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FRACTIONAL STOCHASTIC EVOLUTION EQUATIONS WITH BALAKRISHNAN'S WHITE NOISE

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Abstract. For a fractional stochastic differential equation in a Hilbert space with white noise of the Balakrishnan type, existence and uniqueness theorems for solutions are established. The correlation operator of the stochastic solution is calculated. The results obtained are used in digital signal processing in space communication systems and in the analysis of the profitability of securities.

1. Introduction

A difficult problem in the theory of measure and integral is the analysis of the relationship between finitely additive and countably additive measures. The first fundamental results on this problem are due to A. Aleksandrov [1] and K. Yosida, E. Hewitt [2]. These papers consider real-valued measures that have the property of finite additivity but not necessarily countable additivity. The current state of the problem is detailed in Duanmu and Weiss [3]. In the introduction to the cited work, it was noted that, in contrast to the theorems of Prokhorov [4] and Vitali-Khan-Sachs [5], in which it was established that under regularity conditions a sequence of countably additive measures converges to a countably additive measure, we can prove that for every finitely additive probability measure P on a totally bounded separable metric space there is a sequence of countably additive probability measures $\{P_n\}_{n\in\mathbb{N}}$ such that

$$\int f dP_n = \int f dP$$

for every bounded uniformly continuous real-valued function f. Further, in [3] it was noted that, on the other hand, in contrast to the Portmanteau theorem [6], such convergence does not take place for simply bounded continuous functions. This means that the assumptions of Portmanteau's fundamental theorem are sharp. In the case of infinite-dimensional phase spaces, other mathematical difficulties arise. A. Balakrishnan's monograph [7] describes the structure of measurable sets of Hilbert spaces, consistent with its topology. It is important that the theory of measure in a Hilbert space differs from the classical one in that the

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measure defined on the algebra of cylindrical sets turns out to be only finitely additive.

This article is devoted to the solvability of stochastic differential equations in a Hilbert space with a finitely additive probability measure and with a fractional order of the derivative. In the second section of the work, results are presented related to the finitely additive probability measure on a Hilbert space. A Gaussian measure is constructed to illustrate the singularity of the infinite-dimensional case. The concept of a stochastic integral based on a finitely additive probability measure on an abstract space is also discussed and its relationship with the It integral is explained. In the third section, we study a deterministic fractional evolution equation in a Hilbert space. The results obtained in [8] for a homogeneous equation on the solvability and representation of a solution are brought to the case of an nonhomogeneous equation. In the fourth section, the concept of Balakrishnan's white noise is introduced and its properties are studied. In Section 5, on the basis of the concept of an elementary random variable or a physical random variable, the concept of a weak solution of a fractional stochastic evolution equation is given. Theorems on the existence and uniqueness of a weak random solution of fractional evolution equations are established. The results obtained are a generalization and strengthening of a number of works [9-11]. Next, a probabilistic characteristic is found - a correlation operator corresponding to a stochastic solution.

2. Finitely additive Gauss measures in a Hilbert space

It is well known that the classical measure theory in finite-dimensional spaces originated in the works of A. Lebesgue on the theory of integration and found its application in probability theory, mathematical physics, functional analysis and other branches of mathematics [12]. In particular, A. Kolmogorov's interpretation [13] of the probability of an event as a measure of a set significantly influenced the further development of probability theory. The theory of measure in an infinite-dimensional Hilbert space differs from the classical one in that a measure defined on the algebra of cylindrical sets turns out to be only finitely additive. A.V. Balakrishnan [14] gave a canonical example in the form of a Gaussian measure illustrating this feature of probability measures on infinite-dimensional spaces.

2.1. Algebra of cylindrical sets. Let H-separable real Hilbert space. We choose n elements $x_1, x_2, ..., x_n$ in it and let it be a B-Borel set on the Euclidean n-dimensional space $\mathbb{R}^n$. We call a cylindrical set the set of elements $y \in H$ such that the n-dimensional vector $\{[y, x_i\}, i = 1, 2, ..., n$ belongs to B. Denote by H_n the finite-dimensional space spanned by the elements $x_1, x_2, ..., x_n$. The dimension of the space H_n can be less than n. If P_n is the projection operator from H onto H_n, then together with each element y the cylindrical set also contains $P_n y + (I - P_n)H$. This explains the name "cylindrical set".

This set can be defined differently from more general positions. Consider a finite-dimensional space H_m in H. Let B be a Borel set in H.

A cylindrical set is a set that can be represented as the sum of a Borel set B and an orthogonal complement to H_m. The Borel set B is then called the base of the cylinder, and H_m is called its basis space.

The main properties of cylindrical sets were established in [B], namely

1) the set-theoretic complement of a cylindrical set is also a cylindrical set;
2) the intersection and union of cylindrical sets is also a cylindrical set;
3) two cylindrical sets with bases B_1 in H_1 B_2 in H_2 coincides if and only if $B_1 = B_2$.

Properties 1)-3) show that the class of cylindrical sets $\mathfrak{C}$ forms an algebra. The space H can be represented as the union of a countable number of cylindrical sets. The space H itself and the empty set are cylindrical sets. On the other hand, it is obvious that the union of a countable number from $\mathfrak{C}$ does not necessarily belong to . The smallest σ-algebra of sets containing open (or closed) sets in H is called the Borel σ-algebra of the space H, and the sets belonging to it are called Borel sets. Denote the class of Borel sets by $\mathcal{B}$.

There is an assertion.

Lemma 2.1. *([7]) The class of Borel sets $\mathcal{B}$ coincides with the smallest σ-algebra containing all cylindrical sets.*

It follows from Lemma 2.1 that the $\mathcal{B}$ it least is an σ-algebra containing all closed balls (or, equivalently, all open balls).

So, we have described two objects out of three forming a probability space: the phase (selective) space H and the Borel σ - algebra $\mathcal{B}$ of its subsets. The next subsection is devoted to Gaussian probability measures on finite-dimensional spaces.

2.2. Gauss measure on $\mathbb{R}^n$. We begin by studying the Gaussian probability measure on the Euclidean space $\mathbb{R}^n$ (see [15], [16] for example).

Let $n \in \mathbb{N}$ and let $\mathcal{B}_0(\mathbb{R}^n)$ denote the complete Borel σ-algebra on $\mathbb{R}^n$. Let $\lambda^n : \mathcal{B}_0(\mathbb{R}^n) \to [0, \infty))$ denote the usual n-dimensional Lebesgue measure. Then the standard Gaussian measure $\gamma^n : \mathcal{B}(\mathbb{R}^n) \to [0, 1]$ is defined by the formula

$$\gamma^n(A) = \frac{1}{\sqrt{(2\pi)^n}} \int\limits_A \exp\left(-\frac{1}{2}\|x\|^2_{\mathbb{R}_n}\right) d\lambda^n(x) \tag{2.1}$$

for any measurable set $A \in \mathcal{B}_0(\mathbb{R}^n)$.

In terms of the Radon-Nikodym derivative, equalities (2.1) can be rewritten as

$$\frac{d\gamma^n(x)}{d\lambda^n} = \frac{1}{\sqrt{(2\pi)^n}} \exp\left(-\frac{1}{2}\|x\|^2_{\mathbb{R}^n}\right).$$

In a more general case, the Gaussian measure with mean $\mu \in \mathbb{R}^n$ and variation $\sigma^2, \sigma > 0$, is given as follows

$$\gamma^n_{\mu,\sigma^2}(A) = \frac{1}{\sqrt{(2\pi\sigma^2)^n}} \int\limits_A \exp\left(\frac{1}{2\sigma^2}\|x-\mu\|^2_{\mathbb{R}^n}\right) d\lambda^n(x).$$

The Gaussian measure with mean $\mu > 0$ is called the centered Gaussian measure.

The Dirac measure δ_μ is a weak limit of γ^n_{μ,σ^2} for $\sigma \to 0$ d is considered as a degenerate Gaussian measure. On the contrary, a Gaussian measure with finite nonzero variation is called a degenerate Gaussian measure.

The standard Gaussian measure γ^n on $\mathbb{R}^n$ is a probability measure associated with a normal (Gaussian) probability distribution, that is, if

$$z \sim N(\mu, \sigma^2)$$

then

$$P(z \in A) = \gamma^n_{\mu,\sigma^2}(A).$$

2.3. Gauss measure on cylindrical sets in H. Let be a Z-cylindrical set with base B and basis space H_m. Then by definition:

(1) $\mu(Z) = \chi_m(B)$-countably additive probability measure on the σ-algebra of Borel subsets in H_m. In particular, if $\{Z_k\}$-polarly disjoint cylindrical sets with common base space H_m and corresponding bases $\{B_k\}_\infty$, then

$$\mu\bigg(\sum_{k=1}^{\infty} Z_k\bigg) = \sum_{k=1}^{\infty} \mu(Z_k) = \sum_{k=1}^{\infty} \chi_m(B_k).$$

(2) Consistency conditions. In order for the measure μ to be correctly defined, it is necessary that the following condition be satisfied: if

$$z = B + H_m^c = B + H_p + (H_m + H_p)^c,$$

where H_p-subset, orthogonal to H_m, then

$$\chi_m(B) = \chi_{m+p}(B + H_p),$$

where χ_{m+p}-Borel measure on $H_m + H_p$.

Since χ_m is a countably additive measure on Borel subsets of the finite-dimensional space H_m, it follows that

$$\chi_m(B) = inf \chi_m(G),$$

where G is an arbitrary open set in H_m containing B.

Next, we give an example of a cylindrical finitely additive measure. Let R be a self-adjoint non-negative definite operator mapping H into H. We define a measure χ on Borel sets of a finite-dimensional space H_m as follows. We choose an orthonormal $\{e_1, ..., e_m\}$ in H_m. Borel sets in H_m can be assigned one-to-one correspondence with Borel sets in the "coordinate" space

$$x \leftrightarrow \{[x, e_i] : i = 1, ..., m\}.$$

We now introduce a Gaussian measure on Borel sets with a matrix of second moments $\{r_{ij}\}$, where $r_{ij} = [Re_i, e_i]$, and R is a given operator.

Obviously, this measure does not depend on the chosen basis. Note that the $(m \times m)$ matrix $\{r_{ij}\}$ can be degenerate. It is easy to see that the introduced measure satisfies the consistency condition. We denote this cylindrical measure by μ.

Let N_R denote the null space of the operator R and H_m a subspace of N_R. Then the measure μ of any cylindrical set with base in H_m is equal to 1 or 0, depending on whether this set contains a zero element or not.

The following property of the measure under consideration plays a very important role. Let $\{\varphi_i\}$ be a complete orthonormal system in the set of values of the operator R. Denote by E_n the cylindrical set:

$$E_n = \{x : \sum_{i=1}^{n} [x, \varphi_i]^2 \leq M^2\}, \mu > 0.$$

Since

$$E_n \subset \bigcap_{i=1}^{n} \{x : [x, \varphi_i]^2 \leq M^2\},$$

then

$$\mu(E_n) \leq (\Phi(M)/\lambda_n)^n,$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x exp(-\frac{t^2}{2})dt$$

and

$$\lambda_n^2 = \min_{\varphi \in H_n} \frac{(R\varphi, \varphi)}{[\varphi, \varphi]},$$

where H_n is the space spanned by the vectors $\varphi_1, ..., \varphi_n$.

But in this case

$$\frac{1}{n} log\mu(E_n) \leq log\Phi(\lambda/\lambda_n)$$

and hence

$$\overline{lim} \frac{1}{n} log\mu(E_n) \leq log\Phi(\mu/\lambda),$$

where

$$\lambda = \overline{lim}\lambda_n.$$

In particular, $\mu(E_n) \to 0$, if $\lambda > 0$.

Let $S(0, M)$ be a ball of radius M centered at the origin. Then $S(0, M) \subset E_n$ for every n. This proves that there is no countably additive measure on the class of Borel sets $\mathcal{B}$ that coincides with the measure μ on cylindrical sets. Indeed, if P is such a countably additive measure, then

$$P(S(0, M)) \leq P(E_n)$$

as well as $P(E_n) = \mu(E_n)$, $P(S(0, M)) = 0$ for all M. But

$$H = \bigcup_n S(0, n), n = 1, 2, ...$$

and that's why

$$1 = \lim_{n \to \infty} P(S(0, n)),$$

and we come to a contradiction.

Thus, if an R-positive definite self-adjoint operator for which

$$[Rx, x] \geq m[x, x], m > 0,$$

then the cylindrical measure induced by it cannot be extended in such a way that it becomes countably additive on $\mathcal{B}$, provided that H is infinite-dimensional. And since the prototypes of non-degenerate Gaussian distributions are positive-definite operators, we are forced to restrict ourselves to finitely additive measures.

3. The Cauchy problem for a fractional differential equation in a Hilbert space: a deterministic case

Our goal in this section is to define a weak solution to a class of fractional abstract differential equations. In this case, the fractional derivatives in the equation will be Caputo derivatives. Results will be established for weak and strong solutions of homogeneous and nonhomogeneous equations.

3.1. Operators in fractional Sobolev vector spaces. Let H be a real Hilbert space with inner product $< \cdot, \cdot >$ and norm $\| \cdot \|$. Let A be a linear self-adjoint positive operator on H with dense domain $\mathfrak{D}(A)$. The operator A satisfies the inequalities

$$< Ax, x >\geq a\|x\|^2 \ \forall x \in \mathfrak{D}(A) \tag{3.1}$$

for some $a > 0$. Suppose that the spectrum of A consists of a sequence of positive eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \to \infty$ for $n \to \infty$. Moreover, all λ_n are isolated numbers, and the proper space generated by each prime λ_n has dimension one.

Moreover, the eigenfunctions e_n of the operator A ($Ae_n = \lambda_n e_n$) form an orthonormal basis of the space H.

The fractional powers of A^θ are defined for $\theta > 0$ (see, for example, [8]). The domain $\mathfrak{D}(A^\theta)$ of the operator A^θ consists of those $u \in H$ for which

$$\sum_{n=1}^{\infty} \lambda_n^{2\theta} \big| < u, e_n > \big|^2 < \infty$$

and

$$A^\theta u = \sum_{n=1}^{\infty} \lambda_n^{\theta} < u, e_n > e_n, u \in \mathfrak{D}(A^\theta).$$

It is easy to establish that $\mathfrak{D}(A^\theta)$ is a Hilbert space with the following norm

$$\|u\|_{\mathfrak{D}(A^\theta)} = \|A^\theta u\| = \left(\sum_{n=1}^{\infty} \lambda_n^{2\theta} \big| < u, e_n > \big|^2 \right)^{1/2}, u \in \mathfrak{D}(A^\theta), \tag{3.2}$$

and for any $0 < \theta_1 < \theta_2$ we have

$$\mathfrak{D}(A^{\theta_2}) \subset \mathfrak{D}(A^{\theta_1}).$$

In particular, the norm of the space $\mathfrak{D}(\sqrt{A})$ is defined as follows

$$\|u\|_{\mathfrak{D}(\sqrt{A})} = \|\sqrt{A}\, u\| = \left(\sum_{n=1}^{\infty} \lambda_n^{2\theta} \big| < u, e_n > \big|^2 \right)^{1/2}, u \in \mathfrak{D}(\sqrt{A}). \tag{3.3}$$

If we denote the spaces dual to H by H', then we have

$$\mathfrak{D}(A^{-\theta}) = (\mathfrak{D}(A^{\theta}))^{'} \tag{3.4}$$

whose elements are linear bounded functionals over $\mathfrak{D}(A^{\theta})$.

If $u \in \mathfrak{D}(A^{-\theta})$ and $\varphi \in \mathfrak{D}(A^{\theta})$ then the value $u(\varphi)$ is determined by the formula

$$< u, \varphi >_{-\theta,\theta} = u(\varphi). \tag{3.5}$$

In addition, $\mathfrak{D}(A^{-\theta})$ is a Hilbert space with norm

$$\|u\|_{\mathfrak{D}(A^{-\theta})} = \left(\sum_{n=1}^{\infty} \lambda_n^{-2\theta} | < u, e_n >_{-\theta,\theta} |^2 \right)^{1/2}, u \in \mathfrak{D}(A^{-\theta}), \tag{3.6}$$

and for any $0 < \theta_1 < \theta_2$ we have $\mathfrak{D}(A^{-\theta_1}) \subset \mathfrak{D}(A^{-\theta_2})$. We also recall that

$$< u, \varphi >_{-\theta,\theta} = (u, \varphi) \;\; u \in H, \varphi \in \mathfrak{D}(A^{\theta}). \tag{3.7}$$

We now give the definition of a fractional Sobolev vector space. For $\beta \in (0,1), T > 0$ and a Hilbert space H equipped with the norm $\| \cdot \|_H$ by $H^{\beta}(0,T,H)$ we denote the spaces of all and $u \in L^2(0,T,H)$ such that

$$[u]_{H^{\beta}(0,T,H)} = \left(\int_0^T \int_0^T \frac{\|u(t) - u(\tau)\|_H^2}{|t-\tau|^{2\beta+1}} \right) < +\infty, \tag{3.8}$$

Then $H^{\beta}(0,T,H)$ with norm

$$\| \cdot \|_{H^{\beta}(0,T;H)} = \| \cdot \|_{L^2(0,T;H)} + [\cdot]_{H^{\beta}(0,T;H)}$$

is a Hilbert space.

The following assertion holds.

Theorem 3.1. *(see [8]). Let H be a separable Hilbert space.*

1. Riemann-Liouville operator

$$I^{\beta} : L^2(0,T;H) \to L^2(0,T;H), 0 < \beta \leq 1$$

is injective and its range $\mathfrak{R}(I^{\beta})$ is determined by the formula

$$\mathfrak{R}(I^{\beta}) = \begin{cases} H^{\beta}(0,T,H), & 0 < \beta < 1/2 \\ \{\varphi \in H^{1/2}(0,T;H) : \int_0^T t^{-1}|v(t)|^2 dt < \infty\}, & \beta = 1/2 \\ H_0^{\beta}(0,T,H), & 1/2 < \beta \leq 1, \end{cases} \tag{3.9}$$

where $H_0^{\beta}(0,T,H) = \{u \in H^{\beta}(0,T;H) : u(0) = 0\}$.

2. For the Riemann - Liouville operator I^{β} and the inverse $I^{-\beta}$, the equivalence of the following norms are true

$$\begin{gathered} \|I^{\beta}(u)\|_{H^{\beta}(0,T;H)} \sim \|u\|_{L^2(0,T;H)}, u \in L^2(0,T;H), \\ \|I^{-\beta}(v)\|_{L^2(0,T;H)} \sim \|v\|_{H^{\beta}(0,T;H)}, u \in \mathfrak{R}(I^{\beta}). \end{gathered} \tag{3.10}$$

3.2. Fractional derivatives and Mittag-Leffler functions.

Definition 3.2. For any $\alpha > 0$, we define the Riemann-Liouville integral operator of order α by the formula

$$I^{\alpha}(f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, f \in L^1(0,T) \text{ for a.e. } t \in (0,T),$$

where $T > 0$ and

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt -$$

Euler's gamma function.

The fractional Caputo derivative of order $\alpha \in (1,2)$ is given by

$${}^{c}D_t^{\alpha} f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \frac{d^2 f}{ds^2}(s)ds.$$

The Caputo derivative can be expressed in terms of the Riemann-Liouville integral operator

$${}^{c}D_t^{\alpha} f(t) = I^{2-\alpha}\left(\frac{d^2 f}{dt^2}\right)(t).$$

If $f^{'}$ is absolutely continuous, then

$${}^{c}D_t^{\alpha} f(t) = \frac{d}{dt} I^{2-\alpha}(f' - f'(0))(t).$$

For arbitrary constants $\alpha, \beta > 0$, the Mittag-Leffler function is introduced

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, z \in \mathbb{C}.$$

The function $E_{\alpha,\beta}(z)$ is an entire function $z \in \mathbb{C}$.
Note that $E_{\alpha,1}(0) = 1$.
Next, we introduce the Laplace transform of the function $f(t)$ in the form

$$\mathfrak{L}[f(t)](z) = \int_0^{\infty} e^{zt} f(t)dt, z \in \mathbb{C}.$$

Lemma 3.3. *Let $\alpha \in (1,2)$ and $\beta > 0$. Then for any $\mu \in \mathbb{R}$ such that $\pi\alpha/2 < \mu < \pi$ there exists a constant $C = C(\alpha, \beta, \mu)$ such that what*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1+|z|}, z \in \mathbb{C}, \mu \leq |arg(z)| \leq \pi. \tag{3.11}$$

Lemma 3.4. *For $\alpha, \beta, \lambda > 0$ we have*

$$\mathfrak{L}[t^{\beta-1} E_{\alpha,\beta}(-\lambda t^{\alpha})](z) = \frac{z^{\alpha-\beta}}{z^{\alpha}+\lambda}, Rez > \lambda^{1/2}. \tag{3.12}$$

Lemma 3.5. *If $\alpha, \lambda > 0$, then we have*

$$\frac{d}{dt}E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha), t > 0 \tag{3.13}$$

$$\frac{d}{dt}(t^k E_{\alpha,k+1}(-\lambda t^\alpha)) = t^{k-1}E_{\alpha,k}(-\lambda t^\alpha), k \in \mathbb{N}, t \geq 0 \tag{3.14}$$

$$\frac{d}{dt}(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) = -\lambda t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha), t \geq 0. \tag{3.15}$$

Lemma 3.6. *For any $0 < \beta < 1$, the function $x \to \frac{x^\beta}{1+x}$ reaches its maximum on $[0, +\infty]$ at the point $\frac{beta}{1-\beta}$ and the maximum value as*

$$\max_{x \geq 0} \frac{x^\beta}{1+\beta} = \beta^\beta(1-\beta)^{1-\beta}, \beta \in (0,1). \tag{3.16}$$

3.3. Weak solutions of a homogeneous fractional equation. Let $\alpha \in (1,2)$ and $T > 0$.

Definition 3.7. The function $u(t)$ is called a weak solution of the abstract fractional equation

$${}^cD_t^\alpha u + Au = 0 \tag{3.17}$$

if $u \in C([0,T];\mathfrak{D}(\sqrt{A})), u' \in L^2(0,T;H) \bigcap C([0,T];\mathfrak{D}(A-\beta))$ for some $\beta \in (0,1)$, and for any $v \in \mathfrak{D}(\sqrt{A})$ there is

$$< I^{2-\alpha}(u' - u'(0))(t), v > \in C^1([0,T])$$

and

$$\frac{d}{dt} < I^{2-\alpha}(u' - u'(0))(t), v > + < \sqrt{A}u(t), \sqrt{A}v >= 0, t \in (0,T). \tag{3.18}$$

Remark 3.8. For a weak solution $u(t)$ of equation (3.11) we have

$${}^cD_t^\beta \in H^{1-\beta}(0,T;H), \beta \in (0,1),$$

where the Caputo derivative of order $\beta \in (0,1)$ is given by

$${}^cD_t^\beta u(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}u'(s)ds = I^{1-\beta}(u')(t). \tag{3.19}$$

In fact, since $u' \in L^2(0,T;H)$ we can apply Theorem 3.1 to get $I^{1-\beta}(u') \in H^{1-\beta}(0,T;H)$, so ${}^cD_t^\beta u \in H^{1-\beta}(0,T;H)$.

In particular, for $\beta = \alpha/2$ and $\beta = 1 - \alpha/2$ we have

$${}^cD_t^{\alpha/2}u \in H^{1-\alpha/2}(0,T;H)u$$

$${}^cD_t^{1-\alpha/2}u \in H^{\alpha/2}(0,T;H).$$

Recall (see, for example, [17]) that the Laplace transform is used to solve scalar fractional differential equations.

There is an assertion

Lemma 3.9. *for any $\lambda > 0$ and $x, y \in \mathbb{R}$ the solution to the problem*

$$\begin{cases} {}^cD_t^\alpha u(t) + Au(t) = 0, & t \geq 0 \\ u(0) = x, & u'(0) = y. \end{cases}$$

looks like

$$u(t) = xE_{\alpha,1}(-\lambda t^\alpha) + ytE_{\alpha,2}(-\lambda t^\alpha).$$

Theorem 3.10. *Let $u_0 \in D(\sqrt{A})u$ and $u_1 \in H$, then the function*

$$u(t) = \sum_{n=1}^{\infty} \left[< u_0, e_n > E_{\alpha,1}(-\lambda_n t^\alpha) + < u_1, e_n > tE_{\alpha,2}(-\lambda_n t^\alpha) \right] e_n \tag{3.20}$$

is the only weak solution (3.17) satisfying the initial conditions

$$u(0) = u_0, u'(0) = u_1. \tag{3.21}$$

The proof of the theorem follows the following scheme. First, we need to make sure that the representation of solutions in the form of a series (3.20) is correct.

To do this, we will look for a solution in the form

$$u(t) = \sum_{n=1}^{\infty} u_u(t) e_n \tag{3.22}$$

where the functions $u_n(t) =< u(t), e_n >$ are unknown. It is easy to see that, taking into account the initial conditions (3.21), $u_n(t)$ will be a solution to the problem

$$\begin{cases} {}^cD_t^\alpha u_n(t) + \lambda_n u_n(t) = 0, & t \geq 0 \\ u_n(0) =< u_0, e_n >, & u_n'(0) =< u_1, e_n > \end{cases} \tag{3.23}$$

and, further, with the help of Lemma 3.9 we obtain

$$u_n =< u_n, e_n > E_{\alpha,1}(-\lambda_n t^\alpha) + < u_1, e_n > tE_{\alpha,2}(-\lambda_n t^\alpha), t \geq 0. \tag{3.24}$$

Now, taking $u_0 \in \mathfrak{D}(\sqrt{A}), u_1 \in H$, we show that series (3.22) with $u_n(t)$ given in form (3.23) is a weak solution of (3.17) and satisfies conditions (1.13).

First, note that for any $t \in [0, T]$ we have $u(t) \in D(\sqrt{A})$. Indeed, since

$$\|\sqrt{A}u(t)\|^2 = \sum_{n=1}^{\infty} \lambda_n |u_n(t)|^2 \leq 2 \sum_{n=1}^{\infty} \lambda_n | < u_0, e_n > E_{\alpha,1}(-\lambda_n t^\alpha)| +$$

$$+2 \sum_{n=1}^{\infty} \lambda_n \| < u_1, e_n > tE_{\alpha,2}(-\lambda_n t^\alpha)\|^2$$

then thanks to (3.18) we get

$$\lambda_n | < u_0, e_n > E_{\alpha,1}(-\lambda_n t^\alpha)|^2 \leq C\lambda_n |u_0, e_n|^2,$$

$$\lambda_n | < u_1, e_n > E_{\alpha,2}(-\lambda_n t^\alpha)|^2 \leq Ct^{2-\alpha} |u_1, e_n|^2 \times$$

$$\times \frac{\lambda_n t^\alpha}{(1+\lambda_n t^\alpha)^2} \leq Ct^{2-\alpha} | < u_1 e_n > |^2$$

and, therefore, given $\alpha < 2$, we get

$$\|\sqrt{A}u(t)\|^2 \leq C\|\sqrt{A}u_0\|^2 + CT^{2-\alpha}\|u_1\|^2, \tag{3.25}$$

So, for any $n \in \mathbb{N}$ we have

$$\|\sqrt{A}\sum_{k=u}^{\infty} u_k(t)e_k\|^2 \leq C\sum_{k=n}^{\infty} \lambda_k | < u_0, e_n > |^2 +$$

$$+CT^{2-\alpha}\sum_{k=n}^{\infty} | < u_1, e_k > |^2$$

and in addition,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|\sqrt{A}\sum_{k=n}^{\infty} u_k(t)e_k\| = 0$$

As a result, the series (3.22) converges uniformly in $[0,T]$ in $\mathfrak{D}(\sqrt{A})$ to $u \in C([0,T];\mathfrak{D}(\sqrt{A}))$.

Moreover, $u(0) = \sum\limits_{n=1}^{\infty} < u_0, e_n > e_n = u_0$.

3.4. Weak and strong solutions of an nonhomogeneous equation. Recall (see [17]) that to find a solution to a scalar fractional inhomogeneous equation, one can use the Laplace integral transform of the Mittag-Leffler functions and their derivatives indicated in formulas (3.11) - (3.15).

Lemma 3.11. *Let $f(t)$ be defined on the semiaxis $\mathbb{R}_+$. For any $\alpha, 1 < \alpha < 2, \lambda > 0$ and $x, y \in \mathbb{R}$, the solution to the problem*

$${}^cD_t^\alpha u(t) + \lambda u(t) = f(t), t > 0$$
$$u(0) = x, u'(0) = y$$

presented in the form

$$u(t) = \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)f(s)ds + xE_{\alpha,1}(-\lambda t^\alpha) + yTE_{\alpha,2}(-\lambda t^\alpha).$$

Let us now consider an nonhomogeneous equation in the space H in the form

$${}^cD_t^\alpha u(t) + Au(t) = f(t), t > 0 \tag{3.26}$$

with initial conditions

$$u(0) = u_0, u'(0) = u_1, \tag{3.27}$$

where A is a self-adjoint positively homogeneous operator on H such that $\mathfrak{D}(\bar{A}) = H$.

There is an assertion.

Theorem 3.12. *Assume that $u_0 \in \mathfrak{D}(\sqrt{A})$ and $u_1 \in H$, while the function $f(t)$ and taking values in H is strongly continuously differentiable on compact segments $[kT,(k+1)T] \subset \mathbb{R}_+,\ k \in \mathbb{N}v\{0\}$. Then the function*

$$u(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) < f(s), e_n > e_n ds +$$

$$+\sum_{n=1}^{\infty} [< u_0, e_n > E_{\alpha,\alpha}(-\lambda_n t^\alpha) + < u_1, e_n > t E_{\alpha,2}(-\lambda_n t^\alpha)] e^n \tag{3.28}$$

is the only weak solution of equation (3.26) satisfying the initial conditions (3.27).

Additionally, we note that

$$u'(t) = (\alpha-1) \int_0^t (t-s)^{\alpha-2} E_{\alpha,\alpha}(-\lambda_n (t-s)^2) < f(s), e_n > e_n ds -$$

$$\lambda_n \int_0^t (t-s)^{2\alpha-2} E_{\alpha,\alpha}(-\lambda_n (t-s)^2) < f(s), e_n > e_n ds +$$

$$\sum_{n=1}^{\infty} [-\lambda_n < u_0, e_n > t^{\alpha-1} E_{\alpha,1}(-\lambda_n t^\alpha) + < u_1, e_n > E_{\alpha,2}(-\lambda_n t^\alpha)] e_n \tag{3.29}$$

and $u' \in C([0,T], \mathfrak{D}(A^{-\theta}))$ for $\theta \in (\frac{2-\alpha}{2n}, 1/2)$.

The proof of the theorem consists in the implementation of the Duhamel principle for the class of fractional equations under consideration. The uniqueness of the weak solution follows directly from Theorem 3.10.

Theorem 3.13. *Let $f(t)$ satisfy the conditions of theorem (3.17). For $u_0 \in \mathfrak{D}(A)$ and $u_1 \in \mathfrak{D}(\sqrt{A})$ the weak solution of Eq. (3.28) coincides with the strong solution and has place equality*

$${}^c D_t^\alpha u(t) = -\lambda_n \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) < f(s), e_n > e_n ds -$$

$$-\sum_{n=1}^{\infty} [\lambda_n < u_0, e_n > E_{\alpha,1}(-\lambda_n t^\alpha) + \lambda_n < u_1, e_n > t E_{\alpha,2}(-\lambda_n t^\alpha)] e_n.$$

The central place in the proof of this theorem is occupied by Lerch's theorem on the uniqueness of the inverse Laplace transform in the infinite-dimensional case and the following equality

$$I^{2-\alpha}(u' - u_1)(t) = -\sum_{n=1}^{\infty} \lambda_n [< u_0, e_n > t E_{\alpha,2}(-\lambda_n t^\alpha) +$$
$$+ < u_1, e_n > t^2 E_{\alpha,3}(-\lambda_n t^\alpha)] e^n \tag{3.30}$$

and that $I^{2-\alpha}(u' - u_1)(t) \in C([0,T], H)$.

From (3.29) it follows that

$$\frac{d}{dt}I^{2-\alpha}(u'-u_1)(t) = -\sum_{n=1}^{\infty}\lambda_n[< u_0, e_n > tE_{\alpha,2}(-\lambda_n t^{\alpha})+$$
$$+ < u_1, e_n > t^2 E_{\alpha,2}(-\lambda_n t^{\alpha})]e^n \tag{3.31}$$

$\frac{d}{dt}I^{2-\alpha}(u'-u_1)(t) \in C([0,T],H)$.

4. Balakrishnan's White Noise

Let $(\Omega,\mathcal{B},\rho)$ be a probability triple. In the course of probability theory, a random variable is understood as any function defined on Ω and measurable with respect to the Borel σ-algebra $\mathcal{B}$.

Here we need finitely additive measures on algebras, since in this case we can consider Gaussian random variables with non-nuclear correlation matrices. In the monograph [7], developing the scheme of Dunford and Schwartz, a new approach to the definition of a random variable is proposed.

So, let an Ω-abstract space, a $\mathfrak{C}$-algebra (not necessarily a σ-algebra) of subsets in Ω, a μ-finitely additive probability measure defined on $\mathfrak{C}$. A function $f(\omega)$ mapping Ω into a Hilbert space is called a random variable in the weak sense if for any finite set of elements $\varphi_i, i = 1, ..., n$, from the Hilbert space conditions are met:

1) The set $\{\omega : \{[f(\omega),\varphi_i]\} \in B\}$, where B is a Borel set in Euclidean space, belongs to $\mathfrak{C}$;

2) The measure thus induced on Borel sets is countably additive for every n.

Here the set $\{[f(\omega),\varphi_i]\}$ defines an ordinary random variable. Given a cylindrical probability measure on a Hilbert space, then one can construct the corresponding random variable by setting: $\Omega = H$, the $\mathfrak{C}$-class of cylindrical sets, and $f(\omega) = \omega$. For example, if the cylindrical measure is a Gaussian measure μ such that its characteristic function is

$$\int_H e^{i[\omega,\varphi]}d\mu = exp\Big(-\frac{\|\varphi\|^2}{2}\Big).$$

Then, for an arbitrary orthonormal basis $\{\varphi_k\}$ in H, the scalar products $[f(\omega),\omega_k]$ define independent Gaussian random variables with mean zero and unit variance. However, for all ω

$$\sum_{k=1}^{\infty}[f(\omega),\varphi_k]^2 = \|f(\omega)\|^2 < \infty.$$

This statement contradicts the classical probability theory, according to which a similar sum of squares of independent Gaussian random variables with unit variance should increase infinitely with probability 1. One fundamental point should be noted. In the classical theory, the space of all possible sequences is taken as a phase space, and a certain countably additive measure is defined on the Borel algebra of its subsets. Here we are talking only about finitely additive measures, and they are defined on algebras. In fact, for the case of Gaussian random variables, the subspace of square summable sequences has measure zero. Therefore, it is extremely important how the probability space is arranged.

Everywhere below we assume that $\Omega = H$, $\mathfrak{C}$ is the class of cylindrical sets, and $\mathcal{B}$ is a Borel algebra on H. Note that condition 1) of the definition of a random variable implies that the preimages of Borel sets in E_n (n-fixed) belong to $\mathfrak{C}$. But since the Borel sets in E_n form a σ-algebra, their preimages also form a σ-algebra.

Consequently, the measure μ is also defined on the σ-algebra $\mathcal{B}$ of Borel sets in the space H. Therefore, we can slightly weaken the definition and call $f(\omega)$ a random variable in the weak sense if for any finite set n of elements φ_i from H the set $\{\omega : \{[f(\omega), \varphi_i]\} \in B\}$, where B is a Borel set in E_n, belongs to $\mathcal{B}$, and the measure μ is defined and countably additive on the σ-subalgebra of the preimage algebra of Borel sets.

Let now $(H, \mathfrak{C}, \mu)$-probability space, where H-real separable Hilbert space, $\mathfrak{C}$-class of cylindrical sets with finite-dimensional bases, and μ-cylindrical measure. For a given function $f(\cdot)$ mapping H into another Hilbert space H', the preimages of the Borel sets $\{\omega : f(\omega) \in B, B \in H\}$ do not necessarily belong to the class $\mathfrak{C}$. Therefore, in the general case, it will not be possible to determine the probability of such an event. What class of functions do random variables belong to? To answer this question, we will proceed by analogy with the procedure for completing spaces with the help of Cauchy sequences. Denote by P the projection operator onto a finite-dimensional space. Then the inverse images of the Borel sets of the function $f(P\omega)$ (under the assumption that $f(\cdot)$ is Borel measurable) belong to the class $\mathfrak{C}$.

Therefore, for any Borel set C from the space H' the formula

$$\chi(C) = \mu\{\omega : P(\omega) \in C\}$$

defines a countably additive measure on the σ-algebra of Borel sets in H'. Thus the function $f(P\omega)$ is a random variable. In what follows, each such function will be called an elementary random variable (ESV). A random variable is an arbitrary Cauchy sequence with respect to the measure of elementary random variables.

Let us give an extension of the concept of ESP.

Definition 4.1. Let $f : H \to beanH$-measurable Borel map and let P_n be a sequence of finite-dimensional projections strongly convergent to the identity operator. The mapping $f(\cdot)$ is called a physical random variable (PSV) if:

1) the sequence of tame functions $\{f(P_n\omega)\}$ is a Cauchy sequence in probability for each $\{P_n\}$.

2) a sequence of $\{v_n\}$ probability measures induced by $f \circ P_n$ and defined as

$$v_n = \mu \circ (f \circ P_n)^{-1}$$

converges strongly to the same probability measure on H^1 for every sequence P_n.

Condition 2) is equivalent to the fact that there is a limit

$$C(\omega) = \lim_{n\to\infty} \int\limits_H e^{i[f \circ P_n \omega, \omega']} d\mu(h)$$

independent of P_n.

If f is a FSV, then μ can always be extended to events of the form $f^{-1}(B')$, where B' is a Borel set in H' using the equality

$$\mu(f^{-1}(B')) = \lim_{n\to\infty} \mu((\omega \in H \big| f \circ P_n(\omega) \in B')),$$

where the limit exists by definition. The class of H'-valued FSWs is denoted as $\mathfrak{L}'(H, \mathfrak{C}, \mu, H')$-adjoint to $\mathfrak{L}(H, \mathfrak{C}, \mu, H')$ spaces.

To complete our constructions, we introduce the Balakrishnan's noise model. It is natural to assume that the Gaussian noise has a much larger bandwidth than just the signal. This follows from the fact that the corresponding representation is the identity mapping on H equipped with the Gaussian measure μ_G with the unit correlation operator. Such a mapping is called Balakrishnan's white noise.

Let's give a description of the FSV. To achieve this goal, the following concept of continuity is needed.

Definition 4.2. Let H, H' be real Hilbert spaces. A mapping $F : H \to H'$ is continuous in $x \in H$ with respect to the S-topology if for any $\varepsilon > 0$ there exists a Hilbert-Schmidt operator $L_\varepsilon(x) : H \to H'$ such that from the inequality

$$\|L_\varepsilon(x)(x - x')\| \le 1 \tag{4.1}$$

follows that

$$\|F(x) - F(x')\| \le \varepsilon. \tag{4.2}$$

The mapping F is S-continuous on $U \subset H$ if the Hilbert-Schmidt operator from (4.1) does not depend on $x \in H$.

Let us give a weaker notion of the following form.

Definition 4.3. A map $F : H \to H'$ is said to be uniformly S-continuous in a neighborhood of the origin ($USCNO$) if F is uniformly S-continuous on the sets

$$U_n = \{x \in H : \|L_n x\| \le 1\}$$

where $\{L_n\}_{n\ge1}$ is a sequence of Hilbert-Schmidt operators such that

$$\|L_n\|_{HS} \to 0 \quad \bigcup_{n=1}^{\infty} U_n = H.$$

Obviously, a uniform S-continuous mapping is also a USCNO. In this case $L_n = \frac{1}{n} L$.

Now we have all the necessary tools to formulate the criteria for the FSF.

Theorem 4.4. *A sufficient condition for the mapping $F : H \to H'$ to be FSF is its USCNO-th.*

A useful characterization of the USCNO-ness of a mapping is the following assertion.

Theorem 4.5. *A map $F : H \to H'$ is USCNO if and only if there exists a Hilbert-Schmidt operator $L : H \to H'$ and a continuous map $g : H \to H'$ such that*

$$F = g \circ L.$$

5. The Cauchy problem for a fractional differential equation in a Hilbert space: the stochastic case

In this section, using families of special Mittag-Leffler functions and expansions in eigenfunctions of a non-negative definite self-adjoint operator A with a dense domain in the Hilbert space H, we obtain a generalization of linear states with a finite-dimensional space to the infinite-dimensional case, in particular, it covers the systems described by partial differential equations and fractional time derivatives of order $\alpha, 1 < \alpha < 2$ together with Balakrishnan white noise as input.

To this end, at the very beginning, we define the function space $W = L_2(0, T; H)$ and $0 < T < \infty$. Let H_n be a separable Hilbert space and let $W_n = L_2(0, T; H_n)$ (here the letter n is an abbreviation for the word noise). Denote by A the operator defined in Section 3 and by B the linear bounded operator acting from the space H_n to H.

Consider the fractional stochastic differential equation

$$ {}^cD_t^\alpha u(t) + Au(t) = B\omega(t), t > 0, 1 < \alpha < 2, \tag{5.1}$$

along with initial conditions

$$u(0) = u_0, u'(0) = u_1. \tag{5.2}$$

The results obtained in §3, §4 allow for each $\omega \in W_n$ to rewrite problem (5.1), (5.2) in integral form. Moreover, it can be argued that the integral equation indicated below has a unique weak solution. Since we want to emphasize the dependence of the solution on the input ω, we will use the notation $u(t, \omega)$ for this.

Recall also that e_n and λ_n denote the n-th eigenfunctions and eigenvalues of the operator A, respectively.

The equation

$$< u(t, \omega), e_n >=< u_0, e_n > + < u_1, e_n > t-$$

$$-\int_0^t (t-s)^{\alpha-1} < u(s, \omega), Ae_n > ds + \int_0^t (t-s)^{\alpha-1} < B\omega(s), e_n > ds \tag{5.3}$$

has a solution defined by the formula

$$u(t, \omega) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n (t-s)^\alpha) < B\omega, e_n > e_n ds+$$

$$+\sum_{n=1}^{\infty} [< u_0, e_n > E_{\alpha,1}(-\lambda_n t^\alpha) + < u_1, e_n > tE_{\alpha,2}(-\lambda_n t^\alpha)] e_n,$$

and for each ω this solution is unique in the class of weakly continuous functions satisfying equation (5.3). Let us calculate the correlation operator corresponding to the process $u(t, \omega)$. The process $u(t, \omega)$ is defined at each time t. Assuming u_0 and u_1 are given, we get

$$\mathbb{E}\Bigg(\Bigg[u(t,\omega)-\sum_{n=1}^{\infty}[<u_0,e_n>E_{\alpha,1}(-\lambda_n t^{\alpha})+<u_1,e_n>E_{\alpha,2}(-\lambda_n t^{\alpha})e_n\Bigg]\times$$

$$\Bigg[u(s,\omega)-\sum_{n=1}^{\infty}[<u_0,e_n>E_{\alpha,1}(-\lambda_n t^{\alpha})+<u_1,e_n>E_{\alpha,2}(-\lambda_n t^{\alpha})\Bigg]e_n\Bigg)=$$

$$\mathbb{E}\Bigg(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})<B\omega,e_n>e_n ds\times$$

$$\times\int_0^s (s-\sigma)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(s-\sigma)^{\alpha})<B\omega(\sigma),e_n>e_n d\sigma\Bigg)=$$

$$\mathbb{E}\Bigg(\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})<\omega(s),B^*e_n>e_n ds\times$$

$$\times\int_0^s (s-\sigma)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(s-\sigma)^{\alpha})<\omega(\sigma),B^*e_n>e_n d\sigma\Bigg)=$$

$$\Bigg[\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})ds\cdot$$

$$\int_0^s (s-\sigma)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(s-\sigma)^{\alpha})d\sigma\cdot\|B^*\|\Bigg]e_n.$$

Therefore, the correlation operator $R(t,s)$ is defined by the formulas

$$R(t,s)u_0=\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})ds\cdot$$

$$\int_0^s (s-\sigma)^{\alpha-1}E_{\alpha,1}(-\lambda_n(s-\sigma)^{\alpha})u_0 d\sigma.$$

$$R(t,s)u_1=\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})ds\times$$

$$\times\int_0^s (s-\sigma)^{\alpha-1}E_{\alpha,2}(-\lambda_n(s-\sigma)^{\alpha})u_1 d\sigma.$$

6. Conclusion

Gaussian finitely additive white noise (Balakrishnan's white noise) has a uniform power spectral density, is normally distributed, sums with the useful signal, and is statistically independent of the signal. Most often, such measures are used in digital signal processing in space communication systems and in the analysis of the profitability of securities.

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