Rightarrow (\lambda_{Ak}^{(t)}(w_i) = \lambda_{Ak}(w_i), (\lambda_{Ak}^{(t)}(w_j) = t(\lor) \lambda_{Ak}(w_j)) = (\lambda_{Ak}^{(t)}(w_i) = t\).

\((\lambda_{Ak}^{(t)}(w_j) = \lambda_{Ak}(w_j) \lor (\lambda_{Ak}^{(t)}(w_j) = \lambda_{Ak}(w_j), (\lambda_{Ak}^{(t)}(w_j)) = \lambda_{Ak}(w_j)\)

\[\Rightarrow (\lambda_{Ak}^{(t)}(w_j) \wedge (\lambda_{Ak}^{(t)}(w_j)) = \lambda_{Ak}(w_j) \wedge t = \lambda_{Ak}(w_i) = \lambda_{Ak}(w_i) \wedge \lambda_{Ak}(w_j) (or)

\((\lambda_{Ak}^{(t)}(w_j) \wedge (\lambda_{Ak}^{(t)}(w_j)) = t \wedge \lambda_{Ak}(w_j) = \lambda_{Ak}(w_j) = \lambda_{Ak}(w_i) \wedge \lambda_{Ak}(w_j) (or)\)

\((\lambda_{Ak}^{(t)}(w_i) \wedge (\lambda_{Ak}^{(t)}(w_j)) = \lambda_{Ak}(w_i) \wedge \lambda_{Ak}(w_j)\)

Therefore,

\[(\lambda_{Bk}^{(t)}(w_i,w_j) = \lambda_{Bk}(W_i,W_j) \leq \lambda_{Ak}(w_i) \wedge \lambda_{Ak}(w_j) = (\lambda_{Ak}^{(t)}(w_i) \wedge (\lambda_{Ak}^{(t)}(w_j))\]

\[(\lambda_{Bk}^{(t)}(w_i,w_j) \leq (\lambda_{Ak}^{(t)}(w_i) \wedge (\lambda_{Ak}^{(t)}(w_j))\]

Similarly,

(ii) \((\delta_{Bk}^{(t)}(w_i,w_j) \leq (\delta_{Ak}^{(t)}(w_i) \wedge (\delta_{Ak}^{(t)}(w_j))\)

(iii) \((\varphi_{Bk}^{(t)}(w_i,w_j) \geq (\varphi_{Ak}^{(t)}(w_i) \lor (\varphi_{Ak}^{(t)}(w_j))\) for all \(w_i, w_j \in W, k \in K\).

Hence, \((G^t_{K,W,Y}^*)^{(t)}\) is a upper truncation of picture fuzzy soft graph.

**Example 4.3.** Consider, picture fuzzy soft graph

\(G^t_{K,W,Y} = (W,Y,(K,\lambda_A), (K,\delta_A), (K,\varphi_A), (K,\lambda_B), (K,\delta_B), (K,\varphi_B))\), where

\(W = \{w_1, w_2, w_3\}\) and \(Y = \{(w_1, w_2), (w_2, w_3), (w_1, w_3)\}\). Let \(K = \{k_1, k_2\}\) be the parameter set.

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The lower truncation \((G^t_{K,W,Y}^*)^{(t)}\) and upper truncation \((G^t_{K,W,Y}^*)^{(t)}\) of picture fuzzy soft graph \(G^t_{K,W,Y}\) at \(t = (0.2, 0.3, 0.2)\).

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5. Truncation of the Subdivision picture Fuzzy Soft Graph

**Theorem 5.1.** For any level \( t, 0 < t \leq 1 \), the lower truncation of the subdivision picture fuzzy soft graph \( sd(G_{K,W,Y}^t) \) is same as the subdivision of the lower truncation picture fuzzy soft graph \( (G_{K,W,Y}^t)_{(t)}' \), i.e., \( sd(G_{K,W,Y}^t) = sd((G_{K,W,Y}^t)_{(t)})' \).

**Proof.** Let \( G_{K,W,Y}^t \) be a picture fuzzy soft graph.

Claim: 1
We claim that vertex set \( (sd(G^*_K,W,Y))_{(t)} \) = vertex set \( (sd(G^*_K,W,Y))_{(t)} \).

Take \( w \) to be a vertex in \( (sd(G^*_K,W,Y))_{(t)} \).

By the definition of lower truncation of \( (sd(G^*_K,W,Y))_{(t)} \),

\[
((\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd})_{(t)}(w) = (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd}(w), \text{ if } w \in (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd}
\]

\[
((\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd})_{(t)}(w) = 0, \text{ if } w \notin (\lambda_{Ak})_{sd}(or)w \notin (\delta_{Ak})_{sd}(or)w \notin (\varphi_{Ak})_{sd}
\]

for all \( k \in K \).

By definition of \( sd(G^*_K,W,Y) \),

\[
(\lambda_{Ak})_{sd}(w) = \begin{cases} 
\lambda_{Ak}(w) & \text{if } w \in W \\
\lambda_{BK}(w) & \text{if } w \in Y \end{cases}
\]

(5.3)

\[
(\delta_{Ak})_{sd}(w) = \begin{cases} 
\delta_{Ak}(w) & \text{if } w \in W \\
\delta_{BK}(w) & \text{if } w \in Y \end{cases}
\]

(5.4)

\[
(\varphi_{Ak})_{sd}(w) = \begin{cases} 
\varphi_{Ak}(w) & \text{if } w \in W \\
\varphi_{BK}(w) & \text{if } w \in Y \end{cases}
\]

(5.5)

Case: 1

Let the vertex \( w \) in \( (sd(G^*_K,W,Y))_{(t)} \) be such that \( w \in (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd} \)

i.e., \( (\lambda_{Ak})_{sd}(w) \geq t, (\delta_{Ak})_{sd}(w) \geq t, (\varphi_{Ak})_{sd}(w) \leq t \)

(5.6)

Subcase: 1(a)

If \( w \in W \), by (5.3), (5.4), (5.5) and (5.6) \( \lambda_{Ak}(w) \geq t, \delta_{Ak}(w) \geq t \) and \( \varphi_{Ak}(w) \geq t \).

Hence \( w \in (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak}) \).

By definition of \( (G^*_K,W,Y)_{(t)} \),

\[
(\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)}(w) = (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)}(w), \text{ if } w \in (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})
\]

i.e., \( w \) is a vertex in \( (G^*_K,W,Y)_{(t)} \) such that

\[
(\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)}(w) = (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)}(w).
\]

Subcase: 1(b)

If \( w \in W \), let \( w = y \) (5.3), (5.4) and (5.5) \( \Rightarrow (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd}(y) = (\lambda_{BK}, \delta_{BK}, \varphi_{BK})(y) \)

By (5.6), \( (\lambda_{Ak})_{sd}(w) \geq t, (\delta_{Ak})_{sd}(w) \geq t, (\varphi_{Ak})_{sd}(w) \leq t \) In \( (G^*_K,W,Y)_{(t)} \)

\[
(\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)}(y) = (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)}(y)
\]

i.e., \( w = y \in Y \) is an edge in \( (G^*_K,W,Y)_{(t)} \) such that \( (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)}(y) = (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)}(y) \). As the vertices and edges of \( (G^*_K,W,Y)_{(t)} \) are taken together as the vertices of \( sd((G^*_K,W,Y)_{(t)}) \).
By Sub Case 1(a) and 1(b), \( w \) is a vertex in \( sd \left( (G^*_K, W, Y)_{(t)} \right) \).

\[
((\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd})_{(t)} (w) = (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd} (w) = \left( (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)} \right)_{sd} (w)
\]

**Case: 2**

Let the vertex \( w \) in \( (sd (G^*_K, W, Y))_{(t)} \) be such that \( w \notin (\lambda^t Ak)_{sd} (or) w \notin (\delta^t Ak)_{sd} (or) w \notin (\varphi^t Ak)_{sd} \).

\[
(5.2) \Rightarrow ((\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd})_{(t)} (w) = 0 \text{as} (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd} < t
\]

**Subcase: 2(a)**

If \( w \in W \), by (5.3), (5.4) and (5.5) \( \Rightarrow (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd} (y) = (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd} (y) \).

Hence \( (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak}) (y) < t, w \notin (\lambda^t Ak)_{sd} (or) w \notin (\delta^t Ak)_{sd} (or) w \notin (\varphi^t Ak)_{sd} \).

By definition of \( (G^*_K, W, Y)_{(t)} \), \( (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)} (w) = 0 \). i.e., \( w \) is a vertex in \( (G^*_K, W, Y)_{(t)} \) such that \( (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)} (w) = 0 \).

**Subcase: 2(b)**

If \( w \in W \), let \( w = y \) (5.3), (5.4) and (5.5) \( \Rightarrow (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd} (y) = (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd} (y) \).

Hence \( (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak}) (y) < t, w \notin (\lambda^t Ak)_{sd} (or) w \notin (\delta^t Ak)_{sd} (or) w \notin (\varphi^t Ak)_{sd} \).

By definition of \( (G^*_K, W, Y)_{(t)} \), \( (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)} (y) = 0 \). i.e., \( y \) is an edge in \( (G^*_K, W, Y)_{(t)} \) such that \( (\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)} (y) = 0 \).

Hence, by Sub Case 2(a), 2(b) and by definition of \( sd \left( (G^*_K, W, Y)_{(t)} \right) \), \( w \) is a vertex in \( sd \left( (G^*_K, W, Y)_{(t)} \right) \), such that \( ((\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)} )_{sd} (w) = 0 \).

By Case 1 and Case 2, \( ((\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{sd})_{(t)} (w) = ((\lambda_{Ak}, \delta_{Ak}, \varphi_{Ak})_{(t)} )_{sd} (w) \) for all \( W \in W \cup Y \).

**Claim: 2**

We claim that edge set \( (sd (G^*_K, W, Y))_{(t)} = edge set sd \left( (G^*_K, W, Y)_{(t)} \right) \).

Take \( (w, y) \) to be a edge in \( (sd (G^*_K, W, Y))_{(t)} \).

By the definition of lower truncation of \( sd \left( (G^*_K, W, Y) \right) \),

\[
((\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd})_{(t)} (w, y) = (\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd} (w, y), \text{ if } (w, y) \in (\lambda^t Bk, \delta^t Bk, \varphi^t Bk)_{sd}
\]

\[
(5.7)
\]

\[
((\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd})_{(t)} (w, y) = 0, \text{ if } (w, y) \notin (\lambda^t Bk)_{sd} (or) (w, y) \notin (\delta^t Bk)_{sd} (or) (w, y) \notin (\varphi^t Bk)_{sd}
\]

\[
(5.8)
\]
TRUNCATION OF PICTURE FUZZY SOFT GRAPH

for all $k \in K$.

By definition of $sd \left( G^*_K,W,Y \right)$,

$$ (\lambda_{Bk})_{sd} (w, y) = \begin{cases} (\lambda_{Ak})_{sd} (w) \land (\lambda_{Ak})_{sd} (y), & \text{if } w \in W, y \in Y \text{ and } w \text{ lies on } y \\ 0, & \text{otherwise} \end{cases} \quad (5.9) $$

$$ (\delta_{Bk})_{sd} (w, y) = \begin{cases} (\delta_{Ak})_{sd} (w) \land (\delta_{Ak})_{sd} (y), & \text{if } w \in W, y \in Y \text{ and } w \text{ lies on } y \\ 0, & \text{otherwise} \end{cases} \quad (5.10) $$

$$ (\varphi_{Bk})_{sd} (w, y) = \begin{cases} (\varphi_{Ak})_{sd} (w) \land (\varphi_{Ak})_{sd} (y), & \text{if } w \in W, y \in Y \text{ and } w \text{ lies on } y \\ 0, & \text{otherwise} \end{cases} \quad (5.11) $$

Case: 3

Let $w \in W, y \in Y$ be such that $(w, y) \in (\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd}^t$.

$$(5.7) \Rightarrow ((\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd})_{(t)} = \begin{cases} (\lambda_{Bk})_{sd} (w, y) \geq t \\ (\delta_{Bk})_{sd} (w, y) \geq t \\ (\varphi_{Bk})_{sd} (w, y) \leq t \end{cases} \quad (5.12)$$

As $t > 0$,

$$(5.9) \Rightarrow (\lambda_{Ak})_{sd} (w) \land (\lambda_{Ak})_{sd} (y) \geq t$$

$$(5.10) \Rightarrow (\delta_{Ak})_{sd} (w) \land (\delta_{Ak})_{sd} (y) \geq t$$

$$(5.11) \Rightarrow (\varphi_{Ak})_{sd} (w) \lor (\varphi_{Ak})_{sd} (y) \leq t$$

$$\Rightarrow (\lambda_{Ak})_{sd} (w) \land (\lambda_{Ak})_{sd} (y) \geq t, (\delta_{Ak})_{sd} (w) \land (\delta_{Ak})_{sd} (y) \geq t,$$

$$\Rightarrow (\varphi_{Ak})_{sd} (w) \lor (\varphi_{Ak})_{sd} (y) \leq t.$$
Similarly, $(\delta_{Bk})_{sd}(w, y) = \left( (\delta_{Bk})_{(t)} \right)(y)$, $(\varphi_{Bk})_{sd}(w, y) = \left( (\varphi_{Bk})_{(t)} \right)(y)$. $(w, y)$ to be an edge in $sd\left( (G'_{K,W,Y})_{(t)} \right)$ and

$((\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd})_{(t)}(w, y) = \left( (\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{(t)} \right)_{sd}(w, y)$ if $(w, y) \in (\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd}$.

**Case: 4**

Take $(w, y)$ to be an edge in $sd\left( (G'_{K,W,Y})_{(t)} \right)$ such that $((\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd})_{(t)}(w, y) = 0$ By (5.8) $(w, y) \notin (\lambda_{Bk}Bk)_{sd}(or)(w, y) \notin (\delta_{Bk}Bk)_{sd}(or)(w, y) \notin (\varphi_{Bk}Bk)_{sd} \Rightarrow (\lambda_{Ak})_{sd}(w, y) < t, (\delta_{Ak})_{sd}(w, y) < t, (\varphi_{Ak})_{sd}(w, y) > t$

**Subcase: 4(a)**

If $w$ lies on $y$, (5.9), (5.10), (5.11) $\Rightarrow (\lambda_{Ak})_{sd}(w) \wedge (\lambda_{Ak})_{sd}(y) < t$

$\Rightarrow (\lambda_{Ak})_{sd}(w) \wedge (\lambda_{Ak})_{sd}(y) < t, (\delta_{Ak})_{sd}(w) \wedge (\varphi_{Ak})_{sd}(y) > t$

$\Rightarrow (\lambda_{Ak})_{sd}(w) \wedge (\lambda_{Ak})_{sd}(y) < t, (\delta_{Ak})_{sd}(w) \wedge (\delta_{Ak})_{sd}(y) > t$

$\Rightarrow (\lambda_{Bk})(w) \wedge (\lambda_{Bk})(y) < t, (\delta_{Bk})(w) \wedge (\varphi_{Bk})(y) > t$

$\Rightarrow (\lambda_{Ak})(w) \wedge (\lambda_{Ak})(y) < t, (\delta_{Ak})(w) \wedge (\varphi_{Ak})(y) > t$

$\Rightarrow (\lambda_{Bk})(w) \wedge (\lambda_{Bk})(y) < t, (\delta_{Bk})(w) \wedge (\varphi_{Bk})(y) > t$

$\Rightarrow (\lambda_{Ak})_{sd}(w) \wedge (\lambda_{Ak})_{sd}(y) = 0$

$\Rightarrow (\lambda_{Ak})_{sd}(w) \wedge (\lambda_{Ak})_{sd}(y) = 0$

Similarly, $\left( (\delta_{Bk})_{(t)} \right)_{sd}(w, y) = 0, \left( (\varphi_{Bk})_{(t)} \right)_{sd}(w, y) = 0$.

**Subcase: 4(b)**

If $w$ does not lie on $y$ $\left( (\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd} \right)(w, y) = 0$.

Hence, by Sub Case 4(a) and 4(b)

If $(w, y)$ is an edge in $sd\left( (sd(G'_{K,W,Y})_{(t)} \right)$ such that $((\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd})_{(t)}(w, y) = 0$

then $((\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd} \right)(w, y) = 0$

ie., $((\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd} \right)(w, y) = 0$

if $(w, y) \notin (\lambda_{Bk}Bk)_{sd}(or)(w, y) \notin (\delta_{Bk}Bk)_{sd}(or)(w, y) \notin (\varphi_{Bk}Bk)_{sd}$

Hence, by Case (3) and (4)

$((\lambda_{Bk}, \delta_{Bk}, \varphi_{Bk})_{sd} \right)(w, y) = 0$

ie., $\left( (sd(G'_{K,W,Y})_{(t)} \right) = sd\left( (G'_{K,W,Y})_{(t)} \right)$

\[\square\]

**Theorem 5.2.** For any level $t, 0 < t \leq 1$, the upper truncation of the subdivision picture fuzzy soft graph $sd\left( (G'_{K,W,Y}) \right)$ is same as the subdivision of the upper truncation picture fuzzy soft graph $\left( (G'_{K,W,Y}) \right)$, ie., $sd\left( (G'_{K,W,Y}) \right) = sd\left( (G'_{K,W,Y}) \right)$.
6. Truncation of Strong Picture Fuzzy Soft Graph

Theorem 6.1. If $G^*_{K,W,Y}$ is a strong picture fuzzy soft graph, then $(G^*_{K,W,Y})^{(t)}$ is strong picture fuzzy soft graph.

Proof. Let $G^*_{K,W,Y}$ be a strong picture fuzzy soft graph. Then

$$\lambda_{Bk}(w_i, w_j) = \min (\lambda_{Ak}(w_i), \lambda_{Ak}(w_j)), \quad \delta_{Bk}(w_i, w_j) = \min (\delta_{Ak}(w_i), \delta_{Ak}(w_j))$$

$$\varphi_{Bk}(w_i, w_j) = \max (\varphi_{Ak}(w_i), \varphi_{Ak}(w_j))$$

for every edge $w_iw_j$ of $G^*_{K,W,Y}$.

Let $(w_i,w_j) \in Y^{(t)}$ be any edge of $(G^*_{K,W,Y})^{(t)}$. Then $w_i, w_j \in W^{(t)}$

$$(\lambda_{Bk})^{(t)}(w_i, w_j) = \lambda_{Bk}(w_i, w_j), \quad (\lambda_{Ak})^{(t)}(w_i) = \lambda_{Ak}(w_i),$$

$$(\lambda_{Ak})^{(t)}(w_j) = \lambda_{Ak}(w_j), \quad (\lambda_{Bk})^{(t)}(w_i, w_j) = \lambda_{Ak}(w_i) \land \lambda_{Ak}(w_j)$$

$$= (\lambda_{Ak})^{(t)}(w_i) \land (\lambda_{Ak})^{(t)}(w_j)$$

Similarly,

$$(\delta_{Bk})^{(t)}(w_i, w_j) = (\delta_{Ak})^{(t)}(w_i) \land (\delta_{Ak})^{(t)}(w_j),$$

$$(\varphi_{Bk})^{(t)}(w_i, w_j) = (\varphi_{Ak})^{(t)}(w_i) \lor (\varphi_{Ak})^{(t)}(w_j)$$

Hence, $(G^*_{K,W,Y})^{(t)}$ is strong picture fuzzy soft graph. \qed

Theorem 6.2. If $G^*_{K,W,Y}$ is a strong picture fuzzy soft graph, then $(G^*_{K,W,Y})^{(t)}$ is strong picture fuzzy soft graph.

Proof. Let $G^*_{K,W,Y}$ be a strong picture fuzzy soft graph. Then

$$\lambda_{Bk}(w_i, w_j) = \min (\lambda_{Ak}(w_i), \lambda_{Ak}(w_j)), \quad \delta_{Bk}(w_i, w_j) = \min (\delta_{Ak}(w_i), \delta_{Ak}(w_j))$$

$$\varphi_{Bk}(w_i, w_j) = \max (\varphi_{Ak}(w_i), \varphi_{Ak}(w_j))$$

for every edge $w_iw_j$ of $G^*_{K,W,Y}$.

Let $(w_i,w_j) \in Y^{(t)}$ be any edge of $(G^*_{K,W,Y})^{(t)}$.

Case:1

(i) If $\lambda_{Ak}(w_i) \geq t$, $\lambda_{Ak}(w_j) \geq t$, $\lambda_{Bk}(w_i, w_j) \geq t$. Then

$$(\lambda_{Ak})^{(t)}(w_i) = t, \quad (\lambda_{Ak})^{(t)}(w_j) = t, \quad (\lambda_{Bk})^{(t)}(w_i, w_j) = t$$

$$\Rightarrow (\lambda_{Bk})^{(t)}(w_i, w_j) = (\lambda_{Ak})^{(t)}(w_i) \land (\lambda_{Ak})^{(t)}(w_j).$$

(ii) Similarly, if $\delta_{Ak}(w_i) \geq t$, $\delta_{Ak}(w_j) \geq t$, $\delta_{Bk}(w_i, w_j) \geq t$. Then

$$(\delta_{Bk})^{(t)}(w_i, w_j) = (\delta_{Ak})^{(t)}(w_i) \land (\delta_{Ak})^{(t)}(w_j).$$

(iii) If $\varphi_{Ak}(w_i) \leq t$, $\varphi_{Ak}(w_j) \leq t$, $\varphi_{Bk}(w_i, w_j) \leq t$. Then

$$(\varphi_{Ak})^{(t)}(w_i) = t, \quad (\varphi_{Ak})^{(t)}(w_j) = t, \quad (\varphi_{Bk})^{(t)}(w_i, w_j) = t$$

$$\Rightarrow (\varphi_{Bk})^{(t)}(w_i, w_j) = (\varphi_{Ak})^{(t)}(w_i) \land (\varphi_{Ak})^{(t)}(w_j).$$

Case:2
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lineated truncation of strong picture fuzzy soft graph and truncation of subdivision

Hence,

Case:4
Proof is similar to Case 2.

(iii) If \( \varphi_Ak (w_i) \leq t, \varphi_Ak (w_j) < t, \varphi_Bk (w_i, w_j) < t. \) Then

\[
(\varphi_Bk)^{(t)} (w_i, w_j) = (\varphi_Ak)^{(t)} (w_i) \wedge (\varphi_Ak)^{(t)} (w_j).
\]

Case:3

(i) If \( \lambda_Ak (w_i) < t, \lambda_Ak (w_j) < t, \lambda_Bk (w_i, w_j) < t. \) Then

\[
(\lambda_Ak)^{(t)} (w_i) = (\lambda_Ak) (w_i), \quad (\lambda_Ak)^{(t)} (w_j) = (\lambda_Ak) (w_j). \\
(\lambda_Bk)^{(t)} (w_i, w_j) = (\lambda_Bk) (w_i, w_j)
\]

\[
\Rightarrow (\lambda_Ak)^{(t)} (w_i) \wedge (\lambda_Ak)^{(t)} (w_j) = (\lambda_Ak) (w_i) \wedge (\lambda_Ak) (w_j)
\]

(ii) Similarly, if \( \delta_Ak (w_i) < t, \delta_Ak (w_j) < t, \delta_Bk (w_i, w_j) < t. \) Then

\[
(\delta_Bk)^{(t)} (w_i, w_j) = (\delta_Ak)^{(t)} (w_i) \wedge (\delta_Ak)^{(t)} (w_j).
\]

(iii) If \( \varphi_Ak (w_i) \leq t, \varphi_Ak (w_j) < t, \varphi_Bk (w_i, w_j) < t. \) Then

\[
(\varphi_Bk)^{(t)} (w_i, w_j) = (\varphi_Ak)^{(t)} (w_i) \wedge (\varphi_Ak)^{(t)} (w_j).
\]

Case:4

(i) If \( \lambda_Ak (w_i) < t, \lambda_Ak (w_j) < t, \lambda_Bk (w_i, w_j) < t. \) Then

\[
(\lambda_Ak)^{(t)} (w_i) = (\lambda_Ak) (w_i), \quad (\lambda_Ak)^{(t)} (w_j) = (\lambda_Ak) (w_j),
\]

\[
(\lambda_Bk)^{(t)} (w_i, w_j) = (\lambda_Bk) (w_i, w_j)
\]

\[
\Rightarrow (\lambda_Ak)^{(t)} (w_i) \wedge (\lambda_Ak)^{(t)} (w_j) = (\lambda_Ak) (w_i) \wedge (\lambda_Ak) (w_j)
\]

\[
= (\lambda_Bk) (w_i, w_j) = (\lambda_Bk)^{(t)} (w_i, w_j)
\]

(ii) Similarly, if \( \delta_Ak (w_i) < t, \delta_Ak (w_j) < t, \delta_Bk (w_i, w_j) < t. \) Then

\[
(\delta_Bk)^{(t)} (w_i, w_j) = (\delta_Ak)^{(t)} (w_i) \wedge (\delta_Ak)^{(t)} (w_j).
\]

(iii) If \( \varphi_Ak (w_i) < t, \varphi_Ak (w_j) < t, \varphi_Bk (w_i, w_j) < t. \) Then

\[
(\varphi_Bk)^{(t)} (w_i, w_j) = (\varphi_Ak)^{(t)} (w_i) \wedge (\varphi_Ak)^{(t)} (w_j).
\]

Hence, \((G_{K,W,Y}^*)^{(t)}\) is strong picture fuzzy soft graph. \( \square \)

7. Conclusion

The main objective of this paper is to introduce the thought of picture fuzzy soft graph and truncation of picture fuzzy soft graph. We tend to additionally delineated truncation of strong picture fuzzy soft graph and truncation of subdivision picture fuzzy soft graph.
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