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ON SMOOTH LIE ALGEBRA BUNDLES OF FINITE TYPE

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Abstract. In this paper, we study smooth Lie algebra bundles of finite Type. We discuss tangent bundle and Lie algebra bundle induced by Lie group bundle of finite type. Finite type property of top space bundles and RMS bundle for are also examined.

1. Introduction

For the notions and basic terminologies of fiber bundles and vector bundles, we follow [2], [3] and [6]. When a vector bundle $\zeta = (\zeta, p, X)$ attains a morphism $* : \zeta \times \zeta \to \zeta$, where $*$ induces a Lie algebra structure on every fiber ζ_x, we call ζ as a weak Lie algebra bundle. In the way of achieving a correspondence between, finitely generated projective modules over the ring of continuous real valued functions on the base space and algebraic vector bundle over an affine variety (See [4, 10, 11]), L. N. Vaserstein properly defined vector bundles of finite type in 1986 [12]. Later Ranjitha Kumar et al. [8] defined Lie algebra bundles of finite type as follows: A Lie algebra bundle ζ is said to be of finite type if there exists a finite partition of unity A on X (that is, A is a finite set of non-negative continuous functions on X whose sum equals 1) such that ζ restricted to the set $\{x \in X \,|\, \alpha(x) \neq 0\}$ is a trivial Lie algebra bundle for each $\alpha \in A$. Any Lie algebra bundle over a compact space is a trivial example of a Lie algebra bundle of finite type. Here we consider that all underlying vector spaces are real and finite dimensional.

2. Smooth Lie Algebra Bundles of Finite Type

We begin this section by following definitions:

Definition 2.1. A vector bundle $\zeta = (\zeta, p, M)$ where each fiber is a Lie algebra is called a locally trivial smooth Lie algebra bundle, if it is a smooth vector bundle in which for every x in M we have a diffeomorphism $\Phi : N \times L \to p^{-1}(N)$, where $x \in N$ is an open set in M, L is a Lie algebra and corresponding map $\Phi_m : \{m\} \times L \to p^{-1}(m)$ is an isomorphism of Lie algebras, $\forall m \in N$.

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Definition 2.2. Consider ζ, η, smooth Lie algebra bundles over the same base space M. A map $f : \zeta \to \eta$ which is a smooth morphism between vector bundles ξ and η where for each m in M $f_m : \zeta_m \to \eta_m$ is a Lie algebra homomorphism, will be called as a smooth Lie algebra bundle morphism.

Definition 2.3. If $p : \zeta \to X$ is a Lie algebra bundle, then $s : X \to \zeta$ where $p \circ s = id_M$ is called a section on ζ and we denote the set of all sections by $\Gamma(\zeta)$.

Following the methods of [7, Existence theorem], one can get the existence of a smooth Lie algebra bundles of finite type. By eliminating compactness condition, we are able to extract some conclusions for sections of a Lie algebra bundle of finite type using the results for sections of a vector bundle in [11]. Also, we can find a necessary and sufficient condition for a section to be smooth.

Lemma 2.4. *Suppose X is any arbitrary topological space and U is a neighbourhood of x in X. If s is section of a Lie algebra bundle ξ of finite type over X, then there exists a section s' of ξ over X such that $s' = s$ in some neighbourhood of x.*

Proof. Let $\{f_1, f_2, \dots, f_n\}$ be a finite partition of unity on X. We have s is a section of ξ over U in X. Define $s'(a) = \left(\sum_{i=1}^{n} f_i(a)\right) s(a)$ if $a \in U$ and $s'(a) = 0$ if $a \notin U$, which will satisfy the requirements. □

Corollary 2.5. [11, Corollary 1] *Let ξ be a finite type Lie algebra bundle over X. For any $x \in X$, there are elements $s_1, s_2, ..., s_k \in \Gamma(\xi)$ which form a local base at x.*

Proof. Let $\{f_1, f_2, ..., f_n\}$ be a partition of unity and $U_i = \{y \in X \mid f_i(y) \neq 0\}$. We have $\xi|_{U_i}$ is trivial. Since $\{U_i\}_{i=1}^{n}$ covers X, for any $x \in X$ let $x \in U_1, U_2, \dots U_k$. Then by above lemma, we have $s_1, s_2, \dots s_k \in \Gamma(\xi)$ such that $s = s_i, i = 1, \dots k$. Then $\{s_1, s_2, \dots s_k\}$ span s at x and linear independency follows from local triviality of the bundle over U_i's. □

Corollary 2.6. *Let ξ and η be two Lie algebra bundles of finite type over X. If $f, g : \zeta \to \eta$ and $\Gamma(f) = \Gamma(g) : \Gamma(\zeta) \to \Gamma(\eta)$, then $f = g$.*

Proof. Proof follows from Lemma 2.4 and Corollary 2 of [11]. □

Proposition 2.7. *Let $\pi : \zeta \to M$ be a Lie algebra bundle of finite type over a smooth manifold M. Then the sections $s_1, \dots, s_k$ which form a local base at x for any $x \in M$ are smooth.*

Proof. Since ζ is a Lie algebra bundle of finite type, there exists finite partition of unity $\{f_i\}_{i=1}^{n}$ such that $\zeta|_{U_i}$ is trivial, where $U_i = \{x \in M/f_i(x) \neq 0\}$. Let $\phi_i : U_i \times L_i \to \pi^{-1}(U_i) = \zeta|_{U_i}$ be the the isomorphism. If $x \in M$, $x \in U_i$ for some i. Then $s_1, \dots, s_k$ forms a local base at x. We observe that each section $s_j : U_i \to \zeta|_{U_i}$, $s_j(x) \mapsto \phi_i(x, L_i)$ is smooth (since ϕ_i is a trivialization). □

Remark 2.8. A bundle section $s : M \to \zeta$ is smooth, where ζ is of finite type, if and only if for every open subset U of M and a k-tuple of smooth sections $s_1, ..., s_k : U \to \zeta|_U$ such that $\{s_i(x)\}_i$ is a basis at x, for all $x \in U$, the coefficient functions, $c_1, c_2, ..., c_k : U \to \mathbb{R}$, $(s(x) = c_1(x)s_1(x) + ... + c_k(x)s_k(x), \forall x \in U)$ are smooth.

Let us denote the bundle of tangent vectors in a smooth manifold M by TM. Then $TM = \bigcup_{x\in M} T_xM$, where T_xM is the space of tangents to M at the point x.

Theorem 2.9. *Let $\zeta = (TM, \pi, M)$ be a tangent bundle of finite type over a smooth manifold M. Then there exists a smooth weak Lie algebra bundle.*

Proof. By [8], we know that all smooth sections $\Gamma(\zeta)$ of ζ can be identified as a finitely generated projective module over $C^\infty(M)$. Define a Lie algebra on $\Gamma(\zeta)$ over $C^\infty(M)$ by $[.,.] : \Gamma(\zeta) \times \Gamma(\zeta) \to \Gamma(\zeta), (T_1, T_2) \mapsto [T_1, T_2]$, where $[T_1, T_2] = T_2T_1 - T_1T_2$. Then the result follows from Theorem 4.1 in [7]. □

A Lie group bundle $\pi : \mathcal{G} \to M$ is a smooth bundle in which each fibre $\mathcal{G}_m = \pi^{-1}(m)$ is having a Lie group structure and there exists an atlas which gives an isomorphism between $\mathcal{G}$ and the fiber type F. Let $\mathcal{L}(\mathcal{G})$ denote the tangent bundle, where the fibers are the tangent space at the identity element e_m of each fibre $\mathcal{G}_m$. That is, $\mathcal{L}(\mathcal{G}) = \bigcup_{m\in M} T_{e_m}(\mathcal{G}_m)$. If $\mathcal{G}$ is locally trivial, $\mathcal{L}(\mathcal{G})$ is also locally trivial, called as the Lie algebra bundle of the Lie group bundle $\mathcal{G}$.

Proposition 2.10. *If $\mathcal{G}$ is a Lie group bundle of finite type, then $\mathcal{L}(\mathcal{G})$ is a Lie algebra bundle of finite type.*

Proof. Since $\mathcal{G}$ is a Lie group bundle of finite type, there exists a finite partition of unity $\{f_1, f_2, ..., f_n\}$ where the bundle $\mathcal{G}$ restricted to $U_i = \{x \in M/f_i(x) \neq 0\}$ is trivial. Let $\phi_i : U_i \times G \to \mathcal{G}|_{U_i}$ be the isomorphism. Then

$$d\phi_i : U_i \times T_eG \to \bigcup_{m\in U_i} T_{e_m}(\mathcal{G}_m)$$

is also an isomorphism, where T_eG is a Lie algebra and is the tangent space of G at the identity element e. □

Definition 2.11 (Top space [1])**.** A non-empty smooth d-dimensional manifold $\mathfrak{I}$ is called a top space if there exists an action " . " on $\mathfrak{I}$ such that $p.q = pq \in \mathfrak{I}$, for every $p, q \in \mathfrak{I}$ and satisfies the following conditions:

1. $(wp)q = w(pq)$, $\forall w, p, q \in \mathfrak{I}$.
2. For each $p \in \mathfrak{I}$, there is a unique $e(p) \in \mathfrak{I}$ such that $pe(p) = e(p)p = p$.
3. $\forall p, q \in \mathfrak{I}$, $e(pq) = e(p)e(q)$.
4. For each $p \in \mathfrak{I}$, there exists an $q \in \mathfrak{I}$ such that $pq = qp = e(p)$, we denote q by p^{-1}.
5. The mappings, $m_1 : \mathfrak{I} \times \mathfrak{I} \to \mathfrak{I}$, $(p, q) \mapsto pq$ and $m_2 : \mathfrak{I} \to \mathfrak{I}$, $p \mapsto p^{-1}$ are smooth.

Definition 2.12. A smooth fibre bundle $\mathcal{T} = (\mathcal{T}, \pi, M)$ having a top space $\mathfrak{I}$ as standard fibre, is a top space bundle if there exists a smooth morphism $\mathcal{T} \oplus \mathcal{T} \to \mathcal{T}$ which induces a top space structure on each fibre.

Definition 2.13. Let Ł, $\mathcal{I}$ be two topological bundles. let $\mathcal{G}$ be a topological group bundle and $P : Ł \oplus \mathcal{I} \to \mathcal{G}$ be a bundle morphism. The corresponding Rees matrix Semigroup (RMS) bundle, $M(\mathcal{G}, \mathcal{I}, Ł, P)$ is the topological bundle $\mathcal{I} \oplus \mathcal{G} \oplus Ł$ with a morphism

$$\odot : (\mathcal{I} \oplus \mathcal{G} \oplus Ł) \oplus (\mathcal{I} \oplus \mathcal{G} \oplus Ł) \to (\mathcal{I} \oplus \mathcal{G} \oplus Ł)$$

defined by

$$(i, a, \lambda).(j, b, \mu) = (i, aP(\lambda, j)b, \mu).$$

Theorem 2.14. *Let $Ł$, $\mathcal{I}$ be smooth bundles of finite type over a manifold M. Let $\mathcal{G}$ be a Lie group bundle of finite type over M and $P : Ł \oplus \mathcal{I} \to \mathcal{G}$ be a smooth bundle morphism. then the RMS bundle, $M(\mathcal{G}, \mathcal{I}, Ł, P)$ is a top space bundle of finite type.*

Proof. Since $P_m : Ł_m \times \mathcal{I}_m \to \mathcal{G}_m$ is smooth for each $m \in M$, we observe that $\mathcal{T} = M(\mathcal{G}, \mathcal{I}, Ł, P)$ is a top space bundle. Without loss of generality, we can assume that $\{f_i\}_{i=1}^n$, $\{g_i\}_{i=1}^n$ and $\{h_i\}_{i=1}^n$ are partitions of unity such that $Ł|_{U_i}$, $\mathcal{I}|_{V_i}$, $\mathcal{G}|_{Z_i}$ are trivial, with standard fibres Λ_i, I_i and G_i respectively, where $U_i = \{x \in X/f_i(x) \neq 0\}$, $V_i = \{x \in X/g_i(x) \neq 0\}$ and $Z_i = \{x \in X/h_i(x) \neq 0\}$. Let $\phi_{U_i} : U_i \times \Lambda_i \to Ł|_{U_i}$, $\phi_{V_i} : V_i \times I_i \to \mathcal{I}|_{V_i}$, $\phi_{Z_i} : Z_i \times G_i \to \mathcal{G}|_{Z_i}$ be the isomorphisms. Now we shall choose $W_i = U_i \cap V_i \cap Z_i$. Define

$$\phi_i : W_i \times (I_i \times G_i \times \Lambda_i) \to \bigcup_{m \in W_i} \mathcal{I}_m \times \mathcal{G}_m \times Ł_m$$

by

$$\phi_i(m, a, g, \lambda) = (\phi_{V_i}(m, a), \phi_{Z_i}(m, g), \phi_{U_i}(m, \lambda)),$$

which is an isomorphism (See Theorem 3.1 in [1]). □

Theorem 2.15. *Let $\mathcal{T}$ be a top space bundle over M of finite type with $|e(\mathcal{T}_m)| < \infty$ for each $m \in M$. Define*

$$\mathcal{L}(\mathcal{T}) = \bigcup_{m \in M} \mathcal{L}(\mathcal{T}_m),$$

where $\mathcal{L}(\mathcal{T}_m)$ is the Lie algebra of left invariant vector fields on $\mathcal{T}_m$. Then $\mathcal{L}(\mathcal{T})$ is a Lie algebra bundle of finite type. (We call $\mathcal{L}(\mathcal{T})$, the Lie algebra bundle of the top space bundle $\mathcal{T}$).

Proof. Let $\{f_i\}_{i=1}^n$ be a partition of unity such that $\xi|_{U_i}$ is trivial, where $U_i = \{x \in X/f_i(x) \neq 0\}$. Let $\phi_i : U_i \times T_i \to \bigcup_{m \in U_i} \mathcal{T}_m$ be the isomorphism, where T_i is a Lie algebra. Then we observe that $\phi_{i\mathcal{L}} : U_i \times \mathcal{L}(T_i) \to \bigcup_{m \in U_i} \mathcal{L}(\mathcal{T}_m)$ is also an isomorphism. This proves the result. □

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References

1. Ajaykumar, K. and Kiranagi, B. S.: Lie algebra bundles and generalization of Lie group bundles, *Proc. Jangjeon Math. Soc.*, **22**(4) (2019), 529–542.
2. Atiyah, M. F.: *K- Theory*, W.A. Benjamin, Inc., New York, 1967.
3. Douady, A. and Lazard, M.: Espaces fibrès en algebrès de Lie et en groupes, *Invent. Math.* **1** (1966), 133–151.
4. Goodearl, K. R.: Cancellation of low-rank vector bundles, *Pacific J. Math.* **113** (1984), 289–302.
5. Gundogan, H.: Lie algebras of smooth sections, Preprint, 2007.
6. Husemoller, D.: *Fibre Bundles*, Springer-Verlag New York, Heidelberg Berlin, 1994.
7. Kiranagi, B. S., Madhu, B. and Ajaykumar, K.: On smooth Lie algebra bundles, *Int. J. Algebra*, **11**(5) (2017), 247–254.
8. Kumar, Ranjitha, Prema, G. Kiranagi, B. S.: Lie algebra bundles of finite type, *Acta Univ. Apulensis Math. Inform.*, **39** (2014), 151–160.
9. Mackenzie, K. C. H.: *General theory of Lie groupoids and Lie algebroids*, Cambridge University Press, 2005.
10. Serre, J. P.: Faiseaux Algebriques Coherent, *Ann. of Math.*, **6**(2) (1955), 197–278.
11. Swan, Richard G.: Vector bundles and projective modules, *Trans. Amer. Math. Soc.*, **105** (1962), 264–277.
12. Vaserstein, Leonid N.: Vector bundles and projective modules, *Trans. Amer. Math. Soc.*, **294**(2) (1986), 749–755.
13. Vasilescu, F.-H.: Normed Lie algebras, *Canadian J. Math.*, **24** (1972), 580–591.

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