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ORDERED GENERALIZED $\phi-$CONTRACTION IN ORDERED FUZZY METRIC SPACES WITH AN APPLICATION IN DYNAMIC PROGRAMMING

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ABSTRACT. The common fixed point for ordered generalized $\phi-$contraction in the environment of an ordered fuzzy metric space is determined under minimum possible conditions. A result in ordered metric space is also obtained. The work is supported with a suitable example. Further, as an application, the utility of the present work is shown by solving functional equations in dynamic programming, which are beneficial in mathematical optimization as well as computer programming.

1. Introduction

The notion of fuzzy metric spaces was innovated by Kramosil and Michalek in 1975 and that was later improved by George and Veeramani [7] to obtain Hausdorff topology on these spaces. Recently, in the same framework, Aage et al. [1] innovated $\phi-$contractive mappings, with ϕ being the altering distance function familiarised by Choudhury and Das [5]. On another point of note, Ran and Reurings [13] developed a Banach contraction principle exploiting the ordered set which was further generalized by Nieto and Rodríguez-López [11]. Afterward, fixed point results have been explored by numerous researchers in various frameworks enriched with a partial ordering.
In the current paper, we determine the common fixed point for an ordered generalized $\phi-$contraction in ordered fuzzy metric spaces utilizing the notion of altering distance (Choudhury and Das [5]). Present work accords with George and Veeramani [7] and complements the contemporary work of Aage et al. [1] in the environment of ordered fuzzy metric setting. Further, it broadens the work of Nieto and Rodríguez-López [11] by extending it to an ordered fuzzy metric space.

2. Preliminaries

Next, we state some already existing notions and definitions useful in our work. Suppose that $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{N}$, and $\mathbb{R}^+$ symbolize the set of real numbers, non-negative real numbers, natural numbers, and positive real numbers, respectively.

Definition 2.1. [15] A fuzzy set $\mathcal{U}$ in $\mathcal{Y}$ is a function with domain $\mathcal{Y}$ and range $[0,1]$.

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Definition 2.2. [14] A binary operation $* : [0,1]^2 \longrightarrow [0,1]$ is a continuous $t-$norm if $([0,1],*)$ is a topological abelian monoid with unit 1 so that $\mathfrak{a}*\mathfrak{b} \leq \mathfrak{c}*\mathfrak{d}$, whenever $\mathfrak{a} \leq \mathfrak{c}$ and $\mathfrak{b} \leq \mathfrak{d}$, $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d} \in [0,1]$.

Definition 2.3. [7] The 3-tuple $(\mathcal{Y},\mathcal{M},*)$ is called a fuzzy metric space if $\mathcal{Y}$ is an arbitrary set, $*$ is a continuous $t-$norm and $\mathcal{M}$ is a fuzzy set on $\mathcal{Y}^2 \times \mathbb{R}^+$ satisfying the subsequent postulates:

(FM-1) $\mathcal{M}(\mathfrak{x},\mathfrak{y},t) > 0$;
(FM-2) $\mathcal{M}(\mathfrak{x},\mathfrak{y},t) = 1$ iff $\mathfrak{x} = \mathfrak{y}$;
(FM-3) $\mathcal{M}(\mathfrak{x},\mathfrak{y},t) = \mathcal{M}(\mathfrak{y},\mathfrak{x},t) > 0$;
(FM-4) $\mathcal{M}(\mathfrak{x},\mathfrak{y},t) * \mathcal{M}(\mathfrak{y},\mathfrak{z},s) \leq \mathcal{M}(\mathfrak{x},\mathfrak{z},t+s)$;
(FM-5) $\mathcal{M}(\mathfrak{x},\mathfrak{y},t) : \mathbb{R}_0^+ \longrightarrow [0,1]$ is continuous, $\mathfrak{x},\mathfrak{y},\mathfrak{z} \in \mathcal{Y}$ and $s,t > 0$.

Definition 2.4. [7] Let $(\mathcal{Y},\mathcal{M},*)$ be a fuzzy metric space and $t > 0$. A sequence $\{\mathfrak{x}_n\}$ in $\mathcal{Y}$ is

(i) convergent to a point $\mathfrak{x} \in \mathcal{Y}$ if $\lim_{n\to\infty} \mathcal{M}(\mathfrak{x}_n,\mathfrak{x},t) = 1$;
(ii) Cauchy sequence if $\lim_{n\to\infty} \mathcal{M}(\mathfrak{x}_{n+p},\mathfrak{x}_n,t) = 1$ and $p > 0$.

Definition 2.5. [7] A fuzzy metric space $(\mathcal{Y},\mathcal{M},*)$ is complete iff each Cauchy sequence in $\mathcal{Y}$ is convergent.

Lemma 2.6. [6] *Let $(\mathcal{Y},\mathcal{M},*)$ be a fuzzy metric space. Then $\mathcal{M}(\mathfrak{x},\mathfrak{y},.)$ is non-decreasing, $\mathfrak{x},\mathfrak{y} \in \mathcal{Y}$.*

Definition 2.7. [8] Let $\mathcal{A}$ and $\mathcal{B}$ be self-mappings on a fuzzy metric space $(\mathcal{Y},\mathcal{M},*)$. Then, $\mathcal{A}$ and $\mathcal{B}$ are weakly compatible if $\mathcal{A}\mathfrak{z} = \mathcal{B}\mathfrak{z}$ implies that $\mathcal{A}\mathcal{B}\mathfrak{z} = \mathcal{B}\mathcal{A}\mathfrak{z}$.

Khan et al. [9] innovated the subsequent function to determine a fixed point by altering the distances between them.

Definition 2.8. [9] An altering distance function is a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying:

(ψ_1) ψ is continuous and non-decreasing;
(ψ_2) $\psi(t) = 0$ iff $t = 0$.

Choudhury and Das [5] extended the above notion in complete Menger spaces under the following notion:

Definition 2.9. [5] A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a $\Phi-$function if subsequent postulates hold:

(i) $\phi(t) = 0$ iff $t = 0$;
(ii) $\phi(t) \to \infty$ as $t \to \infty$ and $\phi(t)$ is increasing;
(iii) ϕ is left continuous in $\mathbb{R}_0^+$;
(iv) ϕ is continuous at 0.

Very recently, Aage et al. [1] obtained fixed points for the following $\phi-$contractive mappings.

Definition 2.10. [1] A self-mapping $\mathcal{A} : \mathcal{Y} \to \mathcal{Y}$ of a fuzzy metric space $(\mathcal{Y},\mathcal{M},*)$ is called $\phi-$contractive if $\mathcal{M}(\mathcal{A}\mathfrak{x},\mathcal{A}\mathfrak{y},\phi(t)) \geq \mathcal{M}(\mathfrak{x},\mathfrak{y},\phi(\frac{t}{k}))$, $0 < k < 1$, $\mathfrak{x},\mathfrak{y} \in \mathcal{Y}$, $t > 0$ and ϕ is a $\phi-$function.

3. Main Results

We would be needing subsequent Lemma 3.1 to demonstrate the uniqueness of coincidence and common fixed point of an ordered generalized $\phi-$contractive mapping.

Lemma 3.1. *Let $(\mathcal{Y},\mathcal{M},*)$ be a fuzzy metric space and $\phi:\mathbb{R}^+\to\mathbb{R}^+$ be a $\phi-$function. If $\mathcal{M}(\mathfrak{x},\mathfrak{y},\phi(t))\geq\mathcal{M}(\mathfrak{x},\mathfrak{y},\phi(\frac{t}{k}))$, $\mathfrak{x},\mathfrak{y}\in\mathcal{Y}$, $k\in(0,1)$, $t>0$ then $\mathfrak{x}=\mathfrak{y}$.*

Proof. Since ϕ is strictly increasing and $k\in(0,1)$, utilizing the principle of mathematical induction, we obtain $\mathcal{M}(\mathfrak{x},\mathfrak{y},\phi(t))\geq\mathcal{M}(\mathfrak{x},\mathfrak{y},\phi(\frac{t}{k}))\geq\ldots\mathcal{M}(\mathfrak{x},\mathfrak{y},\phi(\frac{t}{k^n}))$. As $n\to\infty$, we obtain $\mathcal{M}(\mathfrak{x},\mathfrak{y},\phi(t))\geq 1$, i.e., $\mathfrak{x}=\mathfrak{y}$. □

We first define an ordered generalized $\phi-$contraction and then use the notion to establish the common fixed-point.

Definition 3.2. A pair $(\mathcal{A},g)$ of self-mappings in a fuzzy metric space $(\mathcal{Y},\mathcal{M},*)$ equipped with a partial order $\preceq$ is an ordered generalized $\phi-$contraction if the following holds:

$$\begin{aligned}\mathcal{M}(\mathcal{A}\mathfrak{x},\mathcal{A}\mathfrak{y},\phi(t))\geq\min\{&\mathcal{M}(g\mathfrak{x},g\mathfrak{y},\phi(\frac{t}{k})),\mathcal{M}(g\mathfrak{x},\mathcal{A}\mathfrak{x},\phi(\frac{t}{k})),\mathcal{M}(g\mathfrak{y},\mathcal{A}\mathfrak{y},\phi(\frac{t}{k})),\\&\mathcal{M}(g\mathfrak{x},\mathcal{A}\mathfrak{y},2\phi(\frac{t}{k}))*\mathcal{M}(g\mathfrak{y},\mathcal{A}\mathfrak{x},2\phi(\frac{t}{k}))\},\end{aligned}\tag{3.1}$$

$\mathfrak{x},\mathfrak{y}\in\mathcal{Y}$, $g\mathfrak{x}\succeq g\mathfrak{y}$, $k\in(0,1)$, $\phi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a $\Phi-$function and $t>0$.

For g =identity mapping, in inequality (3.1), we derive:

Definition 3.3. A self-mapping $\mathcal{A}$ in a fuzzy metric space $(\mathcal{Y},\mathcal{M},*)$ equipped with a partial order $\preceq$ is an ordered generalized $\phi-$contraction, if the following holds:

$$\begin{aligned}\mathcal{M}(\mathcal{A}\mathfrak{x},\mathcal{A}\mathfrak{y},\phi(t))\geq\min\{&\mathcal{M}(\mathfrak{x},\mathfrak{y},\phi(\frac{t}{k})),\mathcal{M}(\mathfrak{x},\mathcal{A}\mathfrak{x},\phi(\frac{t}{k})),\mathcal{M}(\mathfrak{y},\mathcal{A}\mathfrak{y},\phi(\frac{t}{k})),\\&\mathcal{M}(\mathfrak{x},\mathcal{A}\mathfrak{y},2\phi(\frac{t}{k}))*\mathcal{M}(\mathfrak{y},\mathcal{A}\mathfrak{x},2\phi(\frac{t}{k}))\},\end{aligned}\tag{3.2}$$

$\mathfrak{x},\mathfrak{y}\in\mathcal{Y}$, $k\in(0,1)$, $\mathfrak{x}\succeq\mathfrak{y}$, $\phi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a $\Phi-$function, and $t>0$.

Theorem 3.4. *Let $(\mathcal{Y},\mathcal{M},*)$ be a fuzzy metric space equipped with a partial order $\preceq$ and a continuous $t-$norm satisfying $\mathfrak{a}*\mathfrak{a}\geq\mathfrak{a}$, for every $\mathfrak{a}\in[0,1]$. Suppose that $\mathcal{A}$ and g be the self-mappings on $\mathcal{Y}$ so that $\mathcal{A}$ is $g-$monotonically increasing, $\mathcal{A}(\mathcal{Y})\subseteq g(\mathcal{Y})$, any one of $\mathcal{A}(\mathcal{Y})$ or $g(\mathcal{Y})$ is complete, and a pair $(\mathcal{A},g)$ is an ordered generalized $\phi-$contraction. If $\{g\mathfrak{x}_n\}\subseteq\mathcal{Y}$ is a non-decreasing sequence converging to $g\mathfrak{x}\in g(\mathcal{Y})$, then $g\mathfrak{x}_n\preceq g\mathfrak{x}$ and $g\mathfrak{x}\preceq gg\mathfrak{x}$, $n\in\mathbb{N}$. Moreover, if $\mathcal{A}\mathfrak{x}_0\succeq g\mathfrak{x}_0$, $\mathfrak{x}_0\in\mathcal{Y}$, then $\mathcal{A}$ and g have a coincidence point.*

Proof. Suppose $\mathfrak{x}_0\in\mathcal{Y}$, so that $\mathcal{A}\mathfrak{x}_0\succeq g\mathfrak{x}_0$. Since, $\mathcal{A}(\mathcal{Y})\subseteq g(\mathcal{Y})$, we have some $\mathfrak{x}_1\in\mathcal{Y}$ so that $\mathcal{A}\mathfrak{x}_0=g\mathfrak{x}_1=\mathfrak{x}_0$ (say). Then, $g\mathfrak{x}_1\succeq g\mathfrak{x}_0$. Now, since $\mathcal{A}$ is $g-$monotonically increasing, we get $\mathcal{A}\mathfrak{x}_1\succeq\mathcal{A}\mathfrak{x}_0$. Since $\mathcal{A}(\mathcal{Y})\subseteq g(\mathcal{Y})$, we have some $\mathfrak{x}_2\in\mathcal{Y}$ so that $\mathcal{A}\mathfrak{x}_1=g\mathfrak{x}_2=\mathfrak{y}_1$ (say). Now, we have $\mathfrak{y}_1\succeq\mathfrak{y}_0$. On repeatedly applying this

process, we may obtain the sequence $\{\mathfrak{y}_n\}$ defined as: $\mathfrak{y}_n = \mathcal{A}\mathfrak{x}_n = g\mathfrak{x}_{n+1}$, such that $\mathfrak{y}_{n+1} = g\mathfrak{x}_{n+2} \succeq g\mathfrak{x}_{n+1} = \mathfrak{y}_n$, $n \in \mathbb{N} \cup \{0\}$. If $\mathfrak{y}_n = \mathfrak{y}_{n+1}$, then $g\mathfrak{x}_{n+1} = \mathcal{A}\mathfrak{x}_{n+1}$, i.e., $\mathfrak{x}_{n+1}$ is a coincidence point of $\mathcal{A}$ and g. Thus, the result holds trivially.
Let $\mathfrak{y}_n \neq \mathfrak{y}_{n+1}$. We first assert that $\{\mathfrak{y}_n\}$ is a Cauchy sequence. Let $t > 0$ and $0 < \delta < 1$. Utilizing properties of the $\Phi-$function, we have $s > 0$ so that $t > \phi(s)$. Since, $g\mathfrak{x}_{n+1} \succeq g\mathfrak{x}_n$, $n \in \mathbb{N}$, by inequality (3.1) and using Lemma 2.6,

$$\begin{aligned}
\mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},t) &\geq \mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},\phi(s))\\
&= \mathcal{M}(g\mathfrak{x}_{n+1}, g\mathfrak{x}_{n+2},\phi(s))\\
&= \mathcal{M}(\mathcal{A}\mathfrak{x}_n,\mathcal{A}\mathfrak{x}_{n+1},\phi(s))\\
&\geq \min\{\mathcal{M}(g\mathfrak{x}_n, g\mathfrak{x}_{n+1},\phi(\frac{s}{k})), \mathcal{M}(g\mathfrak{x}_n,\mathcal{A}\mathfrak{x}_n,\phi(\frac{s}{k})), \mathcal{M}(g\mathfrak{x}_{n+1},\mathcal{A}\mathfrak{x}_{n+1},\phi(\frac{s}{k})),\\
&\quad \mathcal{M}(g\mathfrak{x}_n,\mathcal{A}\mathfrak{x}_{n+1},2\phi(\frac{s}{k})) * \mathcal{M}(g\mathfrak{x}_{n+1},\mathcal{A}\mathfrak{x}_n,2\phi(\frac{s}{k}))\}\\
&= \min\{\mathcal{M}(\mathfrak{y}_{n-1},\mathfrak{y}_n,\phi(\frac{s}{k})), \mathcal{M}(\mathfrak{y}_{n-1},\mathfrak{y}_n,\phi(\frac{s}{k})), \mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},\phi(\frac{s}{k})),\\
&\quad \mathcal{M}(\mathfrak{y}_{n-1},\mathfrak{y}_{n+1},2\phi(\frac{s}{k})) * \mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_n,2\phi(\frac{s}{k}))\}\\
&\geq \min\{\mathcal{M}(\mathfrak{y}_{n-1},\mathfrak{y}_n,\phi(\frac{s}{k})), \mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},\phi(\frac{s}{k})),\\
&\quad \mathcal{M}(\mathfrak{y}_{n-1},\mathfrak{y}_n,\phi(\frac{s}{k})) * \mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},\phi(\frac{s}{k}))\}\\
&= \min\{\mathcal{M}(\mathfrak{y}_{n-1},\mathfrak{y}_n,\phi(\frac{s}{k})), \mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},\phi(\frac{s}{k}))\}\\
&= \mathcal{M}_n \text{ say.}
\end{aligned}$$

We claim that $\mathcal{M}_n = \mathcal{M}(\mathfrak{y}_{n-1},\mathfrak{y}_n,\phi(\frac{s}{k}))$, if not, then $\mathcal{M}_n = \mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},\phi(\frac{s}{k}))$. Next, using the previous, we attain that
$\mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},\phi(s)) \geq \mathcal{M}_n = \mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},\phi(\frac{s}{k}))$, which on applying Lemma 3.1 , yields a contradiction to the assumption that $\mathfrak{y}_n \neq \mathfrak{y}_{n+1}$, $n \in \mathbb{N}$. Thus, we get $\mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},t) \geq \mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},\phi(s)) \geq \mathcal{M}(\mathfrak{y}_{n-1},\mathfrak{y}_n,\phi(\frac{s}{k}))$, which on repeatedly using this process, implies that

$$\begin{aligned}
\mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},t) &\geq \mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},\phi(s))\\
&\geq \mathcal{M}(\mathfrak{y}_{n-1},\mathfrak{y}_n,\phi(\frac{s}{k})) \geq \cdots \geq \mathcal{M}(\mathfrak{y}_0,\mathfrak{y}_1,\phi(\frac{s}{k^n})),
\end{aligned}$$

i.e.,

$$\mathcal{M}(\mathfrak{y}_{n-1},\mathfrak{y}_n,t) \geq \mathcal{M}(\mathfrak{y}_0,\mathfrak{y}_1,\phi(\frac{s}{k^n})),\ n \in \mathbb{N}. \tag{3.3}$$

Let $m \geq n$, $m, n \in \mathbb{N}$. Now, using Lemma 2.6 and mathematical induction, we obtain

$$\mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_m,(m-n)t) \geq \min\{\mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_{n+1},t),\ldots,\mathcal{M}(\mathfrak{y}_{m-1},\mathfrak{y}_m,t)\}. \tag{3.4}$$

Using (3.3) in (3.4), we obtain

$$\mathcal{M}(\mathfrak{y}_n,\mathfrak{y}_m,(m-n)t) \geq \min\{\mathcal{M}(\mathfrak{y}_0,\mathfrak{y}_1,\phi(\frac{s}{k^n})),\ldots,\mathcal{M}(\mathfrak{y}_0,\mathfrak{y}_1,\phi(\frac{s}{k^{m-1}}))\}. \tag{3.5}$$

Also, using Lemma 2.6 and increasing property of ϕ, we get

$$\min\{\mathcal{M}(\mathfrak{y}_0,\mathfrak{y}_1,\phi(\frac{s}{k^n})),\ldots,\mathcal{M}(\mathfrak{y}_0,\mathfrak{y}_1,\phi(\frac{s}{k^{m-1}}))\} = \mathcal{M}(\mathfrak{y}_0,\mathfrak{y}_1,\phi(\frac{s}{k^n})).$$

Now, (3.5) becomes

$$\mathcal{M}(\mathfrak{y}_n, \mathfrak{y}_m, (m-n)t) \geq \mathcal{M}(\mathfrak{y}_0, \mathfrak{y}_1, \phi(\frac{s}{k^n})). \tag{3.6}$$

Also, using property (ii) of ϕ, we can get the existence of some $n_0 \in \mathbb{N}$, such that

$$\mathcal{M}(\mathfrak{y}_0, \mathfrak{y}_1, \phi(\frac{s}{k^n})) > 1-\delta, \text{ for } n \geq n_0. \tag{3.7}$$

Using (3.6) and (3.7), for every $m \geq n \geq n_0$, we get

$$\mathcal{M}(\mathfrak{y}_n, \mathfrak{y}_m, (m-n)t) > 1-\delta. \tag{3.8}$$

Since, $0 < \delta < 1$ are chosen arbitrarily, we have that $\{\mathfrak{y}_n\}$ is a Cauchy sequence. Without loss of generality, suppose $g(\mathcal{Y})$ is a complete subspace of $\mathcal{Y}$, we have some $\mathfrak{x} \in \mathcal{Y}$, so that

$$\lim_{n\to\infty} \mathfrak{y}_{n-1} = \lim_{n\to\infty} g\mathfrak{x}_n = \lim_{n\to\infty} \mathcal{A}\mathfrak{x}_{n-1} = g\mathfrak{x} = \mathfrak{z} \text{ (say)}. \tag{3.9}$$

For $t > 0$, there exists $s > 0$ so that $t > \phi(s)$. Now, for $s > 0$ and $0 < \delta < 1$, we have some $n_1 \in \mathbb{N}$, so that

$$\mathcal{M}(\mathfrak{y}_{n-1}, \mathfrak{z}, \phi(\frac{s}{k})) > 1-\delta, \quad \text{for all } n \geq n_1. \tag{3.10}$$

Let $p = \max\{n_0, n_1\}$, then inequalities (3.8) and (3.10) holds for all $n \geq p$. Also, $\{g\mathfrak{x}_n\} \subseteq \mathcal{Y}$ is an increasing sequence converging to $g\mathfrak{x}$ in $g(\mathcal{Y})$, then by assumption, $g\mathfrak{x}_n \preceq g\mathfrak{x}$, $n \in \mathbb{N}$. Using inequalities (3.1), (3.3), (3.8) - (3.10), and Lemma 2.6, for $n \geq p$, we attain

$$\begin{aligned}
\mathcal{M}(\mathcal{A}\mathfrak{x}, \mathfrak{y}_n, t) &\geq \mathcal{M}(\mathcal{A}\mathfrak{x}, \mathfrak{y}_n, \phi(s)) \\
&= \mathcal{M}(\mathcal{A}\mathfrak{x}, \mathcal{A}\mathfrak{x}_n, \phi(s)) \\
&\geq \min\{\mathcal{M}(g\mathfrak{x}, g\mathfrak{x}_n, \phi(\frac{s}{k})), \mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{x}, \phi(\frac{s}{k})), \mathcal{M}(g\mathfrak{x}_n, \mathcal{A}\mathfrak{x}_n, \phi(\frac{s}{k})), \\
&\mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{x}_n, 2\phi(\frac{s}{k})) * \mathcal{M}(g\mathfrak{x}_n, \mathcal{A}\mathfrak{x}, 2\phi(\frac{s}{k}))\} \\
&\geq \min\{\mathcal{M}(g\mathfrak{x}, \mathfrak{y}_{n-1}, \phi(\frac{s}{k})), \mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{x}, \phi(\frac{s}{k})), \mathcal{M}(\mathfrak{y}_{n-1}, \mathfrak{y}_n, \phi(\frac{s}{k})), \\
&\mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{x}, \phi(\frac{s}{k}))\} \\
&= \min\{\mathcal{M}(g\mathfrak{x}, \mathfrak{y}_{n-1}, \phi(\frac{s}{k})), \mathcal{M}(\mathfrak{y}_{n-1}, \mathfrak{y}_n, \phi(\frac{s}{k})), \mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{x}, \phi(\frac{s}{k}))\} \\
&= \min\{\mathcal{M}(\mathfrak{z}, \mathfrak{y}_{n-1}, \phi(\frac{s}{k})), \mathcal{M}(\mathfrak{y}_{n-1}, \mathfrak{y}_n, \phi(\frac{s}{k})), \mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{x}, \phi(\frac{s}{k}))\} \\
&\geq \min\{1-\delta, 1-\delta, \mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{x}, \phi(\frac{s}{k}))\}.
\end{aligned}$$

Since, $0 < \delta < 1$ is arbitrary, we obtain $\mathcal{M}(\mathcal{A}\mathfrak{x}, \mathfrak{y}_n, \phi(s)) \geq \mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{x}, \phi(\frac{s}{k}))$. As $n \to \infty$, $\mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{x}, \phi(s)) \geq \mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{x}, \phi(\frac{s}{k}))$. Using Lemma 3.1, we attain $\mathcal{A}\mathfrak{x} = g\mathfrak{x}$. The uniqueness of a coincidence point follows immediately utilizing Lemma 3.1. □

Theorem 3.5. *If self-mappings $\mathcal{A}$ and g are also weakly compatible in Theorem 3.4, then $\mathcal{A}$ and g have a unique common fixed point in $\mathcal{Y}$.*

Proof. In Theorem 3.4, we obtained some $\mathfrak{x} \in \mathcal{Y}$ such that inequality (3.9) holds and $\mathcal{A}\mathfrak{x} = g\mathfrak{x} = \mathfrak{z}$. Since $\mathcal{A}$ and g are weakly compatible, we obtain $\mathcal{A}g\mathfrak{x} = g\mathcal{A}\mathfrak{x}$, i.e., $\mathcal{A}\mathfrak{z} = g\mathfrak{z}$.
First, we claim that $\mathcal{A}\mathfrak{z} = \mathfrak{z}$. Since $\{g\mathfrak{x}_n\} \subseteq \mathcal{Y}$ is an increasing sequence converging to $g\mathfrak{x}$ in $g(\mathcal{Y})$, then by given assumption, $g\mathfrak{x}_n \preceq g\mathfrak{x}$, $n \in \mathbb{N}$ and $g\mathfrak{x} \preceq gg\mathfrak{x}$, so that $g\mathfrak{x} \preceq g\mathfrak{z}$. Then, using inequality (3.1) and Lemma 2.6, we attain

$$\begin{aligned}
\mathcal{M}(\mathcal{A}\mathfrak{z}, \mathfrak{z}, t) &\geq \mathcal{M}(\mathcal{A}\mathfrak{z}, \mathfrak{z}, \phi(s)) \\
&= \mathcal{M}(\mathcal{A}\mathfrak{z}, \mathcal{A}\mathfrak{x}, \phi(s)) \\
&\geq \min\{\mathcal{M}(g\mathfrak{z}, g\mathfrak{x}, \phi(\frac{s}{k})), \mathcal{M}(g\mathfrak{z}, \mathcal{A}\mathfrak{z}, \phi(\frac{s}{k})), \mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{x}, \phi(\frac{s}{k})), \\
&\quad \mathcal{M}(g\mathfrak{z}, \mathcal{A}\mathfrak{x}, 2\phi(\frac{s}{k})) * \mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{z}, 2\phi(\frac{s}{k}))\} \\
&= \min\{\mathcal{M}(\mathcal{A}\mathfrak{z}, \mathfrak{z}, \phi(\frac{s}{k})), 1, 1, \mathcal{M}(g\mathfrak{z}, \mathfrak{z}, 2\phi(\frac{s}{k})) * \mathcal{M}(\mathfrak{z}, \mathcal{A}\mathfrak{z}, 2\phi(\frac{s}{k}))\} \\
&= \min\{\mathcal{M}(\mathcal{A}\mathfrak{z}, \mathfrak{z}, \phi(\frac{s}{k})), 1\} \\
&= \mathcal{M}(\mathcal{A}\mathfrak{z}, \mathfrak{z}, \phi(\frac{s}{k})),
\end{aligned}$$

i.e., $\mathcal{M}(\mathcal{A}\mathfrak{z}, \mathfrak{z}, \phi(s)) \geq \mathcal{M}(\mathcal{A}\mathfrak{z}, \mathfrak{z}, \phi(\frac{s}{k}))$. Then, using Lemma 3.1, we obtain $\mathcal{A}\mathfrak{z} = \mathfrak{z}$. Hence, $\mathcal{A}\mathfrak{z} = \mathfrak{z} = g\mathfrak{z}$. The uniqueness of a common fixed point of $\mathcal{A}$ and g immediately follows on utilizing Lemma 3.1. □

The following example is given to appreciate the effectiveness of Theorems 3.4 and 3.5 and to validate the results proved herein.

Example 3.6. Let $\mathcal{Y} = [0, \frac{1}{8}) \cup \{\frac{1}{4}\}$ and partial ordering $\preceq$ is the natural ordering $\leq$ of the real numbers. Define $\mathcal{M}(\mathfrak{x}, \mathfrak{y}, t) = \frac{t}{t+|\mathfrak{x}-\mathfrak{y}|}$, $\mathfrak{x}, \mathfrak{y} \in \mathcal{Y}$ and $t > 0$. Let $\mathfrak{x} * \mathfrak{y} = \min\{\mathfrak{x}, \mathfrak{y}\}$, $\mathfrak{x}, \mathfrak{y} \in [0, 1]$. Then $(\mathcal{Y}, \mathcal{M}, *)$ is a non-complete fuzzy metric space equipped with a partial order " $\preceq$ ". Define two self-mappings $\mathcal{A} : \mathcal{Y} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Y}$ respectively as:

$$g(\mathfrak{x}) = \begin{cases} \frac{\mathfrak{x}}{2}, & \mathfrak{x} \in [0, \frac{1}{16}] \\ \mathfrak{x}, & \mathfrak{x} \in (\frac{1}{16}, \frac{1}{8}) \\ \frac{1}{4}, & \mathfrak{x} = \frac{1}{4} \end{cases} \text{ and } \mathcal{A}\mathfrak{x} = \begin{cases} \frac{\mathfrak{x}}{8}, & \mathfrak{x} \in [0, \frac{1}{8}) \\ \frac{1}{64}, & \mathfrak{x} = \frac{1}{4} \end{cases}.$$

Noticeably, g and $\mathcal{A}$ are non-commuting, since $g(\mathcal{A}(\frac{1}{15})) \neq \mathcal{A}(g(\frac{1}{15}))$, $\mathcal{A}(\mathcal{Y}) \subseteq g(\mathcal{Y})$ and $\mathcal{A}(\mathcal{Y})$ is complete. Also, the mappings $\mathcal{A}$ and g are weakly compatible. We claim that the mapping $\mathcal{A}$ is $g-$monotonically increasing. For, let $\mathfrak{x}_1, \mathfrak{x}_2 \in \mathcal{Y}$ such that $g(\mathfrak{x}_1) > g(\mathfrak{x}_2)$, then we have the subsequent possibilities:

Case (i) If $\mathfrak{x}_1, \mathfrak{x}_2 \in [0, 1/16]$, then $g(\mathfrak{x}_1) > g(\mathfrak{x}_2)$ implies that $\frac{\mathfrak{x}_1}{2} > \frac{\mathfrak{x}_2}{2}$, which yields that $\frac{\mathfrak{x}_1}{8} > \frac{\mathfrak{x}_2}{8}$, i.e., $\mathcal{A}\mathfrak{x}_1 > \mathcal{A}\mathfrak{x}_2$.

Case (ii) If $\mathfrak{x}_1 \in (1/16, 1/8)$, $\mathfrak{x}_2 \in [0, 1/16]$, then, clearly $\mathfrak{x}_1 > \mathfrak{x}_2$ which yields that $\frac{\mathfrak{x}_1}{8} > \frac{\mathfrak{x}_2}{8}$, i.e., $\mathcal{A}\mathfrak{x}_1 > \mathcal{A}\mathfrak{x}_2$.

Case (iii) If $\mathfrak{x}_1, \mathfrak{x}_2 \in (1/16, 1/8)$, then $g\mathfrak{x}_1 > g\mathfrak{x}_2$ implies that $\mathfrak{x}_1 > \mathfrak{x}_2$, which yields that $\frac{\mathfrak{x}_1}{8} > \frac{\mathfrak{x}_2}{8}$, i.e., $\mathcal{A}\mathfrak{x}_1 > \mathcal{A}\mathfrak{x}_2$.

Case (iv) If $\mathfrak{x}_1 = \frac{1}{4}$, $\mathfrak{x}_2 \in [0, 1/8)$, then $g\mathfrak{x}_1 > g\mathfrak{x}_2$ implies that $\frac{1}{4} > g\mathfrak{x}_2$ in both the cases whether $\mathfrak{x}_2 \in [0, 1/16]$ or $\mathfrak{x}_2 \in (1/16, 1/8)$, which yields that $\mathcal{A}\mathfrak{x}_1 = \frac{1}{64} > \frac{\mathfrak{x}_2}{8} = \mathcal{A}\mathfrak{x}_2$.

This shows that $\mathcal{A}$ is $g-$monotonically increasing.
Define the function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ as $\phi(t) = \frac{t}{2}$, $t \in \mathbb{R}^+$. Then, ϕ is a $\Phi-$function.
Also,
$\frac{t}{t+\mathcal{U}} \geq \min\{\frac{t}{t+\mathcal{P}}, \frac{t}{t+\mathcal{Q}}, \frac{t}{t+\mathcal{R}}, \frac{t}{t+\mathcal{S}}, \frac{t}{t+\mathcal{T}}\}$ iff $\mathcal{U} \leq \max\{\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}\}$,
$\mathcal{U}, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T} \geq 0,\ t > 0$.
Consider $k = \frac{1}{2} \in (0, 1)$. Now, the inequality (3.1)

$$\mathcal{M}(\mathcal{A}\mathfrak{x}, \mathcal{A}\mathfrak{y}, \phi(t)) \geq \min\{\mathcal{M}(g\mathfrak{x}, g\mathfrak{y}, \phi(\frac{t}{k})), \mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{x}, \phi(\frac{t}{k})), \mathcal{M}(g\mathfrak{y}, \mathcal{A}\mathfrak{y}, \phi(\frac{t}{k})),$$
$$\mathcal{M}(g\mathfrak{x}, \mathcal{A}\mathfrak{y}, 2\phi(\frac{t}{k})), \mathcal{M}(gy, \mathcal{A}\mathfrak{x}, 2\phi(\frac{t}{k}))\},$$

is equivalent to the following

$$2|\mathcal{A}\mathfrak{x} - \mathcal{A}\mathfrak{y}| \leq \max\{|g\mathfrak{x} - g\mathfrak{y}|, |g\mathfrak{x} - \mathcal{A}\mathfrak{x}|, |g\mathfrak{y} - \mathcal{A}\mathfrak{y}|, \frac{1}{2}|g\mathfrak{x} - \mathcal{A}\mathfrak{y}|, \frac{1}{2}|g\mathfrak{y} - \mathcal{A}\mathfrak{x}|\} \quad (3.11)$$

We now verify inequality (3.11) for $\mathfrak{x}, \mathfrak{y} \in \mathcal{Y}$ so that $g\mathfrak{x} \geq g\mathfrak{y}$. Now, consider the subsequent main cases:

Case (1) Let $\mathfrak{x} \in [0, 1/16]$, we discuss the subsequent possibilities:

(1.1) Let $\mathfrak{y} \in [0, 1/16]$. Then, $g\mathfrak{x} \geq g\mathfrak{y}$ implies that $\frac{\mathfrak{x}}{2} \geq \frac{\mathfrak{y}}{2}$, i.e., $\mathfrak{x} \geq \mathfrak{y}$, and
$2|\mathcal{A}\mathfrak{x} - \mathcal{A}\mathfrak{y}| \leq \max\{|g\mathfrak{x} - g\mathfrak{y}|, |g\mathfrak{x} - \mathcal{A}\mathfrak{x}|, |g\mathfrak{y} - \mathcal{A}\mathfrak{y}|, \frac{1}{2}|g\mathfrak{x} - \mathcal{A}\mathfrak{y}|, \frac{1}{2}|g\mathfrak{y} - \mathcal{A}\mathfrak{x}|\}$,
iff $|\frac{\mathfrak{x}}{4} - \frac{\mathfrak{y}}{4}| \leq \max\{|\frac{\mathfrak{x}}{2} - \frac{\mathfrak{y}}{2}|, \frac{3\mathfrak{x}}{8}, \frac{3\mathfrak{y}}{8}, \frac{1}{2}|\frac{\mathfrak{x}}{2} - \frac{\mathfrak{y}}{8}|, \frac{1}{2}|\frac{\mathfrak{y}}{2} - \frac{\mathfrak{x}}{8}|\}$,
iff $|\mathfrak{x} - \mathfrak{y}| \leq \max\{2|\mathfrak{x} - \mathfrak{y}|, \frac{3\mathfrak{x}}{2}, \frac{3\mathfrak{y}}{2}, |\mathfrak{x} - \frac{\mathfrak{y}}{4}|, |y - \frac{\mathfrak{x}}{4}|\}$,
which is true, since $|\mathfrak{x} - \mathfrak{y}| \leq 2|\mathfrak{x} - \mathfrak{y}|$.

(1.2) Let $\mathfrak{y} \in (1/16, 1/8)$. Then, $g\mathfrak{x} \geq g\mathfrak{y}$ implies that $\frac{\mathfrak{x}}{2} \geq \mathfrak{y}$, i.e., $\mathfrak{x} \geq 2\mathfrak{y}$, i.e., which is not possible, since $\mathfrak{x} \in [0, 1/16]$ and $\mathfrak{y} \in (1/16, 1/8)$.

(1.3) Let $\mathfrak{y} = \frac{1}{4}$. Then, $g\mathfrak{x} \geq g\mathfrak{y}$ implies that $\frac{\mathfrak{x}}{2} \geq \frac{1}{4}$, i.e., $\mathfrak{x} \geq \frac{1}{2}$, which is also not possible, since $\mathfrak{x} \in [0, 1/16]$.

Case (2) Let $\mathfrak{x} \in (1/16, 1/8)$, we discuss the subsequent possibilities:

(2.1) Let $\mathfrak{y} \in [0, 1/16]$. Then, $g\mathfrak{x} \geq g\mathfrak{y}$ implies that $\mathfrak{x} \geq \frac{\mathfrak{y}}{2}$, and
$2|\mathcal{A}\mathfrak{x} - \mathcal{A}\mathfrak{y}| \leq \max\{|g\mathfrak{x} - g\mathfrak{y}|, |g\mathfrak{x} - \mathcal{A}\mathfrak{x}|, |g\mathfrak{y} - \mathcal{A}\mathfrak{y}|, \frac{1}{2}|g\mathfrak{x} - \mathcal{A}\mathfrak{y}|, \frac{1}{2}|g\mathfrak{y} - \mathcal{A}\mathfrak{x}|\}$,
iff $2|\frac{\mathfrak{x}}{8} - \frac{\mathfrak{y}}{8}| \leq \max\{|\mathfrak{x} - \frac{\mathfrak{y}}{2}|, |\mathfrak{x} - \frac{\mathfrak{x}}{8}|, |\frac{\mathfrak{y}}{2} - \frac{\mathfrak{y}}{8}|, \frac{1}{2}|\mathfrak{x} - \frac{\mathfrak{y}}{8}|, \frac{1}{2}|\frac{\mathfrak{y}}{2} - \frac{\mathfrak{x}}{8}|\}$,
iff $2|\frac{\mathfrak{x}}{4} - \frac{\mathfrak{y}}{4}| \leq \max\{|\mathfrak{x} - \frac{\mathfrak{y}}{2}|, \frac{7\mathfrak{x}}{8}, \frac{3\mathfrak{y}}{8}\frac{1}{2}|\mathfrak{x} - \frac{\mathfrak{y}}{8}|, \frac{1}{2}|\frac{\mathfrak{y}}{2} - \frac{\mathfrak{x}}{8}|\}$,
iff $|\mathfrak{x} - \mathfrak{y}| \leq \max\{4|\mathfrak{x} - \frac{\mathfrak{y}}{2}|, \frac{7\mathfrak{x}}{2}, \frac{3\mathfrak{y}}{2}, 2|\mathfrak{x} - \frac{\mathfrak{y}}{8}|, |\mathfrak{y} - \frac{\mathfrak{x}}{4}|\}$,
which is true, since $|\mathfrak{x} - \mathfrak{y}| \leq 4|\mathfrak{x} - \frac{\mathfrak{y}}{2}|$.

(2.2) Let $\mathfrak{y} \in (1/16, 1/8)$. Then, $g\mathfrak{x} \geq g\mathfrak{y}$ implies that $\mathfrak{x} \geq \mathfrak{y}$, and
$2|\mathcal{A}\mathfrak{x} - \mathcal{A}\mathfrak{y}| \leq \max\{|g\mathfrak{x} - g\mathfrak{y}|, |g\mathfrak{x} - \mathcal{A}\mathfrak{x}|, |g\mathfrak{y} - \mathcal{A}\mathfrak{y}|, \frac{1}{2}|g\mathfrak{y} - \mathcal{A}\mathfrak{x}|\}$,

iff $2|\frac{\mathfrak{x}}{8} - \frac{\mathfrak{y}}{8}| \leq \max\{|\mathfrak{x} - \mathfrak{y}|, |\mathfrak{x} - \frac{\mathfrak{x}}{8}|, |\mathfrak{y} - \frac{\mathfrak{y}}{8}|, \frac{1}{2}|\mathfrak{x} - \frac{\mathfrak{y}}{8}|, \frac{1}{2}|\mathfrak{y} - \frac{\mathfrak{x}}{8}|\}$,
iff $|\mathfrak{x} - \mathfrak{y}| \leq \max\{4|\mathfrak{x} - \mathfrak{y}|, \frac{7\mathfrak{x}}{2}, \frac{7\mathfrak{y}}{2}, 2|\mathfrak{x} - \frac{\mathfrak{y}}{8}|, 2|\mathfrak{y} - \frac{\mathfrak{x}}{8}|\}$,
which is true, since $|\mathfrak{x} - \mathfrak{y}| \leq 4|\mathfrak{x} - \mathfrak{y}|$.

(2.3) Let $\mathfrak{y} = \frac{1}{4}$. Then, $g\mathfrak{x} \geq g\mathfrak{y}$ implies that $\mathfrak{x} \geq \frac{1}{4}$, which is not possible.

Case (3) Let $\mathfrak{x} = \frac{1}{4}$, we discuss the following possibilities:

(3.1) Let $\mathfrak{y} \in [0, 1/16]$. Then, $g\mathfrak{x} \geq g\mathfrak{y}$ implies that $\frac{1}{4} \geq \frac{\mathfrak{y}}{2}$, i.e., $\frac{1}{2} \geq \mathfrak{y}$ or $\mathfrak{y} \leq \frac{1}{2}$, and
$2|\mathcal{A}\mathfrak{x} - \mathcal{A}\mathfrak{y}| \leq \max\{|g\mathfrak{x} - g\mathfrak{y}|, |g\mathfrak{x} - \mathcal{A}\mathfrak{x}|, |g\mathfrak{y} - \mathcal{A}\mathfrak{y}|, \frac{1}{2|g\mathfrak{y} - \mathcal{A}\mathfrak{x}|}\}$,
iff $2|\frac{1}{64} - \frac{\mathfrak{y}}{8}| \leq \max\{|\frac{1}{4} - \frac{\mathfrak{y}}{2}|, |\frac{1}{4} - \frac{1}{64}|, |\frac{\mathfrak{y}}{2} - \frac{\mathfrak{y}}{8}|, \frac{1}{2}|\frac{1}{4} - \frac{\mathfrak{y}}{8}|, \frac{1}{2}|\frac{\mathfrak{y}}{2} - \frac{1}{64}|\}$,
iff $|\frac{1}{32} - \frac{\mathfrak{y}}{4}| \leq \max\{|\frac{1}{4} - \frac{\mathfrak{y}}{2}|, \frac{15}{64}, \frac{3\mathfrak{y}}{8}, \frac{1}{8}, |1 - \frac{\mathfrak{y}}{2}|, \frac{1}{4}|\mathfrak{y} - \frac{1}{32}|\}$,
iff $|\frac{1}{8} - \mathfrak{y}| \leq \max\{2|\frac{1}{2} - \mathfrak{y}|, \frac{15}{16}, \frac{3\mathfrak{y}}{2}, \frac{1}{2}, |1 - \frac{\mathfrak{y}}{2}|, |\mathfrak{y} - \frac{1}{32}|\}$,
which is true, since $|\frac{1}{8} - \mathfrak{y}| \leq \frac{15}{16}$.

(3.2) Let $\mathfrak{y} \in (1/16, 1/8)$. Then, $g\mathfrak{x} \geq g\mathfrak{y}$ implies that $\frac{1}{4} \geq \mathfrak{y}$ or $\frac{1}{4} \leq \mathfrak{y}$, and
$2|\mathcal{A}\mathfrak{x} - \mathcal{A}\mathfrak{y}| \leq \max\{|g\mathfrak{x} - g\mathfrak{y}|, |g\mathfrak{x} - \mathcal{A}\mathfrak{x}|, |g\mathfrak{y} - \mathcal{A}\mathfrak{y}|, \frac{1}{2}|g\mathfrak{x}\mathfrak{x} - \mathcal{A}\mathfrak{y}|, \frac{1}{2}|g\mathfrak{y} - \mathcal{A}\mathfrak{x}|\}$,
iff $2|\frac{1}{64} - \frac{\mathfrak{y}}{8}| \leq \max\{|\frac{1}{4} - \mathfrak{y}|, |\frac{1}{4} - \frac{1}{64}|, |\mathfrak{y} - \frac{\mathfrak{y}}{8}|, \frac{1}{2}|\frac{1}{4} - \frac{\mathfrak{y}}{8}|, \frac{1}{2}|\mathfrak{y} - \frac{1}{64}|\}$,
iff $|\frac{1}{32} - \frac{\mathfrak{y}}{4}| \leq \max\{|\frac{1}{4} - \mathfrak{y}|, \frac{15}{64}, \frac{7\mathfrak{y}}{8}, \frac{1}{8}, |1 - \frac{\mathfrak{y}}{2}|, \frac{1}{2}|\mathfrak{y} - \frac{1}{64}|\}$,
which is true, since $|\frac{1}{32} - \frac{\mathfrak{y}}{4}| \leq \frac{7\mathfrak{y}}{8}$.

(3.3) Let $\mathfrak{y} = \frac{1}{4}$. Then, we have $g\mathfrak{x} = g\frac{1}{4} = g\mathfrak{y}$, and
$2|\mathcal{A}\mathfrak{x} - \mathcal{A}\mathfrak{y}| \leq \max\{|g\mathfrak{x} - g\mathfrak{y}|, |g\mathfrak{x} - \mathcal{A}\mathfrak{x}|, |g\mathfrak{y} - \mathcal{A}\mathfrak{y}|, \frac{1}{2}|g\mathfrak{x} - \mathcal{A}\mathfrak{y}|, \frac{1}{2}|g\mathfrak{y} - \mathcal{A}\mathfrak{x}|\}$,
which is true, since $\mathcal{A}\mathfrak{x} = \mathcal{A}\frac{1}{4} = \mathcal{A}\mathfrak{y}$.

Hence, in all the cases, the inequality (3.11) is verified. Consequently, inequality (3.1) holds, i.e., the pair $(\mathcal{A}, g)$ is an ordered generalized $\phi-$contraction. Further, we have $\mathfrak{x}_0 = 0 \in \mathcal{Y}$ so that $g\mathfrak{x}_0 = g0 = 0 = \mathcal{A}0 = \mathcal{A}\mathfrak{x}_0$. Now, all the postulates of Theorem 3.5 are verified and a unique common fixed point of $\mathcal{A}$ and g exists in $\mathcal{Y}$, which is indeed 0.

Remark 3.7. It is worth mentioning that Theorem 3.5 is a genuine extension and improvement of Aage et al. [1] to an ordered fuzzy metric space, in view of the fact that, we have neither used the completeness of the entire space nor continuity of mappings under consideration. Rather, we have used a relatively weaker notion like the completeness of any subspace of the entire space. One may also check by simple calculations that in Example 3.6, mappings under consideration are not continuous. Moreover, an ordered generalized $\phi-$contraction for a pair of mappings is a significant generalization of Aage et al. [1], which is presumed to hold only on the elements of the underlying ordered set.

Theorem 3.8. *Let $(\mathcal{Y}, \mathcal{M}, *)$ be a complete fuzzy metric space equipped with a partial order $\preceq$ and a continuous $t-$norm satisfying $\mathfrak{a} * \mathfrak{a} \geq \mathfrak{a}$, $\mathfrak{a} \in [0,1]$. Let a self-mapping $\mathcal{A}$ on $\mathcal{Y}$ be a monotonically increasing ordered generalized $\phi-$contraction. Further, if $\{\mathfrak{x}_n\} \subset \mathcal{Y}$ is an increasing sequence converging to $\mathfrak{x} \in \mathcal{Y}$, then $\mathfrak{x}_n \preceq$*

$\mathfrak{x}$, $n \in \mathbb{N}$. *If we have* $\mathfrak{x}_0 \in \mathcal{Y}$ *so that* $\mathcal{A}\mathfrak{x}_0 \succeq \mathfrak{x}_0$*, then* $\mathcal{A}$ *has a unique fixed point in* $\mathcal{Y}$.

Proof. If g is an identity mapping on $\mathcal{Y}$, the result holds immediately on the pattern of Theorem 3.5. □

Theorem 3.9. *Let* $(\mathcal{Y},\mathcal{M},*)$ *be a complete fuzzy metric space equipped with a partial order* $\preceq$ *and a continuous* $t-$*norm satisfying* $\mathfrak{a} * \mathfrak{a} \geq \mathfrak{a}$, $\mathfrak{a} \in [0,1]$. *Let* $k \in (0,1)$ *be fixed. Let a self-mapping* $\mathcal{A}$ *on* $\mathcal{Y}$ *be monotonically increasing ordered* $\phi-$*contraction, i.e.,*

$$\mathcal{M}(\mathcal{A}\mathfrak{x},\mathcal{A}\mathfrak{y},\phi(t)) \geq \mathcal{M}(\mathfrak{x},\mathfrak{y},\phi(\frac{t}{k})),\ \mathfrak{x} \succeq \mathfrak{y},\ t>0,\ \textit{and}\ \mathfrak{x},\mathfrak{y} \in \mathcal{Y}. \tag{3.12}$$

Further, if $\{\mathfrak{x}_n\} \subset \mathcal{Y}$ *is an increasing sequence converging to* $\mathfrak{x} \in \mathcal{Y}$*, then* $\mathfrak{x}_n \preceq \mathfrak{x}$, $n \in \mathbb{N}$. *If we have* $\mathfrak{x}_0 \in \mathcal{Y}$ *so that* $\mathcal{A}\mathfrak{x}_0 \succeq \mathfrak{x}_0$*, then* $\mathcal{A}$ *has a unique fixed point in* $\mathcal{Y}$.

Proof. Since (3.12) is true for $\mathfrak{x},\mathfrak{y} \in \mathcal{Y}$, $\mathfrak{x} \succeq \mathfrak{y}$ and $t > 0$, we attain

$$\begin{aligned}\mathcal{M}(\mathcal{A}\mathfrak{x},\mathcal{A}\mathfrak{y},\phi(t)) \geq \mathcal{M}(\mathfrak{x},\mathfrak{y},\phi(\frac{t}{k})) \geq \min\{&\mathcal{M}(\mathcal{A}\mathfrak{x},\mathcal{A}\mathfrak{y},\phi(\frac{t}{k})),\mathcal{M}(\mathfrak{x},\mathcal{A}\mathfrak{x},\phi(\frac{t}{k})),\\ &\mathcal{M}(\mathfrak{y},\mathcal{A}\mathfrak{y},\phi(\frac{t}{k})),\mathcal{M}(\mathfrak{x},\mathcal{A}\mathfrak{y},2\phi(\frac{t}{k})),\mathcal{M}(y,\mathcal{A}\mathfrak{x},2\phi(\frac{t}{k}))\}.\end{aligned}$$

Hence, inequality (3.2) is true for $\mathfrak{x},\mathfrak{y} \in \mathcal{Y}$, $t > 0$, and $\mathfrak{x} \succeq \mathfrak{y}$. Utilizing Theorem 3.8, $\mathcal{A}$ has a fixed point in $\mathcal{Y}$. Uniqueness of a fixed point immediately exists on utilizing Lemma 3.1. □

Next, we formulate an interesting result in an ordered metric space:

Theorem 3.10. *Let* $(\mathcal{Y},d)$ *be a complete metric space equipped with a partial order* $\preceq$. *Let* $k \in (0,1)$ *be fixed. Let a self-mapping* $\mathcal{A}$ *on* $\mathcal{Y}$ *be monotonically increasing. Suppose that the following holds:*

$$d(\mathcal{A}\mathfrak{x},\mathcal{A}\mathfrak{y}) \leq kd(\mathfrak{x},\mathfrak{y}),\ \mathfrak{x},\mathfrak{y} \in \mathcal{Y},\ \textit{and}\ \ \mathfrak{x} \succeq \mathfrak{y}. \tag{3.13}$$

Further, if $\{\mathfrak{x}_n\} \subseteq \mathcal{Y}$ *is an increasing sequence converging to* $\mathfrak{x}$*, then* $\mathfrak{x}_n \preceq \mathfrak{x}$, $n \in \mathbb{N}$. *If we have* $\mathfrak{x}_0 \in \mathcal{Y}$ *so that* $\mathcal{A}\mathfrak{x}_0 \succeq \mathfrak{x}_0$*, then* $\mathcal{A}$ *has a unique fixed point in* $\mathcal{Y}$.

Proof. Define $\mathcal{M}(\mathfrak{x},\mathfrak{y},t) = \frac{t}{t+d(\mathfrak{x},\mathfrak{y})}$, $\mathfrak{x},\mathfrak{y} \in \mathcal{Y}$, $t > 0$, and $\mathfrak{a} * \mathfrak{b} = \min\{\mathfrak{a},\mathfrak{b}\}$. Then, $(\mathcal{Y},\mathcal{M},*)$ is a fuzzy metric space. We assert that inequality (3.13) implies (3.12) for $\phi(t) = t$, $t \in \mathbb{R}^+$.
Otherwise, from inequality (3.12), $\frac{t}{t+d(\mathcal{A}\mathfrak{x},\mathcal{A}\mathfrak{y})} < \frac{\frac{t}{k}}{\frac{t}{k}+d(\mathfrak{x},\mathfrak{y})}$ or $t + kd(\mathfrak{x},\mathfrak{y}) < t + d(\mathcal{A}\mathfrak{x},\mathcal{A}\mathfrak{y})$, which implies $kd(\mathfrak{x},\mathfrak{y}) < d(\mathcal{A}\mathfrak{x},\mathcal{A}\mathfrak{y})$, a contradiction to inequality (3.13). Then, the proof follows immediately by the application of Theorem 3.9. □

Remark 3.11. (i) It is interesting to mention that order-theoretic contractions are comparatively weaker than standard contractions since these hold only for the elements in the partially ordered set under consideration (see, Examples 3.6). Consequently, we are able to particularize the existing results to a variety of situations.

(ii) Noticeably, in Rhoades [12], ϕ is taken to be continuous, non-decreasing, and $\lim_{t\to\infty}\phi(t)=\infty$. Consequently, our version is more improved than Rhoades [12].
(iii) Theorem 3.9 may be considered as an analogue of Theorem 19 of Aage et al. [1] for an ordered generalized $\phi-$contraction for a pair of mappings in the set-up of ordered fuzzy metric spaces.
(iv) Theorem 3.10 is actually Theorem 2.1 of Nieto and Rodríguez-López [11].

4. Application

As an application, we solve functional equations emerging in dynamic programming which were first investigated by Bellman [3]-[4] using the famous Banach fixed point theorem. A dynamical process is composed of a state space and a decision space (initial state actions and transition model and possible actions that are allowed). In a way, dynamic programming is a beneficial tool for both mathematical optimizations as well as computer programming. Let $\mathcal{W}\subseteq\mathcal{X}$, $\mathcal{D}\subseteq\mathcal{Y}$, and $\mathcal{B}(\mathcal{W})$ symbolize the state space, decision space, and the set of bounded functions respectively on $\mathcal{W}$. Define, $\mathcal{M}(\mathfrak{x},\mathfrak{y},t)=e^{-\frac{d(\mathfrak{x},\mathfrak{y})}{t}}$, with $t-$norm $\mathfrak{a}*\mathfrak{b}=\min\{\mathfrak{a},\mathfrak{b}\}$, $\mathfrak{a}*\mathfrak{b}\in[0,1]$, where, $d(\mathfrak{x},\mathfrak{y})=\|\mathfrak{x}(\tau)-\mathfrak{y}(\tau)\|_\infty=\sup|\mathfrak{x}-\mathfrak{y}|\tau,\ \tau\in\mathcal{W}$. Then $(\mathcal{B}(\mathcal{W}),\mathcal{M},*)$ is a complete fuzzy metric space. Let it be equipped with a partial order " $\preceq$ ".

Theorem 4.1. *Let $\mathcal{A}$ and g be self-mappings of $(\mathcal{B}(\mathcal{W}),\mathcal{M},*)$ equipped with a partial order $\preceq$. If the following hypotheses hold:*
(a) $u:\mathcal{W}\times\mathcal{D}\times\mathbb{R}\longrightarrow\mathbb{R}$ *and* $\mathcal{H}:\mathcal{W}\times\mathcal{D}\times\mathbb{R}\longrightarrow\mathbb{R}$ *are bounded and* $\tau:\mathcal{W}\times\mathcal{D}\longrightarrow\mathcal{W}$ *denotes transformation of the process.*
(b) $\exists$ *a* $\delta\in(0,1)$ *so that:*
$$|\mathcal{H}(\mathfrak{x},\mathfrak{y},\mathcal{A}h(\tau(\mathfrak{x},\mathfrak{y})))-\mathcal{H}(\mathfrak{x},\mathfrak{y},\mathcal{A}k(\tau(\mathfrak{x},\mathfrak{y})))|\le\delta(h,k,\phi(\frac{t}{k})),\tag{4.1}$$
where, $\delta(h,k,\phi(\frac{t}{k}))=\min\{\mathcal{M}(gx,gy,\phi(\frac{t}{k})),\mathcal{M}(g\mathfrak{x},\mathcal{A}\mathfrak{x},\phi(\frac{t}{k})),$ $\mathcal{M}(g\mathfrak{y},\mathcal{A}\mathfrak{y},\phi(\frac{t}{k})),\mathcal{M}(g\mathfrak{x},\mathcal{A}\mathfrak{y},2\phi(\frac{t}{k})),\mathcal{M}(g\mathfrak{y},\mathcal{A}\mathfrak{x},2\phi(\frac{t}{k}))\}$, *for a* $\phi-$ *function,* $h,k\in\mathcal{B}(\mathcal{W})$ *and* $(\mathfrak{x},\mathfrak{y})\in\mathcal{W}\times\mathcal{D}$, *with* $g\mathfrak{x}\succeq g\mathfrak{y}$ *and* $t>0$,
(c) $\mathcal{A}$ *is* $g-$*monotonically increasing,*
(d) $\mathcal{A}\mathcal{Y}\subseteq g\mathcal{Y}$,
(e) $\mathcal{A}gh=g\mathcal{A}h$, *whenever,* $gh=\mathcal{A}h$.

Then the system
$$\begin{cases}\mathcal{A}h(t)=\sup_{\mathfrak{x}\in\mathcal{W}}\{u(\mathfrak{x},\mathfrak{y})\}+\mathcal{H}(\mathfrak{x},\mathfrak{y},\mathcal{A}h(\tau(\mathfrak{x},\mathfrak{y})))\\ \mathcal{A}k(t)=\sup_{\mathfrak{x}\in\mathcal{W}}\{u(\mathfrak{x},\mathfrak{y})\}+\mathcal{H}(\mathfrak{x},\mathfrak{y},\mathcal{A}k(\tau(\mathfrak{x},\mathfrak{y})))\end{cases},\tag{4.2}$$
has a unique solution in $\mathcal{B}(\mathcal{W})$.

Proof. The system has a unique solution iff $\mathcal{A}$ and g have a single common point. For $h,k\in\mathcal{B}(\mathcal{W})$ and $\epsilon>0$, $\exists\mathfrak{y},\mathfrak{z}\in\mathcal{D}$, so that
$$\mathcal{A}h<u(\mathfrak{x},\mathfrak{y})+\mathcal{H}(\mathfrak{x},\mathfrak{y},\mathcal{A}h(\tau(\mathfrak{x},\mathfrak{y})))+\epsilon,\tag{4.3}$$
$$\mathcal{A}k<u(\mathfrak{x},\mathfrak{z})+\mathcal{H}(\mathfrak{x},\mathfrak{z},\mathcal{A}k(\tau(\mathfrak{x},\mathfrak{z})))+\epsilon,\tag{4.4}$$
and since,
$$\mathcal{A}h\ge u(\mathfrak{x},\mathfrak{z})+\mathcal{H}(\mathfrak{x},\mathfrak{z},\mathcal{A}h(\tau(\mathfrak{x},\mathfrak{z}))),\tag{4.5}$$

$$\mathcal{A}k \geq u(\mathfrak{x}, \mathfrak{y}) + \mathcal{H}(\mathfrak{x}, \mathfrak{y}, \mathcal{A}k(\tau(\mathfrak{x}, \mathfrak{y}))), \tag{4.6}$$

then from inequalities (4.3) and (4.6)

$$\begin{aligned} \mathcal{A}h - \mathcal{A}k &\leq \mathcal{H}(\mathfrak{x}, \mathfrak{y}, \mathcal{A}h(\tau(\mathfrak{x}, \mathfrak{y}))) - \mathcal{H}(\mathfrak{x}, \mathfrak{y}, \mathcal{A}k(\tau(\mathfrak{x}, \mathfrak{y}))) + \epsilon \\ &\leq \delta(h, k, \phi(\frac{t}{k})) + \epsilon. \end{aligned} \tag{4.7}$$

Also from inequalities (4.4) and (4.5)

$$\begin{aligned} \mathcal{A}h - \mathcal{A}k &> \mathcal{H}(\mathfrak{x}, \mathfrak{z}, \mathcal{A}h(\tau(\mathfrak{x}, \mathfrak{z}))) - \mathcal{H}(\mathfrak{x}, \mathfrak{z}, \mathcal{A}k(\tau(\mathfrak{x}, \mathfrak{z}))) - \epsilon \\ &\geq -\delta(h, k, \phi(\frac{t}{k})) - \epsilon. \end{aligned} \tag{4.8}$$

Consequently, inequalities (4.7) and (4.8) implies that

$$\begin{aligned} d(\mathcal{A}h, \mathcal{A}k) &= \sup |\mathcal{A}h - \mathcal{A}k| \tau \\ &\leq \delta(h, k, \phi(\frac{t}{k})) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $d(\mathcal{A}h, \mathcal{A}k) \leq \delta(h, k, \phi(\frac{t}{k}))$.
So, $\mathcal{M}(\mathcal{A}h, \mathcal{A}k, \phi(t)) = e^{-\frac{d(\mathcal{A}h, \mathcal{A}k)}{\phi(t)}} \geq \delta(h, k, \phi(\frac{t}{k}))$.
Exploiting (e), a pair $(\mathcal{A}, g)$ is weakly compatible. Hence, all the postulates of Theorem 3.4 are verified and consequently, $\mathcal{A}$ and g have a single common fixed point, i.e., the system of functional equations (4.2) has a unique solution. □

5. Conclusion

In this work, results for ordered $\phi-$contractive mapping [1] have been extended to an ordered pair of mappings in ordered fuzzy metric space utilizing relatively weaker order theoretic variants. To prove the coincidence of a pair of mapping, we have used the monotonic technique together with the traditional technique. Further, we presented ordered generalized $\phi-$contraction for a pair of self-mappings $(\mathcal{A}, g)$ which is also a novel and sharpened version of celebrated and contemporary contractions existing in the literature (see, [1], [6], [7], [9], [10], [11], [13], [15], and so on) as the underlying contraction is presumed to hold only on the elements of the ordered set. To substantiate the significance of our conclusions, we solved the set of functional equations emerging in dynamic programming, which are being utilized in computer programming and optimization.

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