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AN UNUSUAL QUANTUM ENTANGLEMENT CONSISTENT WITH SCHRÖDINGER'S EQUATION

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ABSTRACT. In this paper, a quantum system is considered to be formed by two free particles in such a manner that for at least one of its states, an Ansatz Ψ specifically modeled to present mathematical properties that could be consistent with an unusual quantum entanglement is proposed. The Ansatz Ψ is defined by an expansion in terms of the eigenfunctions of an arbitrary observable, which is associated, separately, with the considered particles, with coefficients given by the values of functions C_n, initially free and dependent on the quotient of the separation between the particles and their Compton wavelength. It is shown that Ψ satisfies the corresponding Schrödinger equation under the requirement that the C_n be of the square integrable type. The function Ψ corresponds to an entanglement which, in the absence of both a measurement and the performance of entanglement degrading agents, collapses to zero when a condition is satisfied.

1. Introduction

The quantum entanglement, which arouses great interest[1], renewed especially after the quantum computation proposal [4], [5], and in particular the quantum teleportation protocol [6], [7], has as its more general and comprehensive aspects those of its generation, preservation and measurement. More specific aspects like criteria that permit its detection [8]-[11], the methods for its purification [12]-[14], or its consequences for various quantum algorithms and quantum protocols [15]-[17], has been extensively investigated. More recently, investigations are developed to better understand the processes leading to the so-called "entanglement sudden death" [18]-[20].

On the other hand, quantum entanglement admits a simple classification, in two categories, consisting of: (i) *physical entanglement* and (ii) *spurious entanglement*. This classification is consistent with the possibility, or impossibility, of establishing correspondences[2] between mathematical properties (of abstract objects in quantum theory) and physical properties (of a specific quantum system). The category of spurious entanglement (or mathematical entanglement[3]) refers to the possibility of finding quantum states which, although they have tangled form,

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[1]Which began in 1931, with the publication of several works by N. Rosen [1]-[3].
[2]A basis for this can be found in [21].
[3]The idea of mathematical entanglement was first introduced by H. Valqui [22].

cannot be attributed to the physical system in question in the quantum models. Within this category there are some concrete examples, such as those found from an extension of the pseudo pure density matrices [23], [24]. Moreover, within the category of physical entanglement between quantum particles, which has its dynamics described in terms of the pure states, and in the absence of any external degrading influences, it is not known that there are several manifestations of this entanglement. It could be possible, however, that there are some sectors of the solution space of Schrödinger's equation that admits different types of entanglement, each with its own characteristics, besides those that would be common to them.

With the purpose of theoretically probing the possibility of "existing" those sectors (or at least one) for quantum entanglement, it is convenient to identify, within a broad context, functions (Ansatz) that presents mathematical properties that could correspond to a type of entanglement[4] with features not completely coincident with those that are known for a physically entangled state. Considering the case of two identical and free quantum particles, one can imagine, for example, a function (Ansatz) which, in the absence of both a measurement and of external agents or degrading conditions of the entanglement, may present an entanglement that, depending of some quantum property of the particles, *collapses* in the event that they depart "sufficiently".

The work developed here is organized and presented in the following sequence: In section 2 the corresponding physical context is defined, which includes enlightening comments; in section 3 the complete mathematical development is presented; In section 4 we present a discussion of the solution presented, and, finally, the conclusions are presented.

2. Physical context

Consider a non-relativistic physical system formed by two identical quantum particles, which we will call subsystems I and II, which from a given moment are free. The quantum states of subsystem I will be described in terms of the variables $\vec{x}_1$, separately from the states of subsystem II, described in terms of the variables $\vec{x}_2$. In addition, consider an *arbitrary* physical observable $\hat{A}$ associated with these subsystems, as well as the set of their eigenfunctions, which we denote as $\{u_1,\ u_2,\ u_3,\ ...\}$ for subsystem I, and as $\{\psi_1,\ \psi_2,\ \psi_3,\ ...\}$ for subsystem II. These eigenfunctions are, in fact, only formally known in the sense that, because they are associated with an arbitrary observable, we do not have explicit analytic expressions for them. The above context, based partially on that defined by Einstein, Podolsky and Rosen (EPR) in [25], is complemented here by the following Ansatz proposal,

$$\Psi(\vec{x}_1,\vec{x}_2)=\sum_n C_n\Big((\vec{x}_1-\vec{x}_2)/\lambda\Big)\ \psi_n(\vec{x}_2).u_n(\vec{x}_1), \tag{2.1}$$

[4]Which, in principle, is only of the "mathematical" type. A second stage, based on experimental arguments, could reveal whether this entanglement is or is not of the "physical" type.

where the coefficients are defined by the values of functions C_n, initially free[5], dependent on the variable $(\vec{x}_1 - \vec{x}_2)/\lambda$, where λ is a parameter which, in principle, is arbitrary; however, as will be justified further ahead, this is considered equal to the Compton wavelength of the particles of our physical system.

2.1. General comments.

[A] The Ansatz proposed in (1), which represents a tangled function, since it is not possible to write as the product of two factors that correspond separately to each subsystem, has partial mathematical similarity with the EPR function, but is essentially distinct from this not only by the presence of the coefficients C_n. The function EPR, which can be obtained from (1) by $C_n((\vec{x}_1 - \vec{x}_2)/\lambda) = 1$, $\forall n$, becomes known[6] in [25] and corresponds to a series expansion in the orthogonal functions $\{u_n\}$. On the other hand, in our case, we are interested in the opposite situation: given the expansion (2.1), we try to determine, if possible, the conditions that should be imposed on the C_n so that (2.1) can be considered solution of Schrödinger equation for two identical and free quantum particles.

[B] The reason for considering the problem defined in item [A] is that Ansatz (1) would present two interesting properties as long as the functions C_n are of the *square integrable* type. These properties would: (i) naturally collapse in the situation whereby the particles move apart "sufficiently", which would be controlled by the corresponding λ, and (ii) be consistent at the classical limit.

[C] In linear expansion (2.1), written in terms of the base elements of an *arbitrary* physical observable, temporal dependence does not exist ($\partial\Psi/\partial t = 0$). Thus, considering the 3−dimensional case, so that $\vec{x}_1$ and $\vec{x}_2$ represent the coordinate variables x_1, y_1, z_1 and x_2, y_2, z_2, respectively, associated with subsystems I and II, separately, we have to consider the general Schrödinger equation,

$$\frac{-\hbar^2}{2m} \bigtriangledown_1^2 \Psi + \frac{-\hbar^2}{2m} \bigtriangledown_2^2 \Psi = i\hbar \frac{\partial}{\partial t} \Psi = 0 \tag{2.2}$$

and not the stationary equation, which would only be appropriate in the particular case when the functions $\{u_k\}$ and $\{\psi_k\}$ corresponded to the Hamiltonian eigenfunctions of subsystem I and II, separately; a situation that is not considered here.

3. Mathematical development

As it is common in approaches that use Ansatz, we have to identify the conditions that should be imposed on its free parameters (the functions C_n, in our case) so that the function (2.1), thus defined, can be considered a solution of the equation (2.2). Before presenting the development itself, we will introduce, for convenience, the following notation[7]: $\vec{\xi} \equiv (\vec{x}_1 - \vec{x}_2)/\lambda$.

[5] In the sense of not being defined.

[6] For being considered as the solution of Schrödinger's equation for two free particles.

[7] Or by making, equivalently, $\xi_1 = (x_1 - x_2)/\lambda$; $\xi_2 = (y_1 - y_2)/\lambda$; $\xi_3 = (z_1 - z_2)/\lambda$.

We begin by substituting (2.1) in the expression on the left side (which will be isolated) from equation (2.2), with which we write,

$$-\frac{\hbar^2}{2m}\sum_n \psi_n(\vec{x}_2)\nabla_1^2\Big(C_n(\vec{\xi})\ u_n(\vec{x}_1)\Big)-\frac{\hbar^2}{2m}\sum_n u_n(\vec{x}_1)\nabla_2^2\Big(C_n(\vec{\xi})\ \psi_n(\vec{x}_2)\Big) \quad (3.1)$$

In expression (3.1), we can rewrite the Laplacians that refers to the coordinate variables that describe subsystems I and II in the following way[8],

$$\nabla_1^2\Big(C_n(\vec{\xi})\ u_n(\vec{x}_1)\Big)=C_n(\vec{\xi})\nabla_1^2u_n(\vec{x}_1)+\frac{2}{\lambda}\nabla_\xi C_n(\vec{\xi}).\nabla_1u_n(\vec{x}_1)+\frac{1}{\lambda^2}u_n(\vec{x}_1)\nabla_\xi^2C_n(\vec{\xi}) \quad (3.2)$$

and also,

$$\nabla_2^2\Big(C_n(\vec{\xi})\ \psi_n(\vec{x}_2)\Big)=C_n(\vec{\xi})\nabla_2^2\psi_n(\vec{x}_2)-\frac{2}{\lambda}\nabla_\xi C_n(\vec{\xi}).\nabla_2\psi_n(\vec{x}_2)+\frac{1}{\lambda^2}\psi_n(\vec{x}_2)\nabla_\xi^2C_n(\vec{\xi}) \quad (3.3)$$

where, the symbol ∇_ξ represents "Laplacian" in relation to the variables ξ_1, ξ_2, ξ_3. In addition, we have,

$$\nabla_1C_n(\vec{\xi})=\frac{1}{\lambda}\nabla_\xi C_n(\vec{\xi}) \quad \& \quad \nabla_1^2C_n(\vec{\xi})=\frac{1}{\lambda^2}\nabla_\xi^2C_n(\vec{\xi}) \quad (3.4)$$

and also,

$$\nabla_2C_n(\vec{\xi})=-\frac{1}{\lambda}\nabla_\xi C_n(\vec{\xi}) \quad \& \quad \nabla_2^2C_n(\vec{\xi})=\frac{1}{\lambda^2}\nabla_\xi^2C_n(\vec{\xi}) \quad (3.5)$$

Thus, we can rewrite (3.1) as follows,

$$\begin{aligned}&-\frac{\hbar^2}{2m}\sum_n \psi_n(\vec{x}_2)\Big(C_n(\vec{\xi})\nabla_1^2u_n(\vec{x}_1)+2\nabla_1C_n(\vec{\xi}).\nabla_1u_n(\vec{x}_1)+u_n(\vec{x}_1)\nabla_1^2C_n(\vec{\xi})\Big)+\\&-\frac{\hbar^2}{2m}\sum_n u_n(\vec{x}_1)\Big(C_n(\vec{\xi})\nabla_2^2\psi_n(\vec{x}_2)+2\nabla_2C_n(\vec{\xi}).\nabla_2\psi_n(\vec{x}_2)+\psi_n(\vec{x}_2)\nabla_2^2C_n(\vec{\xi})\Big)\end{aligned} \quad (3.6)$$

Recalling that the sets $\{\psi_n\}$ and $\{u_n\}$ are well defined and that, so far, the coefficients C_n are free. Therefore, the most convenient thing to do is to impose these coefficients which verify the equation:

$$\begin{aligned}&\sum_n \psi_n(\vec{x}_2)\Big(C_n(\vec{\xi})\nabla_1^2u_n(\vec{x}_1)+2\nabla_1C_n(\vec{\xi}).\nabla_1u_n(\vec{x}_1)+u_n(\vec{x}_1)\nabla_1^2C_n(\vec{\xi})\Big)+\\&+\sum_n u_n(\vec{x}_1)\Big(C_n(\vec{\xi})\nabla_2^2\psi_n(\vec{x}_2)+2\nabla_2C_n(\vec{\xi}).\nabla_2\psi_n(\vec{x}_2)+\psi_n(\vec{x}_2)\nabla_2^2C_n(\vec{\xi})\Big)=0\end{aligned} \quad (3.7)$$

This is nothing else other than equation (2.2). The former is equivalent to assuming that there are adequate values for the coefficients so that the function (2.1) checks the equation (2.2).

In order to progress in the calculation, considering that we do not have explicit expressions for the eigenfunctions $\{\psi_n\}$ and $\{u_n\}$, we will perform multiple integrations of the equation (3.7) with respect to the independent variables $\vec{x}_1$ (and

[8]The corresponding demonstration is very simple.

later in relation to the variables $\vec{x}_2$) over a space region, which will be conveniently defined[9].

Next, we rewrite the integral of the three terms that appear in parentheses at the top of (3.7), that is, the following sum,

$$\Xi \equiv \int d^3\vec{x}_1 C_n(\vec{\xi})\nabla_1^2 u_n(\vec{x}_1) + 2\int d^3\vec{x}_1 \nabla_1 C_n(\vec{\xi}).\nabla_1 u_n(\vec{x}_1) + \\ + \int d^3\vec{x}_1 u_n(\vec{x}_1)\nabla_1^2 C_n(\vec{\xi}) \tag{3.8}$$

to make use of the following Green identity later,

$$\int d^3x\ \varphi\nabla^2\vartheta = \oint d\vec{s}.\ \varphi\nabla\vartheta - \int d^3x\ \nabla\varphi.\nabla\vartheta \tag{3.9}$$

This will be applied to the integrals that appear with unit coefficients (integrals containing the Laplacians) in (3.8). Thus, according to (3.8) and (3.9), we have,

$$\Xi = \oint d\vec{s}\cdot C_n(\vec{\xi})\ \nabla_1 u_n(\vec{x}_1) - \int d^3\vec{x}_1\ \nabla_1 C_n(\vec{\xi})\cdot\nabla u_n(\vec{x}_1) + \\ +2\int d^3\vec{x}_1\ \nabla_1 C_n(\vec{\xi})\cdot\nabla u_n(\vec{x}_1) + \oint d\vec{s}\cdot u_n(\vec{x}_1)\ \nabla_1 C_n(\vec{\xi}) + \\ - \int d^3\vec{x}_1\ \nabla_1 C_n(\vec{\xi})\cdot\nabla u_n(\vec{x}_1). \tag{3.10}$$

From which it follows that,

$$\Xi = \oint d\vec{s}.\ C_n(\vec{\xi})\ \nabla_1 u_n(\vec{x}_1) + \oint d\vec{s}.\ u_n(\vec{x}_1)\ \nabla_1 C_n(\vec{\xi}). \tag{3.11}$$

Now, if we consider, for example, that the integration surface is spherical, with radius "sufficiently large" (formally considering that "$r \to \infty$") in relation to the value of λ, the integrals in (3.11) will cancel out in the situation where we require that the functions C_n be, $\forall n$, of the square integrable type[10]; thus, we have $\Xi = 0$.

Therefore, after integrating (3.7) with respect to variables $\vec{x}_1$, the expression follows as presented,

$$\frac{-\hbar^2}{2m}\sum_n \int d^3\vec{x}_1\ u_n(\vec{x}_1)\Big(C_n(\vec{\xi})\nabla_2^2\psi_n(\vec{x}_2) + 2\nabla_2 C_n(\vec{\xi}).\nabla_2\psi_n(\vec{x}_2) + \\ + \psi_n(\vec{x}_2)\nabla_2^2 C_n(\vec{\xi})\Big) = 0. \tag{3.12}$$

Then in continuation, we integrate (3.12) with respect to the variables $\vec{x}_2$; in this manner, we have,

$$\frac{-\hbar^2}{2m}\sum_n \int d^3\vec{x}_2\ u_n(\vec{x}_1)\int d^3\vec{x}_1\Big(C_n(\vec{\xi})\nabla_2^2\psi_n(\vec{x}_2) + 2\nabla_2 C_n(\vec{\xi}).\nabla_2\psi_n(\vec{x}_2) + \\ + \psi_n(\vec{x}_2)\nabla_2^2 C_n(\vec{\xi})\Big) = 0. \tag{3.13}$$

[9]It is not convenient to choose a region that has linear dimensions small or comparable to λ.

[10]It is not essential that the surface be spherical, what matters is that on the surface that is considered the coefficients C_n assume null values.

Since the integrations in (3.13) refer to variables that are independent of one another, one can change the order of integration. Taking the integrals with respect to the variables $\vec{x}_2$ that appear in (3.13) we have,

$$\Lambda \equiv \int d^3\vec{x}_2\ C_n(\vec{\xi})\nabla_2^2\psi_n(\vec{x}_2) + 2\int d^3\vec{x}_2\nabla_2 C_n(\vec{\xi}).\nabla_2\psi_n(\vec{x}_2) + \\ + \int d^3\vec{x}_2\ \psi_n(\vec{x}_2)\nabla_1^2 C_n(\vec{\xi}). \tag{3.14}$$

Which, as before, can be rewritten using Green's identity (3.9); with this, one arrives at the expression,

$$\Lambda = \oint d\vec{s}.\ C_n(\vec{\xi})\ \nabla_2\psi_n(\vec{x}_2) + \oint d\vec{s}.\ \psi_n(\vec{x}_2)\ \nabla_2 C_n(\vec{\xi}). \tag{3.15}$$

Considering a region and a spherical surface equivalent to those of the previous case, and assuming that the functions C_n are of the square integrable type, as before, results: $\Lambda = 0$. Thus, the expression (3.13), which resulted from integrating equation (3.7) twice, is identically satisfied for functions C_n of the square integrable type. In this way, the equation (2.2) is satisfied by function (2.1) under the condition that the coefficients $C_n(\vec{\xi})$ correspond to the values of functions C_n that are of the type identified above.

4. Discussion

It is clear that neither the context nor the results presented in this article define any set of specific coefficients $\{C_n\}$ for the function (2.1); consequently, only through experimental considerations (if applicable) can these be defined (modeled). However, just to visualize them better, we present a theoretical example of these coefficients, chosen arbitrarily within the considered context, as follows,

$$C_n\big((\vec{x}_1 - \vec{x}_2)/\lambda\big) = e^{-|\vec{x}_1 - \vec{x}_2|^2/\lambda^2}, \quad \forall n. \tag{4.1}$$

From which it can be noted that,

$$C_n\big((\vec{x}_1 - \vec{x}_2)/\lambda\big) \to 0 \quad \text{if} \quad |\vec{x}_1 - \vec{x}_2|/\lambda \to \infty, \text{ with } \lambda \text{ finite.} \tag{4.2}$$

In the equation above, we have used a classical expression for the "separation" between the quantum particles, namely $|\vec{x}_1 - \vec{x}_2|$. According to Quantum Mechanics, it is the expectation value of the difference of the position operators assigned, separately, to each particle, in the state of interest, that is, $< \hat{\vec{X}}_1 - \hat{\vec{X}}_2 >_\Psi$, which provides statistical information of what could be interpreted as the separation between quantum particles; but the calculation of the corresponding integral cannot be performed because, in (2.1), the explicit dependence of the coefficients in terms of their variables (still) is not defined, as already mentioned. However, the use of the expression $|\vec{x}_1 - \vec{x}_2|$ is justified because, in the situation where the particles are sufficiently far apart, which is the condition that ensures the peculiar manifestation of this entanglement, it must be verified according to the Ehrenfest theorem that $< \hat{\vec{X}}_1 - \hat{\vec{X}}_2 >_\Psi$ behaves as $|\vec{x}_1 - \vec{x}_2|$, which more specifically corresponds to the situation in which the separation between the particles is very many orders of magnitude larger than the Compton wavelength of them. As a consequence

of this, it can be said that the function (2.1) is not completely quantum, it is semi-classical.

On the other hand, the choice of the parameter λ in (2.1) as the Compton wavelength of the particles, $\lambda_C = h/mc$, is consistent with both the type of entanglement revealed here and with the classical boundary. Let's look at the following particular case. Since the Compton wavelength of the electron, λ_e, of the order of 10^{-13}m and considering a separation between the particles of the order of 10^{-3}m, we directly see that $|\vec{x}_1 - \vec{x}_2|/\lambda_e$ is of the order of 10^{10}m, for which the coefficients C_n, which are of the integrable square type, may already have reached a null or quasi-null value. Besides that, when formally we make $\lambda \to 0$, corresponding to the situation of a classical particle (since it results from taking $m \to \infty$, m being the corresponding mass), the function (2.1) collapses everywhere, in particular, when the particles (macroscopic, in this case) are very close, which is consistent with the total absence of entanglement in the classical context.

The difference between the entanglement here revealed and the usual one is best observed in the absence of any external agent that degrades it; furthermore, any measurement on one of the parts of the particle system is excluded. Even so, the entangled function (2.1) collapses when the particles are "sufficiently" spaced from each other, which reveals an objective characteristic incompatible with the normally understood entanglement. The mathematical fact that the coefficients depend on the separation of particles reveals a sector previously unknown from the solution space of the Schrödinger equation for two free particles. In the context under discussion, the coefficients C_n can be interpreted informally as a direct measure of the degree of survival of this unusual entanglement.

A possible physical scenario for the entanglement revealed here would be the one corresponding to the quantum−classical boundary.

Finally, it may be interesting to note that if in the previous calculations we make $C_n\big((\vec{x}_1 - \vec{x}_2)/\lambda\big) = 1$, $\forall n$, and consider the mathematical fact that the eigenfunctions of the physical observables are of the integrable square type, it would be shown that the EPR state itself satisfies the Schrödinger equation, was implicitly assumed, but not demonstrated, in the famous article of 1935 [25].

5. Conclusion

In expression (2.1), we have considered an Ansatz modeled so that it could present two specific mathematical properties: (*i*) one which correspond to a type of entanglement that collapses to zero when the condition $|\vec{x}_1 - \vec{x}_2|/\lambda \to \infty$ is satisfied, which may occur in the absence of both a measurement and external degrading factors of (usual) entanglement, and (*ii*) one which is being consistent at the classical limit. The first property above can be assigned to (2.1) provided that the functions C_n are of the square integrable type. Thus, a type of entanglement is characterized that does not correspond to a manifestation of the usual entanglement. The second property can also be assigned to (2.1) since the parameter λ corresponds to the Compton wavelength of the particles.

The unusual entanglement presented in this paper should be considered, in the meantime, as of the "spurious" type since we have no experimental arguments in favor of the possibility of this manifesting physically.

The considerations presented here are not intended to exhaust the problem of identifying the sectors of the solution space of the Schrödinger's equation that could be consistent with other types of quantum entanglement.

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