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Neumann-Morgenstern Stable Set of a Finite Static Strategy Game

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Abstract: A finite static strategy game is called rational if all the players' common knowledge contains and only contains these: (1) A set of players (at least two players). (2) For each player, a set of actions. (3) Payoffs received by each player for the combinations of the actions, or for each player, preferences over the combinations of the actions. (4) the principle of maximum entropy. In this paper, we put forward a concept of Neumann-Morgenstern stable set(briefly, N-M stable set) which is a subset of set of all pure Nash equilibria in a finite static strategy game. Give an argorithm of finiding N-M statble set . Prove existence and uniqueness of N-M stable set in a finite static strategy game with at least one pure Nash equilibrium.An ideal game is defined as the game whose all players have the same preferences for all pure suations in the N-M stable.For example,a finite rational game is an ideal game. We prove that every finite static strategy game is equivalent to one and only one ideal game.

Keywords: Static finite game of complete information; Completely static finite game; Rational finite game; N-M stable set; Ideal game; the maximal entropy principle

1. INTRODUCTION

The idea of Nash equilibrium is one of the most powerful concepts in game theory. Nash [1,2] proved the existence of mixed strategy equilibria for finite games. From a decision theoretic viewpoint the concept of mixed strategy Nash equilibrium is less compelling than the concept of pure strategy Nash equlibrium (PNE). It is therefore interesting to study PNE in a finite game, under different conditions.

In finite game theory, the simplest model is the so-called bimatrix game. The game is represented as a two-dimensional matrix, in which the rows represent the pure strategies for one player and the columns those for the other. In each cell is placed a pair of numbers representing the payoffs to the two players if the corresponding pair of strategies is chosen. For set of pure Nash equilibria of a bimatrix game, there are, mostly, the hands. The first is existence of pure Nash equilibria, such as [3-5]. The second is number of pure Nash equilibria and its estimation, such as [6-10]. The third is structure of set of pure Nash equilibria, such as [11-13]. The fourth is regularity and stability of equilibrium points of bimatrix games and so forth, such as [14,15].

In the other hand, several authors studied number of pure Nash equilibria of finite games with random payoffs, and studied how the distribution of the random number of PNE's for these games varies under different assumptions [16-27].

Except these, a more important problem is what pure Nash equilibria all the players should prefer. As everyone knows, realization of a pure Nash equilibrium is important and difficult [28]. Our problem can make it is easier that all the players to realize a pure Nash equilibrium.

If a player repeatedly play a zero-sum game, a mixed strategy adds an uncertainty which could confuse his/her enemy [29]. As a result, each player hopes to use his/her mixed strategy with the greatest uncertainty.

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In 1948, Shannon introduced the concept of information to describe an uncertainty of a random variable taking value[30].

In 1999, R. J. Aumann studied the formalization of knowledge and common knowledge in games theory. Call an event common knowledge (in a population) if all know the event, all know that all know it, all know that all know that all know it, and son on ad infinitum [31,32].

The Principle of Maximum Entropy states: When one has only partial information about the possible outcomes one should choose the probabilities so as to maximize the uncertainty about the missing information. In other words, the basic rule is: Use all the information on the parameter that you have, but avoid including any information that you do not have. Therefore one should be as uncommitted as possible about missing information[33,34].

A game is said to be static (or simultaneous move) one of complete information, if all the follow are common knowledge among all the players. (1) A set of players (at least two players). (2) For each player, a set of actions. (3) And payoffs received by each player for the combinations of the actions, or for each player, preferences over the combinations of the actions. But for an actual game, players have other common knowledge relating to their action selection. For instance, in Battle of the Sexes, it is possible that it is their common knowledge that the husband is willing to make any sacrifice for his wife.

A static (or simultaneous move) game of complete information is said to be completely static if all players have no common knowledge except (1)-(3).

By principle of maximum entropy, for a completely static game, Shannon information entropy, that is uncertainty of every player's judgement[35] for the other players' action should be maximal. However, classical game theory does not assume that every player knows the principle of maximum entropy.

A static (or simultaneous move) game of complete information is said to be rational if all players have and only have the common knowledge (1)-(3) and (4) the principle of maximum entropy.

Jiang and Zhang [28] studied a static (or simultaneous move) game of complete information under Strong Common Knowledge System and defined Nash equilibrium, called an expected Nash equilibrium. All players in a static (or simultaneous move) game of complete information under Strong Common Knowledge System are more intelligent than those under Classical Common Knowledge System. Therefore an expected Nash equilibrium situation for the former is easier to realize. By the ideal, we obtained some results. For example, Jiang, Zhang and Ding [36] obtained some perfect results on a matrix game without optimal pure game solution. Jiang, Zhang and Ding [37,38] showed applications of the new game system to environmental science. And Jiang, Zhang and Ding [39] showed applications of the new game system to economic management science.

In this paper, we put forward a concept of Neumann-Morgenstern stable set(briefly, N-M stable set) which is a subset of set of all pure Nash equilibria in a finite static strategy game. Give an argorithm of finiding N-M statble set . Prove existence and uniqueness of N-M stable set in a finite static strategy game with at least one pure Nash equilibrium.An ideal game is defined as the game whose all players have the same preferences for all pure suations in the N-M stable. For example, a finite rational game is an ideal game. We prove that every finite game is equivalent to an ideal game.

2. PREFERENCE RELATIONS ON A SET

Definition 2.1: A binary relation $\prec$ defined on a set X is called inferior relation if it satisfies

(1) nonreflexivity: $\neg(x \prec x), \forall x \in X$, and

(2) dissymmetry: $x \prec y \Rightarrow \neg(y \prec x), \forall x, y \in X$.

Where $x \prec y$ is read as x is inferior than y.

Definition 2.2: The inferior relation $\prec$ derives the four relations

(1) indifference relation: $x \sim y \Leftrightarrow \neg(x \prec y) \wedge \neg(y \prec x) \Leftrightarrow \neg(x \prec y \vee y \prec x)$.

(2) superior relation: $x \succ y \Leftrightarrow y \prec x$.

(3) weakly inferior relation: $x \prec\sim y \Leftrightarrow x \prec y \vee y \sim x$, and

(4) weakly superior relation: $x \succ\sim y \Leftrightarrow x \succ y \vee x \sim y$.

Inferior relation, indifference relation, superior relation, weakly inferior relation, and weakly superior relation are called preference relations.

Theorem 2.1: For any x, $y \in X$, one and only one of the three relations $x \prec y$, $y \prec x$, $x \sim y$ is true.

3. COMPLETE INFORMATION, STATIC, FINITE GAMES

Definition 3.1: The system

$$\Gamma \equiv [N, (A_i), (\prec_i), C] \equiv [N, (A_i), (u_i), C]$$

is called an n person finite game, where N ($|N| = n$) is the set of all players, members in the finite set A_i is the player i's pure strategy (or action). For the player i's inferior relation $\prec_i$ on the set of pure situation $A = \prod_{i \in N} A_i$, the function $u_i: A \to R$ satisfying the condition

$$u_i(a') < u_i(a'') \Leftrightarrow a' \prec_i a''$$

is called player i's utility function. C is set of all the players' common knowledge [31,32].

Definition 3.2: A finite game $\Gamma \equiv [N, (A_i), (\prec_i), C]$ that satisfies the condition $\{N, (A_i), (\prec_i)\} \subseteq C$ is called a static finite game (with complete information).

Definition 3.3: A finite game $\Gamma \equiv [N, (A_i), (\prec_i), C]$ that satisfies the condition $C = \{N , (A_i), (\prec_i)\}$ is called a complete static finite game.

In a static finite game with complete information, the set of players, each player's action space, and each player's preference with regard to all the pure situations are all players' common knowledge. In a complete static game, all players' common knowledge contains and only contains the set of players, each player's action space, and each player's preference with regard to all the pure situations.

4. UNIFORM RELATIONS AND DIVERGENCE RELATIONS ON PURE SITUATIONS

Let a, $b \in$ A, $\varnothing \neq$ S $\subseteq N$. Then

$$a \prec_i b \Leftrightarrow i \in S,\ a \succ_i b \Leftrightarrow i \in S \text{ and } a \sim_i b \Leftrightarrow i \in S$$

are written as $a \prec_S b$, $a \succ_S b$ and $a \sim_S b$, respectively.

Definition 4.1: For a game $\Gamma \equiv [N,(A_i),(\prec_i), C]$, the preference relations on A are defined as the follows

(1) $a \prec b \Leftrightarrow a \prec_i b$, $i = 1, \ldots, n$.

(2) $a \succ b \Leftrightarrow a \succ_i b,\ i = 1, \ldots, n$.

(3) $a \sim b \Leftrightarrow a \sim_i b,\ i = 1, \ldots, n$.

(4) $a \prec_S \succ_T b \Leftrightarrow \exists S \neq \varnothing,\ \exists\, T \neq \varnothing,\ S \bigcup T \subseteq N,\ S \bigcap T = \varnothing,\ (a \prec_S b) \wedge (a \succ_T b)$.

(5) $a \prec_S \sim_T b \Leftrightarrow \exists S \neq \varnothing,\ \exists\, T \neq \varnothing,\ S \bigcup T = N,\ (a \prec_S b) \wedge (a \sim_T b)$.

(6) $a \succ_S \sim_T b \Leftrightarrow \exists S \neq \varnothing,\ \exists\, T \neq \varnothing,\ S \bigcup T = N,\ (a \succ_S b) \wedge (a \sim_T b)$.

Where (1), (2) and (3) are called uniform relations with regard to a, b and (4), (5) and (6) are called divergence relations with regard to a, b.

It is clear that $a \prec b \Leftrightarrow a \prec_N b$, $a \succ b \Leftrightarrow a \succ_N b$, and $a \sim b \Leftrightarrow a \sim_N b$.

Theorem 4.1: For any $a, b \in A$, one and only one of the follow six relations is true. (1) $a \prec b$, (2) $a \succ b$, (3) $a \sim b$, (4) $a \prec_S \succ_T b$, (5) $a \prec_S \sim_T b$ and (6) $a \succ_S \sim_T b$.

Proof: It is obvious that it is impossible that both of the cases (1) to (6) are true. We now show that at least one of (1) to (6) is true.

Suppose first (1) is false. Let $a \succ_i b, \exists i \in N$. If $a \succ_i b, \forall i \in N$ then (2) is true; if not, then $a \succ_S b,\ i \in S \subset N$. If $a \sim_j b, \forall j \in N \backslash S$, then (6) is true; if not, then $a \prec_T b$, $a \sim_{N \backslash (S \bigcup T)} b$, $\varnothing \neq T \subseteq N \backslash S$. At this time,(4) is true. Now let $a \sim_i b, \exists i \in N$. If $a \sim_i b, \forall i \in N$ then (3) is true; if not, then $a \sim_S b$, $\varnothing \neq S \subset N$. If $a \prec_i b, \forall i \in N \backslash S$, then (5) is true; if not, then (4) is true.

Similarly, the case (2) can be shown.

Suppose (3) is false. Without loss of generality, let there exists a nonempty $S \subseteq N$ such that $a \succ_S b$. If $S = N$, then (2) is true. Let $S \neq N$. If $a \sim_{N \backslash S} b$, then (6) is true; if not, then (4) is true.

Suppose (4) is false. For any S, T, $S \bigcup T = N$ satisfying $(a \prec_S b) \wedge (a \sim_T b)$, either $S = \varnothing$, or $T \varnothing$. If $S = \varnothing$, then (3) is true. If $T = \varnothing$, then (1) or (5) is true.

By imitating (5), the case (6) can be shown.

5. N-M STABLE SETS

Definition 5.1: For a game $\Gamma \equiv [N, (A_i), (\prec_i), C]$, let $a^* \in A$ be a pure situation with Nash stability. i.e., for any $i \in N$, and any $a_i \in A_i$, $(a^*_{-i}, a_i) \prec\sim_i a^*$, then a^* is called a pure Nash equilibrium in the game Γ. PNE(Γ) denotes the set of all pure Nash equilibria.

Example 5.1: (1,1), (2,1), (2,2), (2,3) and (3,3) are all the pure Nash equilibria in the game

$$\begin{bmatrix} (\underline{2},\underline{3}) & (2,2) & (\underline{1},2) \\ (\underline{2},\underline{2}) & (\underline{3},\underline{2}) & (\underline{1},\underline{2}) \\ (\underline{2},0) & (2,0) & (\underline{1},\underline{1}) \end{bmatrix}.$$

Obviously, we have that $(3,3) \prec (1,1)$, $(3,3) \prec (2,1)$, $(3,3) \prec (2,2)$ and $(2,3) \prec (1,1)$.

Definition 5.2: For a game $\Gamma \equiv [N, (A_i), (\prec_i), C]$, if $\varnothing \neq V \subseteq PNE(\Gamma)$ satisfies Neumann-Morgenstern stability (or briefly, N-M stability)

(1) internal stability: for any a^*, $a^{**} \in V$, neither of $a^{**} \prec a^*$ and $a^* \prec a^{**}$ is true, and

(2) external stability: for any $a^{**} \in PNE(\Gamma)\backslash V$, there exists an $a^* \in V$ such that $a^{**} \prec a^*$, then V is called an N-M stable set in the game $\Gamma \equiv [N, (A_i), (\prec_i), C]$.

Corollary: Let $\Gamma \equiv [N,(A_i), (\prec_i), C]$ be a zero sum game then $PNE(\Gamma) = V$.

Proof: We have

$$\sum_{i \in N} u_i(a^*) = 0, \quad \forall a^* \in PNE(V).$$

Assume there exist a^*, $a^{**} \in PNE(V)$ such that $a^* \prec a^{**}$. Then $u_i(a^*) < u_i(a^{**})$, $\forall_i \in N$.

And so

$$0 = \sum_{i \in N} u_i(a^*) < \sum_{i \in N} u_i(a^{**}) = 0,$$

a constradiction. This shows

$$\neg(a^* \prec a^{**}), \neg(a^{**} \prec a^*), \forall a^*, a^* \in PNE(V)$$

That is $PNE(\Gamma)$ = V.

Theorem 5.1: Let V be an N-M stable set of the game $\Gamma \equiv [N, (A_i), (\prec_i), C]$, then for any $a^* \in$ V, there exists none $a^{**} \in PNE(\Gamma)$ such that $a^* \prec a^{**}$.

Proof: The conclusion is clear if $a^{**} \in V$. Let $a^{**} \in PNE(\Gamma)\backslash V$ satisfy $a^* \prec a^{**}$. By the internal stability, there exists $a^{\#} \in V$ such that $a^{**} \prec a^{\#}$. And so $a^* \prec a^{\#}$. This contradicts to the internal stability of V.

Remarks about Nash and N-M Stabilities

1. Nash stability: the stability of a pure situation with regard to every player.
2. N-M stability: the stability of a set of pure situations with regard to all the players.

Theorem 5.2: (Algorithm for finding an N-M stable set) Suppose $PNE(\Gamma) \neq \varnothing$. The output V of the follow algorithm is a nonempty N-M stable set and $U = PNE(\Gamma)\backslash V$.

(1) $1 \Rightarrow k$, $\varnothing \Rightarrow U$, $a^* \in PNE(\Gamma)$, $a^* \Rightarrow a^{(k)}$, $\{a^{(k)}\} \Rightarrow V$ and go to (2).

(2) If $PNE(\Gamma)\backslash (U \bigcup V) \neq \varnothing$, then go to (3); if not, then go to (12).

(3) Take $a^* \in PNE(\Gamma)\backslash (U \bigcup V)$ and go to (4).

(4) Set $1 \Rightarrow l$ and go to (5).

(5) If $a^* \prec a^{(l)}$, then go to (6); if not, then go to (7).

(6) Set $a^* \bigcup U \Rightarrow U$ and go to (2).

(7) If $a^{(l)} \prec a^*$, then go to (8); if not, then go to (9).

(8) Set $a^{(l)} \bigcup U \Rightarrow U$, $a^* \Rightarrow a^{(l)}$ and go to (2).

(9) If $l = k$, then go to (11); if not, then go to (10).

(10) Set $l+1 \Rightarrow l$ and go to (5).

(11) Set $k+1 \Rightarrow k$, $a^* \Rightarrow a^{(k)}$, $a^{(k)} \bigcup V \Rightarrow V$ and go to (2).

(12) Output V and end.

Proof: Since $PNE(\Gamma) \neq \varnothing$, there exists an $a^* \in PNE(\Gamma)$. By the step (1), first assign this a^* into V, then either this a^* is kept in V, or a^* is replaced by an element outside V. These new elements in V are repeated finite times as the above. And so $|V| \geq 1$, or $V \neq \varnothing$. By the part "if not" of the step (2), the output V meets $PNE(\Gamma)\backslash(U \bigcup V) = \varnothing$, i.e. $PNE(\Gamma)$ $U \bigcup V$. Since $U \bigcap V = \varnothing$, $U = PNE(\Gamma)\backslash V$. By the steps (4) to (11), V satisfies internal stability. By that "(5) If $a^* \prec a^{(l)}$, then go to (6)", that "(6) Set $a^* \bigcup U \Rightarrow U$ and go to (2)", that "(7) If $a^{(l)} \prec a^*$, then go to (8)" and "(8) Set $a^{(1)} \bigcup U \Rightarrow U$, $a^* \Rightarrow a^{(l)}$", every element in $U = PNE(\Gamma)\backslash V$ uniformly inferior than an element in V. And so V satisfies external stability.

By theorem 4.1, it is obvious that

Theorem 5.3: Any two elements in an *N-M* stable set are either divergent, or indifferent

Theorem 5.4 (existence and uniqueness of N-M stable set) Suppose $PNE(\Gamma) \neq \varnothing$. Then there exists one and only one *N-M* stable set

Proof: By theorem 5.2, the existence has been proved. Now we prove its uniqueness. Let both V and V' are N-M stable sets of a given game and let there exists an $a^{(1)} \in V\backslash V'$. Assume there exists an $a^{(2k-1)} \in V\backslash V'$. By external stability of V', there exists an $a^{(2k)} \in V'$ such that $a^{(2k-1)} \prec a^{(2k)}$. It contradicts internal stability of V that $a^{(2k+1)} \in V$. And so $a^{(2k)} \in V'\backslash V$. By external stability of V, there exists an $a^{(2k+1)} \in V$ such that $a^{(2k)} \prec a^{(2k+1)}$. It contradicts internal stability of V' that $a^{(2k+1)} \in V'$. Therefore $a^{(2k+1)} \in V\backslash V'$. By induction principle, we obtain the two sequences

$$a^{(1)}, a^{(3)}, \ldots, a^{(2k-1)}, \ldots \in V\backslash V' \text{ and } a^{(2)}, a^{(4)}, \ldots, a^{(2k)}, \ldots \in V'\backslash V.$$

Assume there exist $1 \leq k < l$ such that $a^{(2k-1)} = a^{(2l-1)}$ or $a^{(2k)} = a^{(2l)}$. Then by transitivity of the relation $\prec$, we have either

$$a^{(2k-1)} \prec a^{(2k)} \prec a^{(2l-1)} = a^{(2k-1)} \text{ or } a^{(2k)} \prec a^{(2k+1)} \prec a^{(2l)} = a^{(2k)}.$$

They contradict theorem 4.1. Thus $a^{(1)}, a^{(2)}, a^{(3)}, a^{(4)}, \ldots$ are pairwise different. This contradicts that A is a finite set. Hence $V \subseteq V'$. Similarly $V' \subseteq V$. Therefore $V = V'$. This proves uniqueness of the N-M stable set.

6. AN EQUIVALENT RELATION ON THE FINITE GAME CLASS AND IDEAL GAMES

We denote a finite games class by

$$G(A_i | \, i \in N) \equiv \{\Gamma \equiv [N, (A_i), (\prec_i), C] \mid PNE(\Gamma) \neq \varnothing\}.$$

Two games $\Gamma \equiv [N, (A_i), (\prec_i), C]$ and $\Gamma' \equiv [N, (A_i), (\prec'_i), C']$ are equal (written $\Gamma = \Gamma'$) if and only if

$$(1)\ a \prec_i b \Leftrightarrow a \prec'_i b, \forall a, b \in A, \forall i \in N, \text{ and } (2)\ C = C'.$$

On $G(A_i | \, i \in N)$ we introduce a binary relation $R : \Gamma' R \Gamma''$ if and only if

(1) both Γ and Γ'' have the same N-M stable set V, and

(2) $a \prec'_i b \Leftrightarrow a \prec''_i b, \forall \{a, b\} \not\subset V, \forall i \in N,$

By uniqueness of N-M stable set, R is an equivalent relation.

Definition 6.1: If a game $\Gamma \equiv [N, (A_i), (\prec_i), C] \in G\,(A_i| \, i \in N)$ satisfies the condition that $a \sim b$, $\forall\ a, b \in V$, where V is its *N-M* stable set, then Γ is called an ideal game.

Theorem 6.1: Let V be the *N-M* stable set of an ideal game $\Gamma \equiv [N, (A_i), (\prec_i), C]$. Then $a^{**} \prec' a^*$, $\forall\ a^* \in V$, $\forall a^{**} \in PNE(\Gamma)\backslash V$.

Theorem 6.2: Every game in $G\,(A_i| \, i \in N)$ is equivalent to one and only one ideal game.

Proof: For any $\Gamma \equiv [N, A_i), (\prec_i), C] \in G\,(A_i| \, i \in N)$ we define a game $\Gamma' \equiv [N, (A_i), (\prec'_i), C]$, where

$$(1)\ a \sim' b,\ \forall\ a, b \in V \text{ and } (2)\ a \prec'_i b \Leftrightarrow a \prec_i b,\ \forall i \in N,\ \forall\ \{a, b\} \not\subset V.$$

By uniqueness of N-M stable set, we only prove that V is a N-M stable set of Γ'. We first prove that $V \subseteq PNE(\Gamma') \subseteq PNE(\Gamma)$.

For any $a^* \in V$, a^* is a pure Nash equilibrium of the game Γ. For any $i \in N$ and any $a_i^\# \in A_i$, $(a^*_{-i}, a_i^\#) \prec\sim_i (a^*_{-i}, a_i^\#) = a^*$, When $(a^*_{-i}, a_i^\#) \in V$, $(a^*_{-i}, a_i^\#) \sim'_i (a^*_{-i}, a_i^\#) = a^*$. When $(a^*_{-i}, a_i^\#) \in PNE(\Gamma)\backslash V$, there exists an $a^{**} \in V$ such that $(a^*_{-i}, a_i^\#) \prec a^{**}$. By that $\{(a^*_{-i}, a_i^\#), a^{**}\} \not\subset V$, we obtain that $(a^*_{-i}, a_i^\#) \prec' a^{**}$. And by that $a_i^*, a^{**} \in V$, we obtain that $a^* \sim' a^{**}$. Thus $(a^*_{-i}, a_i^\#) \prec' a^*$ and so $(a^*_{-i}, a_i^\#) \prec'_i a^*$. When $(a^*_{-i}, a_i^\#) \in A\backslash PNE(\Gamma)$, it is obvious that either $(a^*_{-i}, a_i^\#) \prec\sim_i a^*$.

Since $\{(a^*_{-i}, a_i^\#), a^*\} \not\subset V$, $(a^*_{-i}, a_i^\#) \prec'\sim'_i a^*$. As a result, we have that

$$(a^*_{-i}, a_i^\#) \prec'\sim'_i (a^*_{-i}, a_i^\#) = a^*,\ \forall a_i^\# \in A_i,\ \forall i \in N.$$

That is $a^* \in PNE(\Gamma')$.

To prove $PNE(\Gamma') \subseteq PNE(\Gamma)$, we sufficiently prove $PNE(\Gamma')\backslash V \subseteq PNE(\Gamma)$ because $V \subseteq PNE(\Gamma)$.Let $a^* \in PNE(\Gamma')\backslash V$ and let $i \in N$ and $a_i \in A_i$ be any element. Then $(a^*_{-i}, a_i) \prec'\sim'_i a^*$. By that $\{a^*, (a^*_{-i}, a_i)\} \not\subset V$, we obtain $(a^*_{-i}, a_i) \prec\sim_i a^*$. Hence $a^* \in PNE(\Gamma)$.

Then, we prove that V is the N-M stable set of the game Γ'. For any $a^*, a^{**} \in V$, we have $a^* \sim' a^{**}$. By theorem 4.1, V satisfies internal stability on $\Gamma' \equiv [N, (A_i), (\prec'_i), C]$. For any $a^* \in V$ and any $a^{**} \in PNE(\Gamma')\backslash V \subseteq PNE(\Gamma)\backslash V$, by external stability of V on Γ, we have $a^{**} \prec a^*$. By that $\{a^*, a^{**}\} \not\subset V$, we obtain that $a^{**} \prec' a^*$. This proves that V satisfies external stability on Γ'. By uniqueness of N-M stable set, V is the N-M stable set of both Γ and Γ'.

Finally we prove the uniqueness. Suppose an ideal game $\Gamma'' \equiv [N, (A_i), (\prec''_i), C]$ is also equivalent to the game Γ. Then we have $\Gamma' R \Gamma''$. It shows that V is the N-M stable set of the ideal games Γ' and Γ'', and we have that

(1) $a \sim' b$, $a \sim''_i b$, $\forall a, b \in V$, $\forall i \in N$, and

(2) $a \prec'_i b \Leftrightarrow a \prec''_i b$, $\forall \{a, b\} \not\subset V$, $i \in N$.

Then for any $a, b \in V$ and $i \in N$, by (1), the two propositions $a \prec'_i b$ and $a \prec''_i b$ are false. This shows that the proposition

$$a \prec'_i b \Leftrightarrow a \prec''_i b,\ \forall a, b \in V,\ \forall i \in N$$

is true. By (2), the proposition $a \prec'_i b \Leftrightarrow a \prec''_i b$, $\forall a, b \in V$, $\forall i \in N$ is true. Hence we obtain that $\Gamma' = \Gamma''$.

7. RATIONAL FINITE GAMES

Principle of Maximal Entropy (PME): To make clear the others' selections, amount of information a player needs is maximal.

Definition 7.1: A finite game $\Gamma \equiv [N, (A_i), (\prec_i), C]$ that satisfies the condition $C = \{N, (A), (\prec_i), PME\}$ is called a rational finite game.

By information theory[30], we have

Lemma: An equal probability distribution has the maximal entropy. And conversely, a probability distribution of a discrete random variable with the maximal entropy is the equal probability distribution.

Theorem 7.1: A rational finite game can be idealized.

Proof: By theorem 5.1, there is no pure Nash equilibrium that is superior than one in the N-M stable set. This shows that it is all players' common knowledge that all the players prefer a pure Nash equilibrium in the N-M stable set. Further, every player hopes that the average utility of pure Nash equilibria in the N-M stable set is maximal. Since no information what pure Nash equilibrium will occur, by the maximal entropy principle and lemma, we have the equality

$$\max\{u_i(a^*)\,|\,a^* \in V\} = \frac{1}{|V|}\sum_{a^* \in V} u_i(a^*), \forall i \in N.$$

For this, we must have that $u_i\, a^* = u_i\, (a^{**})$, $\forall a^*$, $a^{**} \in V$, $\forall i \in N$. Therefore every player should regard all the pure Nash equilibria in the N-M stable set as indifferent. As a result, all the players should regard the preference relation as

$$a^* \sim a^{**}, \forall a^*, a^{**} \in V \text{ and } a \prec'_i b \Leftrightarrow a \prec_i b, \forall \{a, b\} \not\subset V\, A\backslash V, \forall_i \in \mathrm{N}.$$

Example 7.1: The idealization of the game

$$\begin{bmatrix} (\underline{2},\underline{3}) & (2,2) & (\underline{1},2) \\ (\underline{2},\underline{2}) & (\underline{3},\underline{2}) & (\underline{1},\underline{2}) \\ (\underline{2},0) & (2,0) & (\underline{1},\underline{1}) \end{bmatrix}$$

is

$$\begin{bmatrix} (\underline{3},\underline{3}) & (2,2) & (\underline{1},2) \\ (\underline{3},\underline{3}) & (\underline{3},\underline{3}) & (\underline{1},2) \\ (2,0) & (2,0) & (\underline{1},\underline{1}) \end{bmatrix}.$$

And their common N-M stable set is $V = \{(1,1), (2,1), (2, 2)\}$.

Example 7.2[29]: (Battle of the Sexes) A married couple is trying to decide where to go for a night out. She would kike to go to the theatre, and he would like to go to a football match——they have been married a few months! However, they are still very much in love and so they only enjoy the entertainment if their partner is with them. What is their best selection?

The payoff matrix can be written as

$$\begin{array}{llcc} & & \multicolumn{2}{c}{\textit{woman}} \\ & & \textit{football} & \textit{theatre} \\ \textit{man} & \textit{football} & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} (\underline{2},\underline{1}) & (0,0) \\ (0,0) & (\underline{1},\underline{2}) \end{bmatrix}$}} \\ & \textit{theatre} & & \end{array},$$

and its idealization is

$$\begin{array}{llcc} & & \multicolumn{2}{c}{\textit{woman}} \\ & & \textit{football} & \textit{theatre} \\ \textit{man} & \textit{football} & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} (\underline{2},\underline{2}) & (0,0) \\ (0,0) & (\underline{2},\underline{2}) \end{bmatrix}$}} \\ & \textit{theatre} & & \end{array}.$$

The common N-M stable set is V = {(football, football), (theatre, theatre)}. This can be explained as the follows: In general case, they are divergent for go to football together or go to theatre together. But if the maximal entropy principle is their common knowledge, then they should regard them as the same.

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