*Global and Stochastic Analysis*
*Vol. 11 No. 2 (March, 2024)*

*Received: 15th January 2024* *Revised: 27th February 2024* *Accepted: 09th March 2024*

# ON RADIUS DOMINATION IN GRAPHS

M. NANJUNDASWAMY, S. R. NAYAKA, PUTTASWAMY*, P. SIVA KOTA REDDY*, AND S. RAJU

ABSTRACT. A dominating set $S$ in a graph $G$ is called a radius dominating set if for each vertex $v \in V - S$ there is at least one vertex in $u \in S$ such that $d(u, v) = rad(G)$. The cardinality of the minimum radius dominating set is called the radius domination number of $G$, denoted by $\gamma_{rd}(G)$. In this article, we study the properties of this parameter. The radius domination number of some families of standard graphs are obtained and the bounds are estimated.

## 1. Introduction

Let $G$ be a connected simple graph of order $n$ and $u, v$ be any two vertices in $G$. The distance between $u$ and $v$ is the length of the shortest path between $u$ and $v$, denoted by $d(u, v)$. The distance of the farthest vertex from $v$ is called the eccentricity of the vertex $v$, denoted by $e(v)$. That is $e(v) = max\{d(u, v)|u \in V(G)\}$. The minimum and maximum eccentricity taken over all the vertices are called the *radius* and *diameter* of $G$, respectively. The radius and diameter of $G$ are denoted by $rad(G)$ and $diam(G)$), respectively.

For any connected graph, it is obvious that $rad(G) \leq diam(G) \leq 2rad(G)$. A vertex $v$ is called central vertex if $e(v) = r(G)$ and peripheral vertex if $e(v) = diam(G)$. For each vertex $v \in V$, the open neighborhood of $v$ is the set $N(v)$ containing all the vertices adjacent to $v$ and the closed neighborhood of $v$ is the set $N[v]$ containing $v$ and all the vertices adjacent to $v$. For a subset $S$ of $V$, the open neighborhood of $S$ is the set $N(S) = \cup_{v \in S} N(v)$ and the closed neighborhood of $S$ is the set $N[S] = N(S) \cup S$.

The concept of domination [1, 3] was one of the most familiar concept in graph theory that has attracted many mathematicians since it was introduced. The concept of eccentric domination were introduced by T. N. Janakiraman et al. [2].

**Definition 1.1.** A subset $S$ of vertices of a graph $G$ is called a dominating set if every vertex in $V - S$ has a neighbor in $S$. The cardinality of a minimum dominating set is called the domination number of $G$, denoted by $\gamma(G)$.

**Definition 1.2.** A dominating set $S$ is said to be eccentric dominating set if for every vertex $v$ in $V - S$, there exists at least one eccentric point vertex in $S$.

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2000 *Mathematics Subject Classification.* 05C69.
*Key words and phrases.* Dominating set, Radius dominating set, Radius domination number.
*Corresponding authors.

The cardinality of the minimum eccentric dominating set is called the eccentric domination number, denoted by $\gamma_{ed}(G)$.

## 2. Radius Domination

In this section, we shall define and discuss a new type domination, namely, radius domination. Throughout this section $G$ denotes a connected simple graph.

**Definition 2.1.** A subset $S$ of vertices in $G$ is a radius dominating set if $S$ is a dominating set and for each vertex $v$ in $V - S$, there exists a vertex $u$ in $S$ such that $d(u, v) = rad(G)$.

From the above definition, it is clear that every graph possesses a radius dominating set and further, super-set of a radius dominating set is also a radius dominating set whereas a subset of a radius dominating set may not be a radius dominating set.

**Definition 2.2.** If $S$ is a radius dominating set in $G$ such that none of its subset is a radius dominating set, then $S$ is called a minimal radius dominating set.

**Definition 2.3.** For a given graph $G$, the radius domination number $\gamma_{rd}$ is the order of the smallest ordered radius dominating set in $G$. That is

$$\gamma_{rd}(G) = \min\{|S| : S \text{ is a radius dominating set in } G\}.$$

**Definition 2.4.** For any vertex $v \in V(G)$, any vertex in $G$ at a distance $rad(G)$ from $v$ called radius vertex of $v$. The set of all radius vertices of $v$ is called the radii of $v$, denoted by $Rad(v)$. That is, $Rad(v) = \{u \in V(G) | d(u, v) = rad(G)\}$.

**Definition 2.5.** A subset $S \subseteq G$ is called a radius point set of $G$ for each vertex in $V - S$, there is at least one vertex $u$ such that $d(u, v) = rad(G)$. The cardinality of the minimum radius point set is called radius number of a graph and it is denoted by $r_s(G)$.

*Observation* 2.6. From the definition of radius domination and radius point set, it follows that union of a minimum dominating set and the minimum radius point set is always a radius dominating set. That is, $\gamma_{rd}(G) \leq \gamma(G) + r_s(G)$.

*Observation* 2.7. If $G$ is a totally disconnected graph, then $\gamma(G) = \gamma_{rd}(G)$ since each vertex has same radius.

We characterize a minimal radius dominating set of a graph in the following result.

**Theorem 2.8.** *For a graph $G$, a dominating set $S$ is a minimal radius dominating set if and only if for each vertex $v \in S$, one of the following condition is true:*

(1) *$v$ is an isolated vertex of $S$ or it has no vertex at distance $rad(G)$ in $S$.*
(2) *There exists a vertex $u$ in $V - S$ such that $N(u) \cap S = \{v\}$.*
(3) *There exists a vertex $u$ in $V - S$ such that $Rad(u) \cap S = \{v\}$.*

*Proof.* Let $S$ be any minimal radius dominating set in a connected graph $G$. From the definition of minimal set it follows that for any vertex $v \in S$, the set $S - \{v\}$ will not be a radius dominating set which implies that either there is a vertex $u$

in $V(G)$ not dominated by $S$ or having no radius vertex in $S$. Clearly, $u$ must be in $(V - S) \cup \{v\}$.

Now we have two possibilities here: Suppose $u = v$. Then without loss of generality, $v$ will be either an isolated vertex of $S$ or has no radius vertex in $S$. Let us assume, $u \neq v$. Then $u \in V - S$. Suppose $u$ is not dominated by $S - \{v\}$, then it follows that $v$ is the only neighbor of $u$ in $S$ and hence $N(u) \cap S = \{v\}$. Similar argument for radius vertex proves that $Rad(u) \cap S = \{v\}$.

Conversely, we shall on contrary assume that a subset $S$ of vertices satisfies above conditions but not a minimal radius dominating set in $G$. Since it is not minimal there is at least one vertex in $S$ whose removal do not effect the property of $S$. Let $v$ be such vertex in $S$ so that $S - \{v\}$ is also a radius dominating set in $G$. Hence $v$ is adjacent to at least one vertex in $S - \{v\}$ and has a radius vertex in $V - S$. Therefore, condition $(i)$ does not hold.

Now, as $S - \{v\}$ is a radius dominating set, each vertex $w$ not in $S - \{v\}$ will be dominated by $S - \{v\}$. In particular $v$ has a neighbor in $S$ other than itself proving that condition $(ii)$ does not hold.

Similar argument proves that $S$ contains a radius vertex of $v$ other than itself, showing that condition $(iii)$ does not hold. This completes the proof. □

**Theorem 2.9.** *We have*

(1) $\gamma_{rd}(K_n) = 1$.
(2) $\gamma_{rd}(K_{1,n}) = 1$.
(3) $\gamma_{rd}(S_{n,m}) = 2$.
(4) $\gamma_{rd}(K_{m,n}) = 2$.

*Proof.* (1) Let $G \cong K_n$ be a complete graph of order $n$. Then each vertex will have eccentricity one and so any single vertex can dominate the graph. Hence $\gamma_{rd}(K_n) = 1$.

(2) Let $G \cong K_{1,n}$ be a star graph. Vertex at the center will have least eccentricity and suffices to dominate entire vertex set. Thus $\gamma_{rd}(G) = 1$.

(3) Let $S_{n,m}$ be a double star of order $m + n + 2$. Let $D = \{u, v\}$, where $u, v$ are the vertices of degree at least two. Each vertex of $S_{n,m}$ except $u, v$ will have eccentricity 3 and $e(u) = e(v) = 2$. Since $u, v$ are at a distance $rad(G)$ from other vertices and also adjacent to all other vertices, we have $\gamma_{rd}(G) = 2$.

(4) Let $G$ be a complete bi-partite graph with partite sets $V_1, V_2$. From the construction of bipartite graph each vertex of the graph will have same eccentricity and so any dominating set suffices to be a radius dominating set of $G$. Therefore $\gamma_{rd}(G) = 2$.

□

*Remark* 2.10. Let $P_n$ be a path of order $n$. Then we have the followings:

(1) If $n$ is an even integer, then $rad(P_n) = \frac{n}{2}$ and there are two radius vertices. Any minimum dominating set contains at least one of the radius vertex and a leaf.
(2) If $n$ is an odd integer, then $rad(P_n) = \frac{n-1}{2}$ and there is only one radius vertex in $P_n$. Further, any minimum radius dominating set contains the radius vertex and two leaves.

**Theorem 2.11.** *Let $P_n$ be a path of order $n$. Then $\gamma_{rd}(P_n) = rad(P_n)$.*

*Proof.* Let $P_n$ be a path on $n \geq 2$ vertices and let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. First, let us assume that $n$ is odd. Then $rad(G) = \frac{n-1}{2}$ and there is only one vertex of with eccentricity $\frac{n-1}{2}$ and the vertex is obviously $v_{\frac{n+1}{2}}$, which will be a radius vertex of $v_1$ and $v_n$. Now $\{V_1, V_2\}$ will be a partition of $V = V(P_n)$, where $V_1 = \{v_1, v_2, \ldots v_{\frac{n-1}{2}}\}$ and $V_2 = \{v_{\frac{n+3}{2}}, v_{\frac{n+5}{2}}, \ldots, v_{n-1}, v_n\}$ each of order $\frac{n-1}{2}$.

Now we shall choose minimum dominating set choosing radius vertices from partite sets. Without loss of generality, we can observe that the radius vertex of $v_i \in V_1$ will be $v_{\frac{n+2i-1}{2}} \in V_2$. Thus for each vertex in $V_1$ we shall choose a vertex or its radius vertex for a radius dominating set. For minimality, we shall choose vertices alternatively, otherwise which leads to move out of domination. Hence we must select vertices from both $V_1$ and $V_2$. There are again two possibilities here:

Suppose $\frac{n-1}{2}$ is odd. Then we must select minimum of $\frac{n-3}{4}$ vertices from each set along with the central vertex of $P_n$. Hence $\gamma_{rd}(P_n) = 2\left(\frac{n-3}{4}\right) + 1$. That is $\gamma_{rd}(G) = rad(G)$. Next, suppose that $\frac{n-1}{2}$ is an even integer. In such case, select $\frac{n-5}{4}$ vertices from one set and $\frac{n+3}{4}$ from the other set. Therefore, $\gamma_{rd}(P_n) = \frac{n-5}{4} + \frac{n-1}{2} = \frac{n-1}{2}$. Thus, $\gamma_{rd}(P_n) = rad(P_n)$.

Suppose $n$ is an even integer, then $rad(P_n) = \frac{n}{2}$ and there are two radius vertices. Further, each vertex $v_i$ will be a radius vertex of $v_{n-i}$. Thus there must be at least $\frac{n}{2}$ vertices in any radius dominating set. On the other hand, any subset of alternating vertices will be a radius dominating set having at most $\frac{n}{2}$ vertices. Therefore, $\gamma_{rd}(P_n) = \frac{n}{2}$. □

**Theorem 2.12.** *If $G \cong H_n$ is a helm graph with $n \geq 3$, then $\gamma_{rd}(G) = n + 1$.*

*Proof.* Let $G \cong H_n$ be a helm graph of order $2n+1$ with $n \geq 3$. Then $rad(G) = 2$ and for $n > 3$, the singleton set containing central vertex forms a radii of all pendant vertices and further each of the pendant vertex will be radius vertex of a vertex adjacent to its support vertex. Thus, set of support vertices or leaves along with the central vertex forms a radius dominating set of least cardinality. Therefore, $\gamma_{rd}(G) = n + 1$. □

**Theorem 2.13.** *Let $G_1$ and $G_2$ be two connected graphs. We have*

(1) $\gamma_{rd}(G_1 \cup G_2) = \gamma_{rd}(G_1) + \gamma_{rd}(G_2)$.
(2) $\gamma_{rd}(G_1 \vee G_2) = \begin{cases} 1, & \text{if } \gamma_{rd}(G_1) = 1 \text{ or } \gamma_{rd}(G_2) = 1; \\ 2, & \text{otherwise.} \end{cases}$.

*Proof.* Let $G_1$ and $G_2$ be two connected graphs. Proof of (1) is trivial. From the definition of join of graphs, it follows that every pair of vertices from $G_1$ and $G_2$ are adjacent. Thus $rad(G_1 \vee G_2) = 2$. Since $G_1$ or $G_2$ contains a dominating vertex then this vertex will be a dominating vertex of $G_1 \vee G_2$ and so $\gamma_{rd}(G_1 \vee G_2) = 1$.

On the other hand, if none of $G_1$ and $G_2$ contains a dominating vertex, then clearly a pair of vertices from $V(G_1)$ and $V(G_2)$ will be a minimum radius dominating set and so $\gamma_{rd}(G_1 \vee G_2) = 2$. □

**Corollary 2.14.** *For the complete bipartite graph $K_{m,n}$ with $m, n \geq 2$, $\gamma_{rd}(K_{m,n}) = 2$. In particular, $\gamma_{rd}(K_{m_1,m_2,\ldots,m_k}) = 2$, if $m_i \geq 2$ for each $i$.*

As a consequence of Theorem 2.13, if $G$ is a wheel or a Dutch-windmill graph, then $\gamma_{rd}(G) = 1$.

**Theorem 2.15.** *For a graph $G \cong P_2 \times P_n$,*

$$\gamma_{rd}(G) = \begin{cases} n & \text{if } n, \text{ is even;} \\ n-1, & \text{if } n \text{ is odd.} \end{cases}$$

*Proof.* Let $G \cong P_2 \times P_n$, a graph of order $2n$. Suppose $n$ is even. Then $rad(G) = \frac{n+2}{2}$ and there will be four radius vertices in $G$. Each vertex in $G$ can dominate maximum of three vertices and will be a radius vertex for at least two vertices. For minimality let us fix a vertex $v$ which is a end vertex of a path and choose minimum dominating set. Choose alternate vertices of other copy of $P_n$ each of which is a radius vertex of two other vertices. Thus, we must choose at least $n-1$ vertices along with $v$ and so $\gamma_{rd}(P_n) = n$.

Suppose $n$ is odd. Then $rad(G) = \frac{n+1}{2}$ and there will be two radius vertices of $G$. Let us choose a vertex $v$ of degree 2 in $G$ and choose vertices from other copy of $P_n$. Since each vertex will have two radius vertices, selection of alternating vertices gives a minimum radius dominating set having $n-2$ vertices along with $v$. Therefore, $\gamma_{rd}(G) = n-1$. □

**Theorem 2.16.** *For $G \cong P_3 \times P_n$, $\gamma_{rd}(G) = n$.*

*Proof.* Let $G \cong P_3 \times P_n$, a graph of order $3n$. Since it contains three copies of path $P_n$, the middle copy of $P_n$ is sufficient to dominate entire graph $G$ and that vertices themselves will be radius vertices of all other vertices. Since it forms a minimum dominating set, we can conclude that $\gamma_{rd}(G) = n$. □

**Definition 2.17.** A firefly graph is a graph on $n$ vertices having $s$ triangles, $t$ pendant paths of length 2 and $n-2s-2t-1$ pendant edges sharing a common vertex. It is denoted by $F_{s,t,n-2s-2t-1}$.

**Theorem 2.18.** *For $G \cong F_{s,t,n-2s-2t-1}$, a firefly graph of order $n$, we have*

$$\gamma_{rd}(G) = n - 2s - 2t.$$

*Proof.* Let $G \cong F_{s,t,n-2s-2t-1}$, a firefly graph of order $n$ having $n-2s-2t-1$ pendant paths. Then $rad(G) = 1$ and $diam(G) = 3$. Since the central vertex form radii of all pendant edges and triangles including support vertices of pendant edges, the minimum dominating set of pendant paths containing support vertices along with this central vertex leads to minimum radius dominating set of $G$. As all the pendant paths are independent, it follows that $\gamma_{rd}(G) = n - 2s - 2t$. □

The following results are immediate from the definition of radius domination:

**Proposition 2.19.** *For any connected graph $G$ of order $n$, $1 \leq \gamma_{rd}(G) \leq n$. Equality holds if $\Delta(G) = 0$ or $n - 1$.*

**Proposition 2.20.** *For any connected graph $G$ of order $n$, $1 \leq \gamma(G) \leq \gamma_{rd}(G)$.*

*Remark* 2.21. We can not compare eccentric domination number and radius domination number by an inequality. For instance, for a star graph, we have $\gamma_{rd} = 1$ and $\gamma_{ed} = 2$ resulting $\gamma_{rd} < \gamma_{ed}$; but for a path $P_{16}$, we have $\gamma_{rd} = 8$ and $\gamma_{ed} = 6$ resulting $\gamma_{ed} < \gamma_{rd}$.

**Acknowledgment.** We would like to express our gratitude to anonymous reviewers for their valuable comments.

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M. Nanjundaswamy: Department of Mathematics, Government First Grade College for Women, Byrapura, T. Narasipura-571 124, INDIA & Department of Mathematics, PET Research Foundation (Affiliated to University of Mysore), P.E.S College of Engineering, Mandya-571 401, INDIA.
*Email address*: mnswamy1974@gmail.com

S. R. Nayaka: Department of Mathematics, PET Research Foundation (Affiliated to University of Mysore), P.E.S College of Engineering, Mandya-571 401, INDIA.
*Email address*: nayaka.abhi11@gmail.com

Puttaswamy: Department of Mathematics, PET Research Foundation (Affiliated to University of Mysore), P.E.S College of Engineering, Mandya-571 401, INDIA.
*Email address*: prof.puttaswamy@gmail.com

P. Siva Kota Reddy: Department of Mathematics, JSS Science and Technology University, Mysuru-570 006, Karnataka, INDIA.
*Email address*: pskreddy@jssstuniv.in; pskreddy@sjce.ac.in

S. Raju: Department of Mathematics, PET Research Foundation (Affiliated to University of Mysore), P.E.S College of Engineering, Mandya-571 401, INDIA.
*Email address*: rajus24@gmail.com