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INVERSE POWER FRÈCHET DISTRIBUTION: STATISTICAL PROPERTIES, ESTIMATION AND APPLICATION

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Abstract. In this paper, we propose an extension of the inverse Frèchet distribution called the inverse power Frèchet distribution, which offers more reliability and flexibility in modeling lifetime data. A comprehensive account of the statistical properties such as reliability characteristics, moments, quantiles, mean deviation, generating function, stochastic ordering, mean residual lifetime function and various entropy measures have been derived. Bonferroni, Lorenz Curves and Gini index are also computed for the proposed model. Different estimation methods are studied to estimate the proposed model parameters. Simulation studies is done to present the performance and behavior of the different estimates of the proposed model parameters. The real data application for the proposed distribution is modeled to illustrate its applicability, and it is shown that our distribution fits much better than some other existing distributions.

1. Introduction

Extreme value theory and its applications plays an important role in statistical analysis and one of the important distributions used to describe extreme data is the Frèchet distribution (which is also known as the extreme value distribution of type II). The Frèchet distribution is named according to French mathematician Maurice Renè Frèchet, who developed it as a maximum value distribution.

Some new important extensions of the Fréchet distribution have been proposed in the literature. For example, Beta Fréchet disrribution by Barreto-Souza et al. [9], The modified Fréchet distribution and its properties by Tablada and Cordeiro [24], a new three parameter Fréchet by Al-Babtain et al. [3] and the generalized transmuted Fréchet distribution by Nofal and Ahsanullah [21]. De Gusmão et al. [12] proposed a three parameter generalized inverse Weibull distribution in which includes the Fréchet distribution. Krishna et.al. [18] proposed Marshall-Olkin Fréchet and applications of Marshall-Olkin Fréchet have been studied by Krishna et.al. [19]. Silva et al. [23] defined the gamma extended Fréchet. Afify et al. [2] investigated the Weibull Fréchet distribution and its applications. Logistic Fréchet distribution has been proposed by Tahir et al [25]. For more details about the

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Fréchet distribution and its applications, see Kotz and Nadarajah. [17]. Furthermore, applications of this model in various fields are given in Harlow. [16].

The Fréchet (Fr) distribution can be seen as the inverse Weibull distribution with shape parameter θ and scale parameterλ. Thus, its probability density function(pdf) is given by

$$f\left(y;\theta,\lambda\right)=\theta\lambda^{\theta}y^{-(\theta+1)}\exp\left[-\left(\frac{\lambda}{y}\right)^{\theta}\right] \tag{1.1}$$

where $y>0, \theta>0$, and $\lambda>0$. The cumulative distribution function (cdf) is given by

$$F\left(y;\theta,\lambda\right)=\exp\left[-\left(\frac{\lambda}{y}\right)^{\theta}\right] \tag{1.2}$$

Researchers have more interest to generate inverted distributions under inverse transformation for example, the inverse power two-parameter weighted Lindley distribution by Abd El-Monsef and Al-Kzzaz [1], the inverse power Lindley distribution was discussed by Barco et al [8] using power transformation $X=Y^{1/\alpha}$ for the inverse Lindley distribution and Alkarni [4] proposed the extended inverse two-parameter Lindley distribution as a statistical inverse model for upside-down bathtub survival data that uses the transformation $X=Y^{-1/\alpha}$.

Starting from the Fréchet (Fr) distribution, we develop and study a new generalization model called an inverse power Frèchet (IPFr) distribution. The objective of this article is to study the statistical properties of the IPFr distribution, and then estimate the unknown parameters using maximum likelihood method. We aim that comprehensive description of mathematical and statistical properties of this distribution will attract wider applications in biology, engineering, medicine, economics and other areas of research.

Here, this paper generates another generalization of the Frèchet distribution using inverse transformation of Frèchet random variates. Let us consider a transformation $X=Y^{-1/\alpha}$ where $Y\sim Fr\left(\theta,\lambda\right)$. Then the resulting distribution of X is called the inverse power Frèchet distribution and denoted by $X\sim IPFr\left(\alpha,\theta,\lambda\right)$. The probability density function and cumulative distribution function of the IPFr are given by

$$f\left(x;\alpha,\theta,\lambda\right)=\alpha\theta\lambda^{\theta}x^{\alpha\theta-1}\exp\left[-\left(\lambda x^{\alpha}\right)^{\theta}\right] \tag{1.3}$$

and

$$F\left(x;\alpha,\theta,\lambda\right)=1-\exp\left[-\left(\lambda x^{\alpha}\right)^{\theta}\right] \tag{1.4}$$

respectively, where $\alpha,\theta>0$ are shape parameter and $\lambda>0$ is scale parameter . It can be noticed that the inverse Fr distribution (Weibull distribution) is a special case of the IPFr distribution when $\alpha = 1$.

The paper is organized as follows. The introduction of the proposed study including the methodological details is given in Section 1. Section 2 provides some statistical properties related to the proposed model. Entropy such as Renyi entropy, Shannon entropy, β-entropy and generalized entropy are derived in Section 3. Different methods of estimation of the model parameters are constructed in Section 4. In Section 5, simulation study has been carried out to see the performance of the estimates of the model parameters using the methods of estimation discussed in Section 4. In Section 6, we illustrate the application and usefulness of the proposed model by applying it to one data set. Finally, Section 7 offers some concluding remarks.

2. Statistical properties

The different mathematical and statistical properties such as Asymptotic behavior, reliability measures, moments, mean deviation, generating functions, quantile function, Bonferroni and Lorenz curves, stochastic ordering and order statistics of the IPFr distribution have been derived in following subsections.

2.1. Asymptotic behavior. By omitting the dependence on the positive parameters α, θ and λ in (1.3) and (1.4), we have $f(x; \alpha, \theta, \lambda) = f(x)$ and $F(x; \alpha, \theta, \lambda) = F(x)$. The behaviors of the pdf of IPFr distribution $f(x)$ at $x = 0$ and $x = \infty$, respectively, are given by

$$f(0) = \begin{cases} \infty, & if\, \alpha\theta<1 \\ \lambda^{\theta}, & if\, \alpha\theta=1, \\ 0, & if\, \alpha\theta>1 \end{cases} \qquad f(\infty) = 0$$

The following theorem shows that there are two shapes for the pdf of IPFr distribution.

Theorem 2.1. *The pdf $f(x)$ of the IPFr distribution is*

(1) *decreasing if $\{\alpha\,\theta \leq 1\}$.*
(2) *unimodal if $\{\alpha\,\theta>1\}$.*

Proof. The first derivative of $f(x)$ is given by

$$f'(x) = -\frac{\psi(x)}{x} f(x)$$

where $\psi(x) = 1 + \alpha\,\theta\left(\lambda^{\theta} x^{\alpha\,\theta} - 1\right)$

- If $\alpha\,\theta \leq 1$, then $\psi(x)>0$. Hence $f'(x) < 0$ which implies that $f(x)$ is decreasing.
- $f'(x) = 0$ if $\psi(x) = 0$ which occurs at the point

$$x_0 = \left(\frac{\alpha\,\theta - 1}{\alpha\,\theta\,\lambda^{\theta}}\right)^{\frac{1}{\alpha\,\theta}}$$

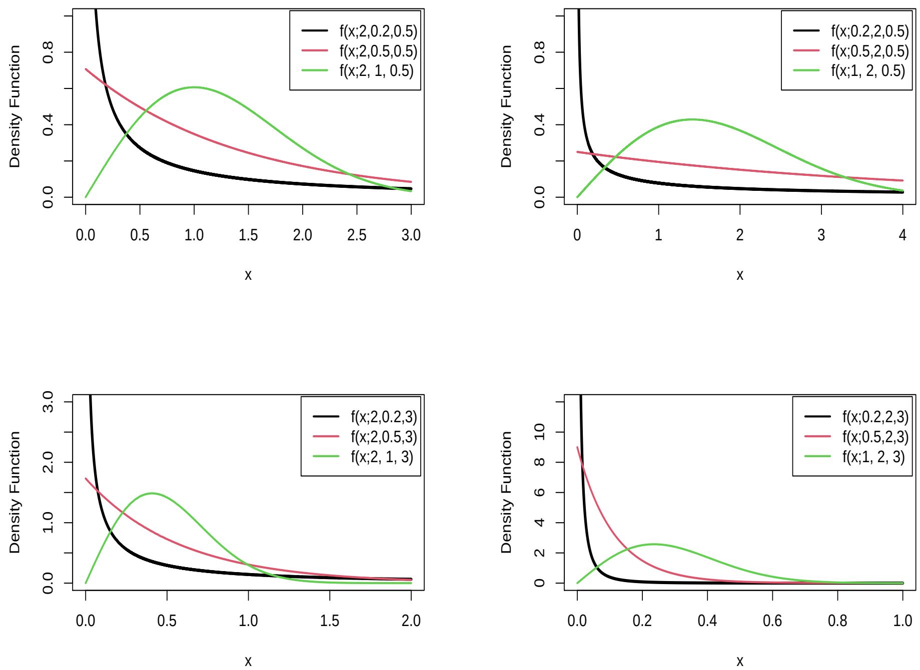

FIGURE 1. Plots of the IPFr pdf distribution for different parameter values.

The second derivative of $f\left(x\right)$ given by

$$f''\left(x\right)=-\frac{1}{x}\left[\left(1+\psi\left(x\right)\right)f'\left(x\right)+\psi'\left(x\right)f\left(x\right)\right]$$

where $\psi'\left(x\right)=\lambda^{\theta}\left(\alpha\,\theta\right)^{2}x^{\alpha\,\theta-1}$.

Clearly, at $\alpha\,\theta>1$, $\psi\left(x\right)$ is a unimodal function with maximum value at the point x_0 since, $f''\left(x_0\right)<0$, $f\left(x\right)$ has a global maximum at x_0; hence, the mode of $f\left(x\right)$ is given by

$$x_0=\left(\frac{\alpha\,\theta-1}{\alpha\,\theta\,\lambda^{\theta}}\right)^{\frac{1}{\alpha\,\theta}}$$

□

The behavior of IPFr distribution density can be illustrated as in the Fig. 1.

2.2. Reliability measures. The characteristics based on reliability analysis play an important role to study the pattern of any lifetime phenomenon. Here, we present the survival (or reliability) function and study the hazard (or failure) rate function. Also, the mean time to system failure and the mean residual lifetime are obtained for the IPFr distribution.

2.2.1. *The survival.* The survival function of the IPFr distribution, denoted by $S(x)$, is given by

$$S(x) = 1 - F(x) = \exp\left[-\left(\lambda x^{\alpha}\right)^{\theta}\right], \quad x>0 \tag{2.1}$$

2.2.2. *The hazard rate function.* The other characteristic of interest of a random variable is the hazard (failure) rate function. $h(x)$, also known as the (instantaneous) rate of failure for the survivors to time x during the next instant of time. The hazard rate function for the IPFr distribution is given by

$$h(x) = \frac{f(x)}{R(x)} = \alpha\theta\lambda^{\theta}x^{\alpha\theta-1} \tag{2.2}$$

Now, we study the behavior of $h(x)$, of the Log-IL distribution and show its different shapes. According to Glaser [15], the behavior of $h(x)$ has the same meanings as behavior of $\eta(x)$ where $\eta(x) = -\frac{d}{dx}\ln f(x)$. The following theorem shows the shapes of the hazard rate function of the IPFr distribution

Theorem 2.2. *Hazard rate function of the IPFr distribution is*

- *Decreasing if* $\{\alpha\,\theta<1\}$.
- *Constant if* $\{\alpha\,\theta = 1\}$.
- *Increasing if* $\{\alpha\,\theta>1\}$.

Proof. Since

$$\eta(x) = -\frac{d}{dx}\ln f(x) = \alpha\theta\lambda^{\theta}x^{\alpha\theta-1} - \frac{\alpha\theta-1}{x}$$

It follows that

$$\eta'(x) = (\alpha\theta-1)\left(\alpha\theta\lambda^{\theta}x^{\alpha\theta-2} + \frac{1}{x^2}\right)$$

- If $\alpha\,\theta<1$, then $\eta'(x)<0$. Hence $\eta(x)$ is decreasing which implies that $h(x)$ is decreasing.
- If $\alpha\,\theta=1$, then $\eta'(x)=0$. Hence $\eta(x)$ is constant which implies that $h(x)$is constant.
- If $\alpha\,\theta>1$, then $\eta'(x)>0$. Hence $\eta(x)$ is increasing which implies that $h(x)$ is increasing.

□

Fig. 2 illustrates the behavior of the hazard rate function of the IPFr distribution at different values of the parameters involved.

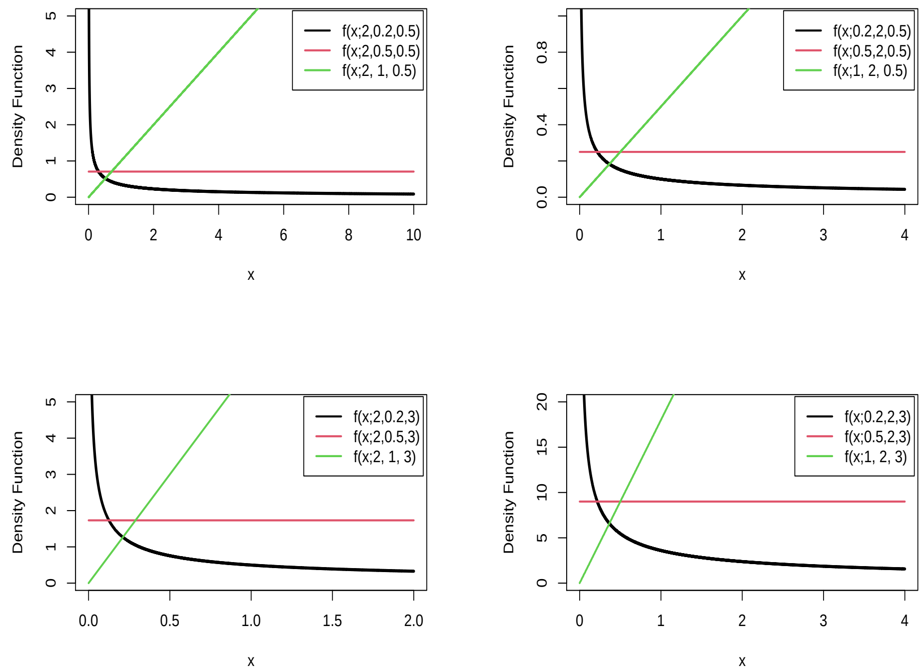

FIGURE 2. Plots of hazard rate function of the IPFr distribution for different parameter values.

2.2.3. *The mean residual life time.* The mean residual life (also known as the mean remaining life) function represents the expected additional life length for a unit that is alive at age x, denoted by $E\left(X-x|X\geq x\right)$ which can be given as

$$\begin{aligned} m\left(x\right) = E\left(X-x|X\geq x\right) &= \frac{1}{S\left(x\right)}\int_{x}^{\infty} S\left(t\right)dt \\ &= \frac{1}{\lambda^{\frac{1}{\alpha}}\ \exp\left(\lambda x^{\alpha\theta}\right)}\Gamma\left(\frac{\alpha\theta+1}{\alpha\theta},\lambda x^{\alpha\theta}\right) \end{aligned} \tag{2.3}$$

2.3. Moments. Let $x_1, x_2, ...x_n$ be the random observation from the IPFr distribution, the rth raw moment (about the origin) is given by

$$\begin{aligned} \mu_r^{'} &= E\left(x^r\right) \\ &= \lambda^{-\frac{r}{\alpha}}\Gamma\left(\frac{\alpha\,\theta+r}{\alpha\,\theta}\right); r=1,2, \end{aligned} \tag{2.4}$$

The mean and variance are, respectively

$$\mu = \mu_1^{'} = \lambda^{-\frac{1}{\alpha}} \Gamma\left(\frac{\alpha\,\theta + 1}{\alpha\,\theta}\right)$$

$$\sigma^2 = \mu_2^{'} - \left(\mu_1^{'}\right)^2 = \lambda^{-\frac{2}{\alpha}} \left[\Gamma\left(\frac{\alpha\,\theta + 2}{\alpha\,\theta}\right) - \left\{\Gamma\left(\frac{\alpha\,\theta + 1}{\alpha\,\theta}\right)\right\}^2\right]$$

The nth central moment μ_n can be obtained from the rth raw moments through the relation

$$\mu_n = E\left(X - \mu\right)^n = \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array}\right) (-1)^{n-r} \left(\mu\right)^{n-r} \mu_r^{'};\ n = 1, 2, ...$$

Then, the nth central moment is given by

$$\begin{aligned} \mu_n &= E\left(X - \mu\right)^n \\ &= \sum_{r=0}^{n} \left(\begin{array}{c} n \\ r \end{array}\right) (-1)^{n-r} \lambda^{-\frac{n}{\alpha}} \left(\Gamma\left(\frac{\alpha\,\theta + 1}{\alpha\,\theta}\right)\right)^{n-r} \Gamma\left(\frac{\alpha\,\theta + r}{\alpha\,\theta}\right) \end{aligned} \tag{2.5}$$

The coefficient of skewness and kurtosis measure convexity of the curve and its shape. It is obtained by moments based relations suggested by Pearson and given by;

$$\beta_1 = \frac{\left(\mu_3\right)^2}{\left(\mu_2\right)^3} = \frac{\left(\mu_3^{'} - 3\mu_2^{'}\mu + 2\mu^3\right)^2}{\left(\mu_2^{'} - \mu^2\right)^3}$$

and

$$\beta_2 = \frac{\mu_4}{\left(\mu_2\right)^2} = \frac{\mu_4^{'} - 4\mu_3^{'}\mu + 6\mu_2^{'}\mu^2 - 3\mu^4}{\left(\mu_2^{'} - \mu^2\right)^2}$$

The coefficient of variation (CV) is calculated by

$$CV = \frac{\sqrt{\left[\Gamma\left(\frac{\alpha\,\theta+2}{\alpha\,\theta}\right) - \left\{\Gamma\left(\frac{\alpha\,\theta+1}{\alpha\,\theta}\right)\right\}^2\right]}}{\Gamma\left(\frac{\alpha\,\theta+1}{\alpha\,\theta}\right)} \times 100$$

Some measures are calculated in Table 1 for different combination of model parameters and it is observed that the shape of the IPFr distribution is right skewed for some choices of α, θ, and λ. Also, there is no change in β_1, β_2 and CV when α and θ are fixed.

2.4. Mean Deviation. The mean deviation (MD) about mean is defined by

$$\begin{aligned} MD &= \int_x |x - \mu|\, f\left(x, \alpha, \theta, \lambda\right) dx \\ &= \int_0^{\mu} \left(\mu - x\right) f\left(x, \alpha, \theta, \lambda\right) dx + \int_{\mu}^{\infty} \left(x - \mu\right) f\left(x, \alpha, \theta, \lambda\right) dx \end{aligned}$$

After simplification, we get

TABLE 1. Values of mean, variance, skewness, kurtosis, mode and coefficient of variation for the IPFr distribution at different parameter combinations.

Moments	μ	σ^2	β_1	β_2	x_0	CV
λ	$\alpha = 1$ and $\theta = 0.5$					
0.5	4.0000	80.0000	43.8080	87.7200	No mode	223.6070
1	2.0000	20.0000	43.8080	87.7200	No mode	223.6070
1.5	1.3333	8.8889	43.8080	87.7200	No mode	223.6070
2	1.0000	5.0000	43.8080	87.7200	No mode	223.6070
λ	$\alpha = 2$ and $\theta = 0.5$					
0.5	1.4142	2.0000	4.0000	9.0000	No mode	100
1	1.0000	1.0000	4.0000	9.0000	No mode	100
1.5	0.8165	0.6667	4.0000	9.0000	No mode	100
2	0.7071	0.5000	4.0000	9.0000	No mode	100
λ	$\alpha = 2$ and $\theta = 1$					
0.5	1.2533	0.4292	0.3983	3.2451	1.0000	52.2723
1	0.8862	0.2146	0.3983	3.2451	0.7071	52.2723
1.5	0.7236	0.1431	0.3983	3.2451	0.5773	52.2723
2	0.6267	0.1073	0.3983	3.2451	0.5000	52.2723

$$\begin{aligned} MD &= 2\mu F(\mu) - 2\int_0^{\mu} x f(x,\alpha,\theta,\lambda)\,dx \\ &= 2\lambda^{-\frac{1}{\alpha}}\left(\Gamma\left(\frac{\alpha\theta+1}{\alpha\theta}, \lambda^{\theta}\mu^{\alpha\,\theta}\right) - \Gamma\left(\frac{\alpha\theta+1}{\alpha\theta}\right)\exp\left[-\left(\lambda x^{\alpha}\right)^{\theta}\right]\right) \end{aligned} \tag{2.6}$$

2.5. Generating Functions. In distribution theory, the role of generating functions is very useful to generate the respective moments of the distribution and also these functions are uniquely determining the distribution. The different generating function of the IPFr distribution have been calculated as follows;

- Moment generating function $M_X(t)$for a random variable X is obatined as

$$M_X(t) = E\left(e^{tx}\right) = \sum_{j=0}^{\infty}\frac{1}{j\,!}\left(t\lambda^{-\frac{1}{\alpha}}\right)^j\Gamma\left(\frac{\alpha\,\theta+j}{\alpha\,\theta}\right) \tag{2.7}$$

- Characteristics function $\phi_X(t)$ for random variable X is obtained by replacing t by $i\,t$,

$$\phi_X(t) = E\left(e^{itx}\right) = \sum_{j=0}^{\infty}\frac{1}{(j)\,!}\left(i\,t\lambda^{-\frac{1}{\alpha}}\right)^j\Gamma\left(\frac{\alpha\,\theta+j}{\alpha\,\theta}\right) \tag{2.8}$$

where, $i^2 = -1$.

- The kumulants generating function $K_X(t)$ is obtained as

$$K_X(t) = \ln\left(\sum_{j=0}^{\infty}\frac{1}{j\,!}\left(t\lambda^{-\frac{1}{\alpha}}\right)^j\Gamma\left(\frac{\alpha\,\theta+j}{\alpha\,\theta}\right)\right) \tag{2.9}$$

2.6. Quantile function and median. Let X be a random variable follows the IPFr distribution then, the quantile function, say $Q(x)$ is

$$Q(x) = \lambda^{-\frac{1}{\alpha}} \left(-\ln(1-x)\right)^{\frac{1}{\alpha\theta}} \quad (2.10)$$

The median can be derive from (2.10) by letting $x = 0.5$

$$Q(0.5) = \lambda^{-\frac{1}{\alpha}} \left(\ln(2)\right)^{\frac{1}{\alpha\theta}}$$

2.7. Bonferroni, Lorenz Curves and Gini index. The Bonferroni, Lorenz curves and Gini index have more comprehensive applications to describe inequality distribution in economics, reliability, demography, insurance and medicine. For the IPFr distribution, the Bonferroni curve defined by

$$\begin{aligned} B_F(p) &= \frac{1}{p\mu} \int_0^p Q(x)\, dx \\ &= \frac{\Gamma\left(\frac{\alpha\theta+1}{\alpha\theta}\right) - \Gamma\left(\frac{\alpha\theta+1}{\alpha\theta}, -\ln(1-p)\right)}{p\Gamma\left(\frac{\alpha\theta+1}{\alpha\theta}\right)} \end{aligned} \quad (2.11)$$

the Lorenz curve is given by

$$\begin{aligned} L_F(p) &= \frac{1}{\mu} \int_0^p Q(x)\, dx \\ &== \frac{\Gamma\left(\frac{\alpha\theta+1}{\alpha\theta}\right) - \Gamma\left(\frac{\alpha\theta+1}{\alpha\theta}, -\ln(1-p)\right)}{\Gamma\left(\frac{\alpha\theta+1}{\alpha\theta}\right)} \end{aligned} \quad (2.12)$$

and, the Gini index is given by

$$\begin{aligned} G &= 1 - 2\int_0^1 L_F(p)\, dp \\ &= 1 - (2)^{-\frac{1}{\alpha\theta}} \end{aligned} \quad (2.13)$$

2.8. Stochastic Ordering. Stochastic ordering of a positive continuous random variables is an important property for judging the comparative behavior of continuous distributions. A random variable X is said to be smaller than a random variable Y in the

(1) stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ for all x.
(2) hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x.
(3) mean residual life order $(X \leq_{mrl} Y)$ if $(m_X(x) \leq_{hr} m_Y(x))$ for all x.
(4) likelihood ratio order$(X \leq_{lr} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar [22] are well known for establishing stochastic ordering of distributions

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y$$

and hence

$$X \leq_{hr} Y \Rightarrow X \leq_{st} Y$$

The following theorem shows that distribution is ordered with respect to the strongest likelihood ratio ordering.

Theorem 2.3. *Let* $X \sim IPFr(\alpha_1, \theta_1, \lambda_1)$ *and* $Y \sim IPFr(\alpha_2, \theta_2, \lambda_2)$. *Then, the following results hold true*

(1) *If* $\alpha_1\theta_1 = \alpha_2\theta_2$ *and* $\lambda_1 > \lambda_2$, *then* $X \leq_{lr} Y$, $X \leq_{hr} Y, X \leq_{mrl} Y$ *and* $X \leq_{st} Y$.
(2) *If* $\alpha_1\theta_1 > \alpha_2\theta_2$, *then* $X \leq_{lr} Y$, $X \leq_{hr} Y, X \leq_{mrl} Y$ *and* $X \leq_{st} Y$.

Proof. We have

$$r(x) = \frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1\theta_1{\lambda_1}^{\theta_1}}{\alpha_2\theta_2{\lambda_2}^{\theta_2}} x^{\alpha_1\theta_1 - \alpha_2\theta_2} e^{\lambda_2 x^{\alpha_2\theta_2} - \lambda_1 x^{\alpha_1\theta_1}} \quad ; x>0$$

Taking logarithm both sides, we can write

$$\ln r(x) = \ln\left(\frac{\alpha_1\theta_1{\lambda_1}^{\theta_1}}{\alpha_2\theta_2{\lambda_2}^{\theta_2}}\right) + (\alpha_1\theta_1 - \alpha_2\theta_2)\ln x + \lambda_2 x^{\alpha_2\theta_2} - \lambda_1 x^{\alpha_1\theta_1}$$

This gives

$$\begin{aligned} \frac{\partial \ln r(x)}{\partial x} &= \frac{1}{r(x)} r'(x) \\ &= \frac{(\alpha_1\theta_1 - \alpha_2\theta_2)}{x} + \alpha_2\theta_2\lambda_2 x^{\alpha_2\theta_2 - 1} - \alpha_1\theta_1\lambda_1 x^{\alpha_1\theta_1 - 1} \end{aligned}$$

Now, if $\alpha_1\theta_1 = \alpha_2\theta_2$ and $\lambda_1 > \lambda_2$ or $(\alpha_1\theta_1 > \alpha_2\theta_2)$, then $\frac{\partial \ln r(x)}{\partial x} < 0$. Hence, $r(x)$ decreases in x this implies that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y, X \leq_{mrl} Y$ and $X \leq_{st} Y$. □

2.9. Order Statistics. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ are the n ordered random sample drawn from (1.3). Then, the density of the rth order statistic follows from Arnold et al. [7], with the pdf of $X_{(r)}$ is given

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} \sum_{k=0}^{n-r} \binom{n-r}{k} (-1)^k [F(x)]^{r+k-1} f(x), \qquad x>0$$

and the rth order cdf $F_{r:n}(x)$ is

$$F_{r:n}(x) = \sum_{j=0}^{n} \sum_{k=0}^{n-r} \binom{n}{j} \binom{n-j}{k} (-1)^k [F(x)]^{j+k}$$

Hence, using (1.3) and (1.4), the pdf and the cdf of rth order statistics are, respectively, given by

$$f_{r:n}(x) = \frac{\alpha\theta\lambda^{\theta} x^{\alpha\theta-1}}{B(r, n-r+1)} \sum_{k=0}^{n-r} \sum_{i=0}^{r+k-1} \binom{n-r}{k} \binom{r+k-1}{i} (-1)^{k+i} e^{-(i+1)(\lambda x^{\alpha})^{\theta}} \tag{2.14}$$

$$F_{r:n}(x) = \sum_{j=0}^{n} \sum_{k=0}^{n-r} \sum_{i=0}^{j+k} \binom{n}{j} \binom{n-j}{k} \binom{j+k}{i} (-1)^{k+i} e^{-i(\lambda x^{\alpha})^{\theta}} \tag{2.15}$$

The distributions (pdf & cdf) of the smallest and the largest order statistics of the IPFr distribution are obtained by putting $r = 1$ and $r = n$ in (2.14) and (2.15) respectively.

3. Entropy Measurement

In information theory, entropy is a measure of variation of the uncertainty and large value of entropy indicates the greater uncertainty in the data. In this section, we discuss the different measure of change.

3.1. Renyi Entropy. The Rényi entropy $I_R(\rho)$ of a random variable X is defined as

$$I_R(\rho) = \frac{1}{1-\rho} \ln \left(\int_0^\infty [f(x)]^\rho \, dx \right)$$

where $\rho > 0$ and $\rho \neq 1$. Suppose X follows the IPFr distribution, we obtain

$$I_R(\rho) = \frac{1}{1-\rho} \ln \left(\left(\alpha\theta\lambda^\theta\right)^\rho \int_0^\infty x^{(\alpha\theta-1)\rho} e^{-\rho(\lambda x^\alpha)^\theta} dx \right)$$

Let $\xi(\rho) = \frac{\rho(\alpha\theta-1)+1}{\alpha\theta}$, after solving the internal, we get the following

$$I_R(\rho) = \frac{\ln(\xi(\rho))}{1-\rho} - \ln(\alpha\theta) - \xi(\rho) \frac{\ln(\rho)}{1-\rho} - \frac{1}{\alpha} \ln(\lambda) \tag{3.1}$$

3.2. Shannon Entropy. The Shannon entropy of a random variable X is defined by

$$I_S(\rho) = -\int_0^\infty f(x) \ln[f(x)] \, dx$$

Using the IPFr pdf, we obtain

$$I_S(\rho) = \frac{\alpha\theta + (\alpha\theta - 1)\gamma + \theta \ln(\lambda) - \alpha\theta \ln\left(\alpha\theta\lambda^\theta\right)}{\alpha\theta} \tag{3.2}$$

where γ is Euler-Mascheroni constant.

3.3. β-Entropy. The β-entropy is defined as

$$I_H(\beta) = \frac{1}{\beta-1} \left(1 - \int_0^\infty [f(x)]^\beta \, dx \right)$$

where $\beta > 0$ and $\beta \neq 1$. Using (1.3) and after simplification the expression for -entropy is given by

$$I_H(\beta) = \frac{1}{\beta-1} \left(1 - \frac{(\alpha\theta)^{\beta-1} \Gamma(\xi(\beta))}{\rho^{\xi(\beta)} \lambda^{\frac{1-\beta}{\alpha}}} \right) \tag{3.3}$$

3.4. Generalized Entropy. The generalized entropy is obtained by;

$$I_G = \frac{\upsilon_\omega \mu^{-\omega} - 1}{\omega(\omega-1)}; \omega \neq 0, 1$$

where, $\upsilon_\omega = \int_0^\infty x^\omega f(x)\,dx$. From (2.4) the value of υ_ω can be written as

$$\upsilon_\omega = \lambda^{-\frac{\omega}{\alpha}} \Gamma\left(\frac{\alpha\,\theta + \omega}{\alpha\,\theta}\right)$$

The generalized entropy can be written as

$$I_G = \frac{\Gamma\left(\frac{\alpha\,\theta+\omega}{\alpha\,\theta}\right)\left(\Gamma\left(\frac{\alpha\,\theta+1}{\alpha\,\theta}\right)\right)^{-\omega} - 1}{\omega(\omega-1)} \tag{3.4}$$

4. Estimation and inference of the parameters

The main aim of this section is to study different estimation methods of the unknown parameters of the IPFr distribution.

4.1. Maximum Likelihood method. Here, we discuss maximum likelihood estimation method and their fisher information matrix as well as asymptotic confidence intervals for estimating the unknown parameters α, θ, and λ of the IPFr distribution. The estimators obtained under these methods are not in nice closed form; thus, numerical approximation techniques are used to get the solution. Further, the performances of these estimators are studied through Monte Carlo simulation. Consider the random sample $x_1, x_2, ..., x_n$ of size n from the IPFr distribution and $\Theta = (\alpha, \theta, \lambda)^T$ be the parameter vector. The sample likelihood function is written as

$$\prod_{i=1}^{n} f(x; \alpha, \theta, \lambda) = \alpha^n \theta^n \lambda^{n\theta} e^{-\lambda^\theta \sum_{i=1}^n {x_i}^{\alpha\theta}} \left(\prod_{i=1}^{n} {x_i}^{\alpha\theta-1}\right) \tag{4.1}$$

The log-likelihood function is given by

$$L(\alpha, \theta, \lambda) = n\ln\alpha + n\ln\theta + n\theta\ln\lambda + (\alpha\theta - 1)\sum_{i=1}^{n}\ln x_i - \lambda^\theta \sum_{i=1}^{n} {x_i}^{\alpha\theta} \tag{4.2}$$

The maximum likelihood estimators (MLEs) $\widehat{\alpha}$, $\widehat{\theta}$ and $\widehat{\lambda}$ are obtained by solving the non-linear equations:

$$\frac{\partial}{\partial\alpha} L(\alpha, \theta, \lambda) = \frac{n}{\alpha} + \theta\sum_{i=1}^{n}\ln(x_i) - \theta\lambda^\theta \sum_{i=1}^{n}\left[{x_i}^{\alpha\theta}\ln(x_i)\right] \tag{4.3}$$

$$\frac{\partial}{\partial\theta} L(\alpha, \theta, \lambda) = \frac{n}{\theta} + n\ln(\lambda) + \alpha\sum_{i=1}^{n}\ln(x_i) - \lambda^\theta\ln(\lambda)\sum_{i=1}^{n} {x_i}^{\alpha\theta} - \alpha\lambda^\theta\sum_{i=1}^{n}\left[{x_i}^{\alpha\theta}\ln(x_i)\right] \tag{4.4}$$

$$\frac{\partial}{\partial \lambda} L(\alpha, \theta, \lambda) = \frac{n\theta}{\lambda} - \theta \lambda^{\theta-1} \sum_{i=1}^{n} x_i^{\alpha\theta} \tag{4.5}$$

The exact solution of the MLEs $\widehat{\alpha}$, $\widehat{\theta}$ and $\widehat{\lambda}$ in (4.3-4.5) is genuinely not possible. Hence, we use the non-linear optimization algorithms for maximizing the likelihood function numerically such as a Newton-Raphson algorithm for maximizing the likelihood function numerically.

From standard large-sample theory of maximum likelihood estimators (Theorem 5.1) Lehmann and Casella [20], the expected Fisher information matrix $I = [I_{ij}]$, $i, j = 1, 2, 3$, from a single observation for constructing $100(1-\psi)\,\%$ asymptotic confidence interval for the parameters using is given by

$$I = -E \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

where,

$$I_{11} = -E\left[\frac{\partial^2}{\partial \alpha^2} L(\alpha, \theta, \lambda)\right] = \frac{n}{\alpha^2} + \theta^2 \lambda^\theta \sum_{i=1}^{n} \left[x_i^{\alpha\theta} \left[\ln(x_i)\right]^2\right]$$

$$I_{22} = -E\left[\tfrac{\partial^2}{\partial \theta^2} L(\alpha, \theta, \lambda)\right] = \tfrac{n}{\theta^2} + \lambda^\theta \left[\ln(\lambda)\right]^2 \sum_{i=1}^{n} x_i^{\alpha\theta} + 2\alpha\lambda^\theta \ln(\lambda) \sum_{i=1}^{n} x_i^{\alpha\theta} \ln(x_i) \\ + \alpha^2 \lambda^\theta \sum_{i=1}^{n} \left[x_i^{\alpha\theta} \left[\ln(x_i)\right]^2\right]$$

$$I_{33} = -E\left[\frac{\partial^2}{\partial \lambda^2} L(\alpha, \theta, \lambda)\right] = \frac{n\theta}{\lambda^2} + (\theta - 1)\,\theta \lambda^{\theta-2} \sum_{i=1}^{n} x_i^{\alpha\theta}$$

$$I_{12} = -E\left[\tfrac{\partial^2}{\partial \alpha \theta} L(\alpha, \theta, \lambda)\right] = \theta \lambda^\theta \ln(\lambda) \sum_{i=1}^{n} \left[x_i^{\alpha\theta} \ln(x_i)\right] - \sum_{i=1}^{n} \ln(x_i) \\ + \lambda^\theta \sum_{i=1}^{n} \left[x_i^{\alpha\theta} \ln(x_i) + \alpha\theta x_i^{\alpha\theta} \left[\ln(x_i)\right]^2\right]$$

$$I_{23} = -E\left[\tfrac{\partial^2}{\partial \theta \lambda} L(\alpha, \theta, \lambda)\right] = \lambda^{\theta-1} \sum_{i=1}^{n} x_i^{\alpha\theta} - \tfrac{n}{\lambda} + \theta \lambda^{\theta-1} \ln(\lambda) \sum_{i=1}^{n} x_i^{\alpha\theta} \\ + \lambda^\theta \sum_{i=1}^{n} \left[x_i^{\alpha\theta} \ln(x_i) + \alpha\theta x_i^{\alpha\theta} \left[\ln(x_i)\right]^2\right] \\ + \alpha\theta \lambda^{\theta-1} \sum_{i=1}^{n} x_i^{\alpha\theta} \ln(x_i)$$

We have as $n \to \infty$, $\sqrt{n}\left(\widehat{\Theta} - \Theta\right)$ is asymptotically normal with (vector) mean zero and variance matrix I^{-1}, and $\widehat{\Theta}$ is asymptotically efficient in the sense that

$$\sqrt{n}\left(\widehat{\Theta} - \Theta\right) \xrightarrow{d} N_3\left(0, I^{-1}\right)$$

where $\xrightarrow{d}$ denotes convergence in the distribution and I^{-1} is the inverse of the Fisher information matrix I. The asymptotic variances and covariance of the MLEs $\widehat{\alpha}$, $\widehat{\theta}$ and $\widehat{\lambda}$ are given by:

$$V(\widehat{\alpha}) = \frac{I_{22}I_{33} - I_{23}^2}{n\Delta}, \quad V\left(\widehat{\theta}\right) = \frac{I_{11}I_{33} - I_{13}^2}{n\Delta}, \quad V\left(\widehat{\lambda}\right) = \frac{I_{11}I_{22} - I_{12}^2}{n\Delta}$$

$$Cov\left(\widehat{\alpha}, \widehat{\theta}\right) = \frac{I_{13}I_{23} - I_{12}I_{33}}{n\Delta}, \quad Cov\left(\widehat{\alpha}, \widehat{\lambda}\right) = \frac{I_{12}I_{23} - I_{13}I_{22}}{n\Delta},$$

$$Cov\left(\widehat{\theta}, \widehat{\lambda}\right) = \frac{I_{13}I_{12} - I_{11}I_{23}}{n\Delta}$$

where $\Delta = \det(I)$ is the determinant of the matrix I. The corresponding asymptotic $100(1-\psi)$ %confidence interval of Θ, are given by

$$\widehat{\Theta} \pm z_{\psi/2}\sqrt{\widehat{Var\left(\widehat{\Theta}\right)}}$$

where $\widehat{Var\left(\widehat{\Theta}\right)}$ is the MLE of $Var\left(\widehat{\Theta}\right)$ and $z_{\psi/2}$ is the upper $\psi/2$ quantile of the standard normal distribution.

4.2. Least squares and weighted least squares methods. The least squares (LSE) and the weighted least squares (WLSE) methods are used to find the minimum distance between theoretical cumulative distribution and the empirical cumulative distribution. Let $F\left(X_{(i)}\right)$ be the distribution function of the ordered random variables $X_{(1)}<X_{(2)}<...<X_{(n)}$ where $\{X_1, X_2, ..., X_2\}$ is a random sample of size n from a distribution function $F(.)$. Then, the expectation of the empirical cumulative distribution function is defined as

$$E\left[F\left(X_{(i)}\right)\right] = \frac{i}{n+i}; \; i = 1, 2, ..., n$$

The LSEs of α, θ and λ denoted by $\widehat{\alpha}_{LSE}$, $\widehat{\theta}_{LSE}$and $\widehat{\lambda}_{LSE}$ can be obtained by minimizing the following function

$$LS(\alpha, \theta, \lambda) = \sum_{i=1}^{n}\left(F\left(x_{(i)}; \alpha, \theta, \lambda\right) - \frac{i}{n+i}\right)^2 \tag{4.6}$$

with respect to α , θ and λ. Therefore $\widehat{\alpha}_{LSE}$,$\widehat{\theta}_{LSE}$ and $\widehat{\lambda}_{LSE}$ can be obtained as the solution of the following system of non-linear equations:

$$\frac{\partial LS(\alpha, \theta, \lambda)}{\partial \alpha} = \sum_{i=1}^{n}\left(F\left(x_{(i)}; \alpha, \theta, \lambda\right) - \frac{i}{n+i}\right) F_{\alpha}^{'}(x; \alpha, \theta, \lambda) = 0 \tag{4.7}$$

$$\frac{\partial LS(\alpha, \theta, \lambda)}{\partial \theta} = \sum_{i=1}^{n}\left(F\left(x_{(i)}; \alpha, \theta, \lambda\right) - \frac{i}{n+i}\right) F_{\theta}^{'}(x; \alpha, \theta, \lambda) = 0 \tag{4.8}$$

$$\frac{\partial LS(\alpha,\theta,\lambda)}{\partial\lambda}=\sum_{i=1}^{n}\left(F\left(x_{(i)};\alpha,\theta,\lambda\right)-\frac{i}{n+i}\right)F_{\lambda}^{'}(x;\alpha,\theta,\lambda)=0 \tag{4.9}$$

The WLSEs of α, θ and λ denoted by $\widehat{\alpha}_{WLSE}$,$\widehat{\theta}_{WLSE}$and $\widehat{\lambda}_{WLSE}$ can be obtained by minimizing

$$WLS(\alpha,\theta,\lambda)=\sum_{i=1}^{n}\frac{(n+1)^{2}(n+2)}{i(n-i+1)}\left(F\left(x_{(i)};\alpha,\theta,\lambda\right)-\frac{i}{n+i}\right)^{2} \tag{4.10}$$

with respect to α , $\theta$and λ, therefore these estimators can also be obtained by solving:

$$\frac{\partial WLS(\alpha,\theta,\lambda)}{\partial\alpha}=\sum_{i=1}^{n}\frac{(n+1)^{2}(n+2)}{i(n-i+1)}\left(F\left(x_{(i)};\alpha,\theta,\lambda\right)-\frac{i}{n+i}\right)F_{\alpha}^{'}(x;\alpha,\theta,\lambda)=0 \tag{4.11}$$

$$\frac{\partial WLS(\alpha,\theta,\lambda)}{\partial\theta}=\sum_{i=1}^{n}\frac{(n+1)^{2}(n+2)}{i(n-i+1)}\left(F\left(x_{(i)};\alpha,\theta,\lambda\right)-\frac{i}{n+i}\right)F_{\theta}^{'}(x;\alpha,\theta,\lambda)=0 \tag{4.12}$$

$$\frac{\partial WLS(\alpha,\theta,\lambda)}{\partial\lambda}=\sum_{i=1}^{n}\frac{(n+1)^{2}(n+2)}{i(n-i+1)}\left(F\left(x_{(i)};\alpha,\theta,\lambda\right)-\frac{i}{n+i}\right)F_{\lambda}^{'}(x;\alpha,\theta,\lambda)=0 \tag{4.13}$$

where

$$F_{\alpha}^{'}(x;\alpha,\theta,\lambda)=\theta(\lambda x^{\alpha})^{\theta}\ln(x)\exp\left[-(\lambda x^{\alpha})^{\theta}\right]$$

,

$$F_{\theta}^{'}(x;\alpha,\theta,\lambda)=(\lambda x^{\alpha})^{\theta}\ln(\lambda x^{\alpha})\exp\left[-(\lambda x^{\alpha})^{\theta}\right]$$

and

$$F_{\lambda}^{'}(x;\alpha,\theta,\lambda)=\frac{\theta(\lambda x^{\alpha})^{\theta}\exp\left[-(\lambda x^{\alpha})^{\theta}\right]}{\lambda}$$

4.3. Cramer-von-Mises method. The Cramer-von Mises statistics can be given by

$$C(\alpha,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F\left(x_{(i)};\alpha,\theta\right)-\frac{2i-1}{2n}\right)^{2} \tag{4.14}$$

Then the CME estimators $\widehat{\alpha}_{CME}$, $\widehat{\theta}_{CME}$ and $\widehat{\lambda}_{CME}$ are obtained by minimizing (4.14) with respect to α, θ and λ. These estimators can also be obtained by solving the following non-linear equations:

$$\frac{\partial C(\alpha,\theta,\lambda)}{\partial\alpha}=\sum_{i=1}^{n}\left(F\left(x_{(i)};\alpha,\theta,\lambda\right)-\frac{2i-1}{2n}\right)F_{\alpha}^{'}(x;\alpha,\theta,\lambda)=0 \tag{4.15}$$

$$\frac{\partial C(\alpha,\theta,\lambda)}{\partial\theta}=\sum_{i=1}^{n}\left(F\left(x_{(i)};\alpha,\theta,\lambda\right)-\frac{2i-1}{2n}\right)F_{\theta}^{'}(x;\alpha,\theta,\lambda)=0 \tag{4.16}$$

$$\frac{\partial C(\alpha,\theta,\lambda)}{\partial\lambda}=\sum_{i=1}^{n}\left(F\left(x_{(i)};\alpha,\theta,\lambda\right)-\frac{2i-1}{2n}\right)F_{\lambda}^{'}(x;\alpha,\theta,\lambda)=0 \tag{4.17}$$

where $F_{\alpha}^{'}(x;\alpha,\theta,\lambda)$, $F_{\theta}^{'}(x;\alpha,\theta,\lambda)$ and $F_{\lambda}^{'}(x;\alpha,\theta,\lambda)$ are defined above .

4.4. Maximum product spacing method. Cheng and Amin [11] introduced the maximum product spacing (MPS) and showed that the MPS method can be used as an alternative to MLE to estimate the parameters of continuous univariate distributions. Let the difference is defined as

$$D_i(\alpha,\theta,\lambda)=F\left(x_{(i)};\alpha,\theta,\lambda\right)-F\left(x_{(i-1)};\alpha,\theta,\lambda\right),\qquad i=1,2,...,n \tag{4.18}$$

where $F\left(x_{(0)};\alpha,\theta,\lambda\right)=0$ and $F\left(x_{(n+1)};\alpha,\theta,\lambda\right)=1$. The geometric mean of the differences can be written as

$$G(\alpha,\theta,\lambda)=\sqrt[n+1]{\prod_{i=1}^{n+1}D_i(\alpha,\theta,\lambda)} \tag{4.19}$$

Substituting (1.4) in (4.18) and maximizing the above expression, we have

$$\frac{\partial\log G(\alpha,\theta,\lambda)}{\partial\alpha}=\frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{F_{\alpha}^{'}\left(x_{(i)};\alpha,\theta,\lambda\right)-F_{\alpha}^{'}\left(x_{(i-1)};\alpha,\theta,\lambda\right)}{F\left(x_{(i)};\alpha,\theta,\lambda\right)-F\left(x_{(i-1)};\alpha,\theta,\lambda\right)}\right)=0 \tag{4.20}$$

The MPSEs $\widehat{\alpha}_{MPS}$, $\widehat{\theta}_{MPS}$ and $\widehat{\lambda}_{MPS}$ are obtained as the simultaneous solution of the following non linear equations:

$$\frac{\partial\log G(\alpha,\theta,\lambda)}{\partial\theta}=\frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{F_{\theta}^{'}\left(x_{(i)};\alpha,\theta,\lambda\right)-F_{\theta}^{'}\left(x_{(i-1)};\alpha,\theta,\lambda\right)}{F\left(x_{(i)};\alpha,\theta,\lambda\right)-F\left(x_{(i-1)};\alpha,\theta,\lambda\right)}\right)=0 \tag{4.21}$$

$$\frac{\partial\log G(\alpha,\theta,\lambda)}{\partial\lambda}=\frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{F_{\lambda}^{'}\left(x_{(i)};\alpha,\theta,\lambda\right)-F_{\lambda}^{'}\left(x_{(i-1)};\alpha,\theta,\lambda\right)}{F\left(x_{(i)};\alpha,\theta,\lambda\right)-F\left(x_{(i-1)};\alpha,\theta,\lambda\right)}\right)=0 \tag{4.22}$$

where $F_{\alpha}^{'}(x;\alpha,\theta,\lambda)$, $F_{\theta}^{'}(x;\alpha,\theta,\lambda)$ and $F_{\lambda}^{'}(x;\alpha,\theta,\lambda)$are defined above .

4.5. Anderson–Darling and right-tail Anderson–Darling methods. The method of Anderson-Darling (AD) was introduced by Anderson and Darling [5,6] and is based on an Anderson-Darling statistic. The Anderson-Darling statistic is given by

$$A^2=n\int_0^{\infty}\frac{\left(F(x_i)-E\left[F\left(x_{(i)}\right)\right]\right)^2}{F(x_i)\left(1-F(x_i)\right)}dF(x_i)$$

The Anderson-Darling statistic is given by

$$A(\alpha,\theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F\left(x_{(i)};\alpha,\theta\right) + \log\left(1 - F\left(x_{(i)};\alpha,\theta\right)\right)\right] \quad (4.23)$$

Therefore, the ADs $\widehat{\alpha}_{AD}$,$\widehat{\theta}_{AD}$ and $\widehat{\lambda}_{AD}$ can be determined by minimizing (4.23) with respect to α, θ and λ. These estimators can also be obtained by solving the non-linear equations

$$\frac{\partial A(\alpha,\theta,\lambda)}{\partial\alpha} = -\frac{1}{n}\sum_{i=1}^{n+1}(2i-1)\left(\frac{F_{\alpha}^{'}\left(x_{(i)};\alpha,\theta,\lambda\right)}{F\left(x_{(i)};\alpha,\theta,\lambda\right)} - \frac{F_{\alpha}^{'}\left(x_{(n-i+1)};\alpha,\theta,\lambda\right)}{1-F\left(x_{(n-i+1)};\alpha,\theta,\lambda\right)}\right) = 0 \quad (4.24)$$

$$\frac{\partial A(\alpha,\theta,\lambda)}{\partial\theta} = -\frac{1}{n}\sum_{i=1}^{n+1}(2i-1)\left(\frac{F_{\theta}^{'}\left(x_{(i)};\alpha,\theta,\lambda\right)}{F\left(x_{(i)};\alpha,\theta,\lambda\right)} - \frac{F_{\theta}^{'}\left(x_{(n-i+1)};\alpha,\theta,\lambda\right)}{1-F\left(x_{(n-i+1)};\alpha,\theta,\lambda\right)}\right) = 0 \quad (4.25)$$

$$\frac{\partial A(\alpha,\theta,\lambda)}{\partial\lambda} = -\frac{1}{n}\sum_{i=1}^{n+1}(2i-1)\left(\frac{F_{\lambda}^{'}\left(x_{(i)};\alpha,\theta,\lambda\right)}{F\left(x_{(i)};\alpha,\theta,\lambda\right)} - \frac{F_{\lambda}^{'}\left(x_{(n-i+1)};\alpha,\theta,\lambda\right)}{1-F\left(x_{(n-i+1)};\alpha,\theta,\lambda\right)}\right) = 0 \quad (4.26)$$

Also, the Right-tail AD is given by

$$RA(\alpha,\theta) = \frac{n}{2} - 2\sum_{i=1}^{n}F\left(x_{(i)};\alpha,\theta\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(1 - F\left(x_{(i)};\alpha,\theta\right)\right) \quad (4.27)$$

Hence, the RADs $\widehat{\alpha}_{RAD}$,$\widehat{\theta}_{RAD}$ and $\widehat{\lambda}_{RAD}$ are obtained by minimizing (4.27) with respect to α,θ and λ. These estimators can also be determined by solving the non-linear equations

$$\frac{\partial RA(\alpha,\theta,\lambda)}{\partial\alpha} = -n\sum_{i=0}^{n}\frac{F_{\alpha}^{'}\left(x_{(i)};\alpha,\theta,\lambda\right)}{F\left(x_{(i)};\alpha,\theta,\lambda\right)} + \frac{1}{n}\sum_{i=1}^{n+1}(2i-1)\frac{F_{\alpha}^{'}\left(x_{(n-i+1)};\alpha,\theta,\lambda\right)}{1-F\left(x_{(n-i+1)};\alpha,\theta,\lambda\right)} = 0 \quad (4.28)$$

$$\frac{\partial RA(\alpha,\theta,\lambda)}{\partial\theta} = -n\sum_{i=0}^{n}\frac{F_{\theta}^{'}\left(x_{(i)};\alpha,\theta,\lambda\right)}{F\left(x_{(i)};\alpha,\theta,\lambda\right)} + \frac{1}{n}\sum_{i=1}^{n+1}(2i-1)\frac{F_{\theta}^{'}\left(x_{(n-i+1)};\alpha,\theta,\lambda\right)}{1-F\left(x_{(n-i+1)};\alpha,,\lambda\right)} = 0 \quad (4.29)$$

$$\frac{\partial RA(\alpha,\theta,\lambda)}{\partial\lambda} = -n\sum_{i=0}^{n}\frac{F_{\lambda}^{'}\left(x_{(i)};\alpha,\theta,\lambda\right)}{F\left(x_{(i)};\alpha,\theta,\lambda\right)} + \frac{1}{n}\sum_{i=1}^{n+1}(2i-1)\frac{F_{\lambda}^{'}\left(x_{(n-i+1)};\alpha,\theta,\lambda\right)}{1-F\left(x_{(n-i+1)};\alpha,,\lambda\right)} = 0 \quad (4.30)$$

5. Simulation Study

Here, a simulation study is performed to examine the performance of the different estimates presented above. The following procedure for evaluating the efficiency of the estimators is adopted as follow:

- Generate random sample with size n from the IPFr distribution.
- The values obtained in step 1are used to compute the $\widehat{\Theta} = \left(\widehat{\alpha}, \widehat{\theta}, \widehat{\lambda}\right)$ considering the MLE, LSE,WLSE, CME, MPS, AD and RAD estimators.
- Repeat the steps 1 and 2 Ntimes.
- Using $\widehat{\Theta} = \left(\widehat{\alpha}, \widehat{\theta}, \widehat{\lambda}\right)$ and $\Theta = (\alpha, \theta, \lambda)$, compute the Bias and the mean square errors (MSE).

The results are computed using the nlminb function (in the stat package) and Nelder-Mead method in R software. From the IPFr distribution , 5,000 samples were generated, where $n = \{50, 100, 150\}$, and by choosing$\alpha = \{1, 1.5, 0.5\}$,$\theta = \{0.5, 0.6, 1.5\}and\lambda = \{0.8, 0.2, 2\}$, for each parameters combination and each sample. Bias and mean square error (MSE) of the MLE, LSE, WLSE, CME, MPS, AD and RAD were evaluated in tables 2-4 and the ranks (partial and overall) of the estimators were calculated in table 5. From tables 2-5, we can observe that:

- All estimates show the property of consistency i.e., the MSEs decrease as sample size increase for all parameter combinations.
- According to MSEs, the MPS estimation method has the best performance of estimators for all parameter combinations.

TABLE 2. Simulation results for $\alpha = 1, \theta = 0.5, \lambda = 0.8$.

Est. Meth.	S.Size	n=50			n=100			n=150		
	Est. Par.	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\lambda}$
MLE	Bias	-0.0153	0.0256	0.0213	-0.0080	0.0140	0.0058	-0.0048	0.0100	0.0007
	MSE	0.0028	0.0030	0.0450	0.0008	0.0012	0.0175	0.0005	0.0007	0.0091
LSE	Bias	-0.0200	0.0082	0.0427	-0.0095	0.0044	0.0228	-0.0073	0.0035	0.0109
	MSE	0.0026	0.0036	0.0726	0.0010	0.0016	0.0317	0.0007	0.0011	0.0205
WLSE	Bias	-0.0271	0.0177	0.0438	-0.0151	0.0100	0.0194	-0.0100	0.0072	0.0159
	MSE	0.0058	0.0039	0.0624	0.0022	0.0016	0.0301	0.0013	0.0010	0.0200
CME	Bias	-0.0108	0.0203	0.0548	-0.0070	0.0111	0.0275	-0.0050	0.0062	0.0186
	MSE	0.0028	0.0040	0.0745	0.0011	0.0018	0.0341	0.0006	0.0011	0.0213
MPS	Bias	-0.0115	-0.0043	0.0167	-0.0052	-0.0006	-0.0025	-0.0026	0.0003	-0.0025
	MSE	0.0011	0.0017	0.0484	0.0004	0.0007	0.0175	0.0002	0.0004	0.0098
AD	Bias	-0.0120	0.0144	0.0245	-0.0073	0.0086	0.0070	-0.0052	0.0056	0.0041
	MSE	0.0022	0.0023	0.0488	0.0009	0.0009	0.0189	0.0005	0.0005	0.0102
RAD	Bias	-0.0031	0.0108	0.0432	-0.0030	0.0061	0.0179	-0.0052	0.0059	0.0194
	MSE	0.0039	0.0034	0.0690	0.0016	0.0018	0.0329	0.0010	0.0012	0.0220

6. Real Data Illustration

In this section, we perform the practical applicability of the proposed model using maximum likelihood estimate of the parameter to represents the potentiality of the new model as compared to some other existing life-time models by using the real data set. This data set represents vinyl chloride data obtained from clean upgradient ground-water monitoring wells in mg/L; this data set is used by Bhaumik et al.

TABLE 3. Simulation results for $\alpha = 1.5, \theta = 0.6, \lambda = 0.2$.

Est. Meth.	S.Size	n=50			n=100			n=150		
	Est. Par.	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\lambda}$
MLE	Bias	0.0041	0.0169	0.0080	0.0027	0.0076	0.0039	0.0015	0.0048	0.0018
	MSE	0.0009	0.0042	0.0029	0.0003	0.0018	0.0013	0.0002	0.0012	0.0008
LSE	Bias	-0.0145	0.0052	0.0094	-0.0057	0.0013	0.0040	-0.0027	0.0003	0.0025
	MSE	0.0028	0.0064	0.0029	0.0010	0.0029	0.0013	0.0004	0.0018	0.0009
WLSE	Bias	-0.0109	0.0076	0.0081	-0.0026	0.0034	0.0043	-0.0012	0.0019	0.0027
	MSE	0.0023	0.0058	0.0027	0.0009	0.0023	0.0013	0.0004	0.0015	0.0009
CME	Bias	-0.0075	0.0205	0.0112	-0.0017	0.0092	0.0058	-0.0012	0.0050	0.0036
	MSE	0.0027	0.0070	0.0031	0.0009	0.0031	0.0015	0.0004	0.0018	0.0009
MPS	Bias	-0.0100	-0.0222	0.0069	-0.0057	-0.0143	0.0028	-0.0042	-0.0111	0.0028
	MSE	0.0006	0.0040	0.0027	0.0002	0.0019	0.0012	0.0001	0.0012	0.0008
AD	Bias	-0.0071	0.0072	0.0095	-0.0018	0.0028	0.0038	-0.0009	0.0020	0.0021
	MSE	0.0020	0.0049	0.0028	0.0006	0.0022	0.0012	0.0003)	0.0014	0.0008
RAD	Bias	-0.0026	0.0104	0.0085	-0.0022	0.0059	0.0046	-0.0014	0.0035	0.0038
	MSE	0.0011	0.0051	0.0030	0.0005	0.0024	0.0014	0.0002	0.0016	0.0009

TABLE 4. Simulation results for $\alpha = 0.5, \theta = 1.5, \lambda = 2$.

Est. Meth.	S.Size	n=50			n=100			n=150		
	Est. Par.	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\lambda}$	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\lambda}$
MLE	Bias	0.0143	0.0069	0.0326	0.0080	0.0028	0.0169	0.0052	0.0040	0.0119
	MSE	0.0029	0.0113	0.0165	0.0014	0.0049	0.0071	0.0008	0.0031	0.0046
LSE	Bias	0.0054	-0.0190	-0.0034	0.0025	-0.0089	0.0004	0.0014	-0.0052	-0.0005
	MSE	0.0034	0.0184	0.0199	0.0014	0.0085	0.0098	0.0009	0.0056	0.0067
WLSE	Bias	0.0117	-0.0198	0.0212	0.0083	-0.0159	0.0174	0.0059	-0.0132	0.0148
	MSE	0.0038	0.0163	0.0226	0.0015	0.0067	0.0116	0.0010	0.0044	0.0076
CME	Bias	0.0105	0.0116	0.0291	0.0048	0.0060	0.0146	0.0032	0.0019	0.0073
	MSE	0.0038	0.0195	0.0216	0.0014	0.0084	0.0098	0.0009	0.0055	0.0066
MPS	Bias	-0.0057	-0.0340	-0.0337	-0.0031	-0.0205	-0.0207	-0.0019	-0.0148	-0.0151
	MSE	0.0021	0.0111	0.0135	0.0010	0.0051	0.0062	0.0006	0.0032	0.0039
AD	Bias	0.01126	-0.0119	0.0173	0.0063	-0.0087	0.0089	0.0045	-0.0037	0.0102
	MSE	0.0034	0.0150	0.0157	0.0013	0.0069	0.0071	0.0008	0.0040	0.0045
RAD	Bias	0.01196	-0.0071	0.0179	0.0048	-0.0043	0.0095	0.0038	-0.0039	0.0067
	MSE	0.0049	0.0177	0.0145	0.0015	0.0069	0.0075	0.0010	0.0043	0.0052

[10] in testing parameters of a Gamma distribution for small samples and they are:

5.1	1.2	1.3	0.6	0.5	2.4	0.5
0.4	2	0.5	5.3	3.2	2.7	2.9
1.8	0.9	2	4	6.8	1.2	0.4
8	2.3	0.9	0.8	1	0.1	0.6
1.1	2.5	0.2	0.4	0.2	0.1	

The proposed distribution is compared with other five alternative distributions such as:

- Inverse power Lindley (IPL) distribution (Barco et al [8]) given by the pdf

$$f(x) = \frac{\alpha\theta^2}{\theta+1}\left(\frac{1+x^\alpha}{x^{2\alpha+1}}\right)\exp\left(\frac{-\theta}{x^\alpha}\right)$$

  where $x, \alpha, \theta > 0$

- Inverted xgamma (IXG) distribution (Yadav et al [27]) given by the pdf

TABLE 5. Ranks (partial and overall) of all the estimation methods for various combination of α,θ and λ.

Est. Meth.	Int. Val.	$\alpha=1,\theta=0.5,\lambda=0.8$			$\alpha=1.5,\theta=0.6,\lambda=0.2$			$\alpha=0.5,\theta=1.5,\lambda=2$			Sum	Overall Rank
	n	50	100	150	50	100	150	50	100	150		
MLE	$\widehat{\alpha}$	4.5	2	2.5	2	2	2.5	2	4	2.5	24	2
	$\widehat{\theta}$	3	3	3	2	1	1.5	2	1	1	17.5	2
	$\widehat{\lambda}$	1	1.5	1	4.5	4	2	4	2.5	3	23.5	3
LSE	$\widehat{\alpha}$	3	4	5	7	7	6	3.5	4	4.5	44	4
	$\widehat{\theta}$	5	4.5	5.5	6	6	6.5	6	7	7	53.5	6
	$\widehat{\lambda}$	6	5	5	4.5	4	5.5	5	5.5	6	46.5	5
WLSE	$\widehat{\alpha}$	7	7	7	5	5.5	6	5.5	3	6.5	52.5	7
	$\widehat{\theta}$	6	4.5	4	5	4	4	4	7	5	43.5	4
	$\widehat{\lambda}$	4	5	4	1.5	4	5.5	7	6.5	7	44.5	4
CME	$\widehat{\alpha}$	4.5	6	4	6	5.5	6	5.5	6	4.5	48	6
	$\widehat{\theta}$	7	6.5	4.5	7	7	6.5	7	5.5	6	57	7
	$\widehat{\lambda}$	7	7	6	7	7	5.5	6	4.5	5	55	7
MPS	$\widehat{\alpha}$	1	1	1	1	1	1	1	2	1	10	1
	$\widehat{\theta}$	1	1	1	1	2	1.5	1	1	2	11.5	1
	$\widehat{\lambda}$	2	1.5	2	1.5	1.5	2	1	1	1	13.5	1
AD	$\widehat{\alpha}$	2	3	2.5	4	4	4	3.5	4.5	2.5	30	3
	$\widehat{\theta}$	2	2	2	3	3	3	3	2.5	3	23.5	3
	$\widehat{\lambda}$	3	3	3	3	1.5	2	3	2.5	2	23	2
RAD	$\widehat{\alpha}$	6	6	6	3	3	2.5	7	4.5	6.5	44.5	5
	$\widehat{\theta}$	4	6.5	7	4	5	5	5	4	4	44.5	5
	$\widehat{\lambda}$	5	6	7	6	6	5.5	2	6.5	4	48	6

$$f\left(x\right)=\frac{\theta^{2}}{\left(1+\theta\right)}.\frac{1}{x^{2}}\left(1+\frac{\theta}{2}.\frac{1}{x^{2}}\right)\exp\left(\frac{-\theta}{x}\right)$$

where $x,\theta>0$

- Exponentiated inverse Rayleigh (EIR) distribution (Ul Haq [26]) given by the pdf

$$f\left(x\right)=\frac{2\alpha\theta}{x^{3}}\exp\left(\frac{-\alpha\theta}{x^{2}}\right)$$

where $x,\alpha,\theta>0$

- Exponentiated inverse Weibull (EIW) distribution (Flaih et al [14])given by the pdf

$$f\left(x\right)=\theta\beta x^{-(\beta+1)}\exp\left(-x^{-\beta}\right)^{\theta}$$

where $x,\beta,\theta>0$

- Inverse Gompertz (IG) distribution (Eliwa et al [26]) given by the pdf

$$f\left(x\right)=\frac{\alpha}{x^{2}}\exp\left(\frac{-\alpha}{\beta}\left(\exp\left(\frac{\beta}{x}\right)-1\right)+\frac{\beta}{x}\right)$$

where $x,\alpha,\beta>0$.

For these models the method of maximum likelihood is used to estimate of the parameter (s). We perform goodness of fit measures selection tools such as log-likelihood (-L), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hanna-Quinn Information Criterion (HQIC) and Consistent Akaike Information Criterion (CAIC). Goodness of fit tests statistics such as Cram-von

Mises W_n^2, Anderson–Darling A_n^2, Watson U_n^2, Liao-Shimokawa $L_n$and Kolmogrov-Smirnov $K-S$ with its respective P-value are considered in order to verify which distribution fits better to these data. In general, the smaller values of these statistics indicate, the better fit to the data.

TABLE 6. The goodness of fit measures for the data set.

Models	Measures							
	MLEs			- L	AIC	BIC	HQIC	CAIC
$IPFr(\alpha,\theta,\lambda)$	1.1618	0.8695	0.4779	55.4496	116.8992	121.4783	118.4608	124.4783
$IPL(\alpha,\theta)$	0.7772	—	1.0729	59.1204	122.2409	125.2936	123.2819	127.2936
$IXG(\theta)$	—	1.0675	—	62.6554	129.3108	132.3636	130.3519	134.3636
$EIR(\alpha,\theta)$	0.1000	—	1.1486	93.3510	190.7021	193.7548	191.7432	195.7548
$EIW(\beta,\theta)$	—	0.6540	0.8804	58.6266	121.2532	124.3059	122.2942	126.3059
$IG(\alpha,\beta)$	—	0.42935	0.1000	63.0348	130.0695	133.1223	131.1106	135.1223

TABLE 7. The goodness-of-fit test statistics for the data set.

Models	Statistics					
	W_n^2	A_n^2	U_n^2	L_n	$K-S$	p-value
$IPFr(\alpha,\theta,\lambda)$	0.0433	0.2826	2.9048	0.6589	0.0918	0.9366
$IPL(\alpha,\theta)$	0.1072	0.8256	8.1014	1.0161	0.1130	0.7777
$IXG(\theta)$	0.4092	2.3257	8.3411	1.9872	0.2022	0.1241
$EIR(\alpha,\theta)$	3.2125	25.1005	10.9716	11.9177	0.4854	2.197E-07
$EIW(\beta,\theta)$	0.1027	0.7719	8.0938	0.9963	0.1134	0.7745
$IG(\alpha,\beta)$	0.5355	2.9530	8.45024	2.2547	0.2214	0.0713

Table 5, indicates that the inverse power Fret distribution presents a better fit model to the data than the other models. The tests shown in Table 6 observe that the EIR distribution not fit the data (p-value < 0.05) and the inverse power Fret distribution shows the lowest test statistics with the largest p-values. Therefore, our proposed distribution can be recommended as a good alternative to the existing family of Fret distribution.
Furthermore, seven estimation methods are used to estimate the unknown parameters of the proposed distribution. Table 7 display the estimates of the IPFr parameters using these estimation methods and the values of $K-S$ and its p-value for the data set. We can conclude that the AD estimation method is recommended to estimate the IPFr parameters for the data set.

7. Conclusion

In this paper, we proposed inverse power Fret distribution (IPFr) as an extension of Fret distribution which offers the upside-down bathtub shape for its hazard rate. Different mathematical and statistical properties, such as reliability measures, moments, generating functions, quantile, mean deviation, stochastic ordering, order statistics, some entropy measures, Bonferroni, Lorenz curves and Gini index are studied. We also perform the behavior of the estimated parameters by using seven estimation methods and these methods are performed through the simulation study. Real data set has also studied for the demonstration of flexibility and better fit of the observed model as compared to some other existing models.

TABLE 8. The parameter estimates of IPFr distribution, $K-S$ and p-value for the data set.

Est. Meth.	Est. Par.			$K-S$	p-value
	$\widehat{\alpha}$	$\widehat{\theta}$	$\widehat{\lambda}$		
MLE	1.1618	0.8695	0.4779	0.0918	0.9366
LSE	1.1072	0.8372	0.5136	0.0995	0.8893
WLSE	0.5445	1.7424	0.7119	0.0893	0.9489
CME	1.1190	0.8641	0.5188	0.0909	0.9416
MPS	1.1254	0.8243	0.4782	0.0895	0.9482
AD	1.1177	0.8683	0.5053	0.0856	0.9645
RAD	1.0990	0.8365	0.5167	0.1015	0.8751

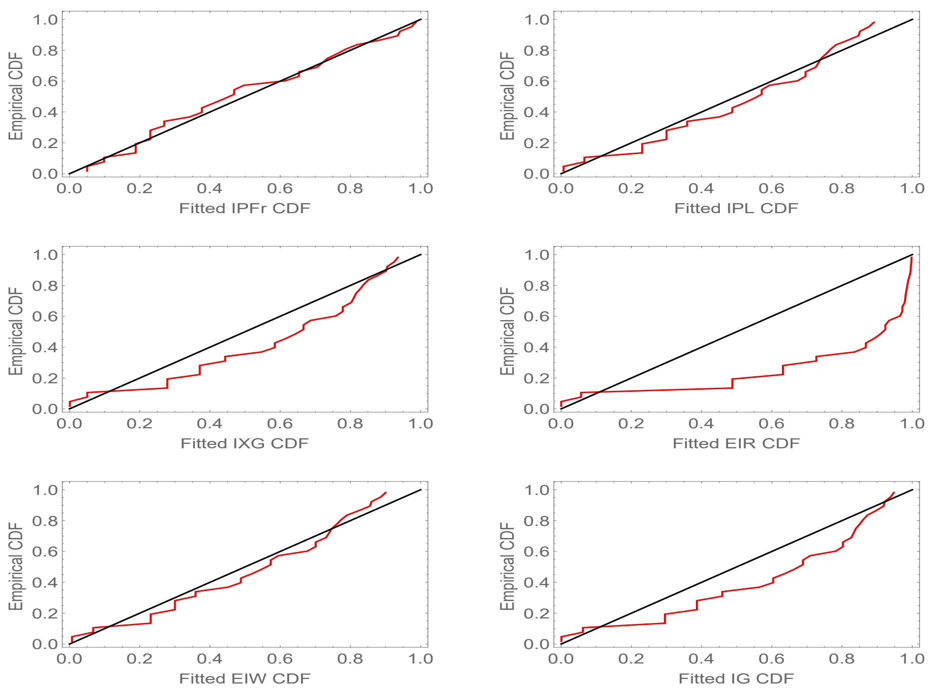

FIGURE 3. Plots of (P-P) for the fitted distributions.

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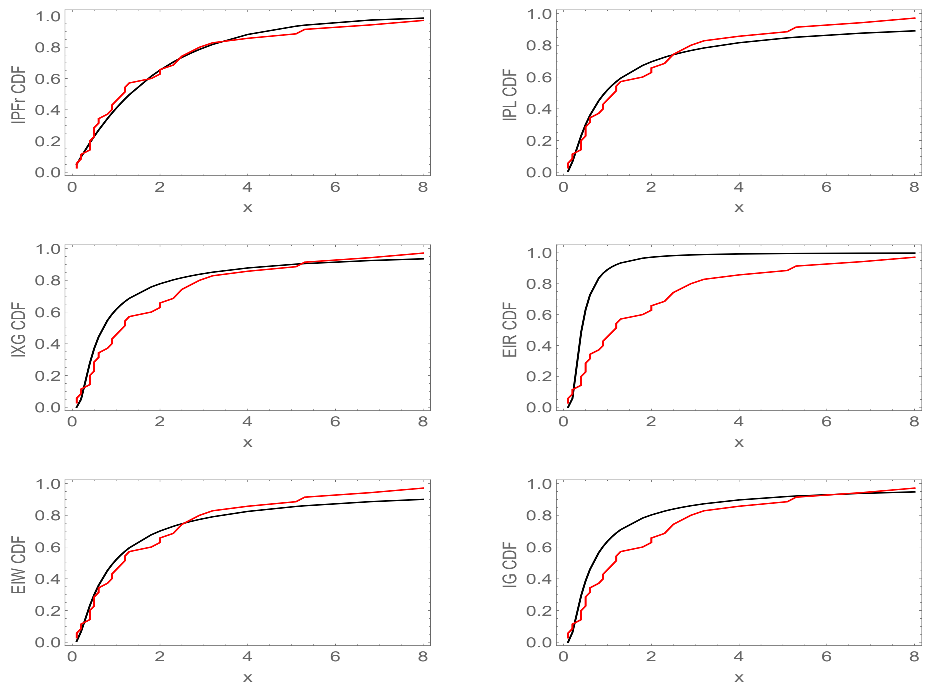

FIGURE 4. Plots of fitted cdf for the fitted distributions.

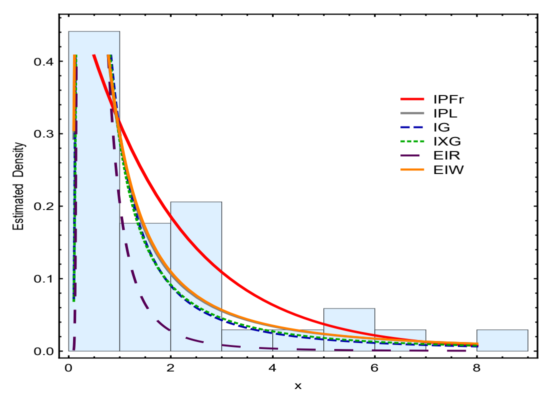

FIGURE 5. Estimated pdfs for the data set 1.

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