Global and Stochastic Analysis
Vol. 10 No. 2 (July-December, 2023)

Received: 20th May 2023 *Revised: 16th July2023* *Accepted: 07st August 2023*

MATRIX MAPS OVER SEMINEARRINGS

KUNCHAM S.P., TAPATEE S.*, RAJANI S., KEDUKODI B.S., AND HARIKRISHNAN P.K*

Abstract. In this paper, we introduce the notion of a matrix seminearring (abbr. $M_n(S)$) over an arbitrary seminearring S. A (right) seminearring is a generalization of a semiring and a nearring, wherein $(S,+)$ and $(S,\cdot)$ are semigroups; with only one distributive law is assumed. We prove various properties of matrix maps over a seminearring and obtain a one-one correspondence between the ideals of a seminearring and that of full ideals of matrix seminearring. Furthermore, we introduce prime ideal in matrix seminearring and prove that the ideal P^*, induced by a prime ideal P in S is prime in $M_n(S)$.

1. Introduction and preliminaries

Nearrings are generalized rings where the addition need not be abelian and only one distributive property is assumed. Rings can be viewed as algebraic systems of 'linear' functions on groups, while nearrings describe the general non-linear case [3, 9]. Matrix nearrings over arbitrary nearrings were introduced by Meldrum & Van der Walt [10], wherein the correspondence between the two-sided ideals in nearring N and those of matrix nearring $M_n(N)$ were obtained. Some developments in matrix nearrings over arbitrary nearrings were due to Meyer [11], Booth, and Groenewald [2]. Juglal et.al (see, [6]) studied different prime N-ideals and prime relations between generalized matrix nearring and multiplication modules over a nearring. Furthermore, Juglal and Groenewald [7] studied the class of strongly prime nearring modules and shown that it forms a τ-special class. For more literature on matrix nearrings, we refer to [5, 4, 12, 13, 15, 14].

We introduce the notion of matrix seminearring $M_n(S)$ over a seminearring S with 1. We prove various properties of matrix maps over a seminearring and obtain a one-one correspondence between the ideals of a seminearring and that of a matrix seminearring. Furthermore, we introduce prime ideal in matrix seminearring and prove that the ideal P^*, induced by a prime ideal P in S is prime in $M_n(S)$.

Definition 1.1. [8] A set S together with two binary operations $+$ and $\cdot$ is called a (right) seminearring if

(1) $(S,+)$ and $(S,\cdot)$ are semigroups;
(2) $(p+q)r = pr + qr$, for every $p,q,r \in S$;
(3) There exists $0 \in S$ such that $0+s = s+0 = s$ for every $s \in S$.

2000 *Mathematics Subject Classification.* 16Y30, 16Y60.
Key words and phrases. Seminearring, matrix nearring, prime ideal.
* Corresponding author.

(4) $0 \cdot s = 0$ for every $s \in S$.

Moreover, a right seminearring is said to be zero-symmetric if $p0 = 0$ for all $p \in S$.

Example 1.2. Let $(S, +)$ be a semigroup. Then the set of maps from S to S with respect to usual addition and substitution of maps becomes a seminearring (we denote it by $(M(S), +, \circ)$), which is not a nearring.

Definition 1.3. [1] A subset I of a seminearring S is called a right (left) s-ideal if

(1) $p + q \in I$, and
(2) rp $(pr) \in I$,

for all $p, q \in I$ and $r \in S$.

Definition 1.4. A subset I of a seminearring S is called a right (left) invariant if $IS \subseteq I$ $(SI \subseteq I)$.

Throughout, S denote a right seminearring having an absorbing zero.

Analogous to the notion given in [3], for any $u \in S$, the ideal generated by u is denoted by $\langle u \rangle$ and defined as, $\langle u \rangle = \bigcup_{i=1}^{\infty} S_{i+1}$, where $S_{i+1} = S_i^0 \cup S_i^+ \cup S_i^\dagger$ with $S_0 = \{u\}$, and $S_i^0 = \{p + q : p, q \in S_i\}$, $S_i^+ = \{as : s \in S, a \in S_i\}$, $S_i^\dagger = \{sa : s \in S, a \in S_i\}$.

2. Matrix Seminearring

For a right seminearring S with identity 1, let S^n will be the direct sum of n copies of $(S, +)$. The elements of S^n are written as $(s_1, \cdots, s_n)$ as column vectors. The symbols i_i and τ_j respectively, denote the i^{th} coordinate injective and j^{th} coordinate projective maps.

For an element $a \in S$, $i_i(a) = (0, \cdots, \underbrace{a}_{i^{th}}, \cdots, 0)$, and $\tau_j(a_1, \cdots, a_n) = a_j$, for any $(a_1, \cdots, a_n) \in S^n$. The seminearring of $n \times n$ matrices over S, denoted by $M_n(S)$, is defined as $M_n(S) = \langle \{\delta_{ij}^r : S^n \to S^n \mid r \in S, 1 \le i, j \le n\} \rangle$, where $\delta_{ij}^r(p_1, \cdots, p_n) := (s_1, s_2, \cdots, s_n)$ with $s_i = rp_j$ and $s_k = 0$ if $k \neq i$. Clearly, $\delta_{ij}^r = i_i \delta^r \tau_j$, where $\delta^r(s) = rs$, for all $r, s \in S$. $M_n(S)$ is a subseminearring of $M(S^n)$. If S is a semiring, then δ_{kl}^s corresponds to the $n \times n$-matrix with s in position (k, l) and zeros elsewhere. We denote e_i as 1 in the i^{th} component and 0 elsewhere; and $\underline{e} = e_1 + e_2 + \cdots + e_n$.

Definition 2.1. For $1 \le i, j \le n$, δ_{ij}^1 are defined as the matrix units.

Definition 2.2. For the identity matrix I is defined as $I = \delta_{11}^1 + \delta_{22}^1 + \cdots + \delta_{nn}^1$.

Definition 2.3. The i^{th} row of matrix A is the function $\tau_i A : S^n \to S$. It is denoted by $A(i)$.

Definition 2.4. The product of a scalar $s \in S$ and a given matrix A is sA, defined as $\sum_{i=1}^{n} i_i \delta^s A(i)$.

Definition 2.5. Scalar multiplication on the right of a matrix A by an element $s \in S$ is defined by

$$As = A(\delta_{11}^{s} + \delta_{22}^{s} + \cdots + \delta_{nn}^{s})$$

For any matrix $B \in M_n(S)$, we use $w(B)$ to denote minimum number of δ_{kl}^{s} therein.

Proposition 2.6. *If S is zero-symmetric, then $s\delta_{ij}^{1} = \delta_{ij}^{s}$, for all $s \in S$.*

Proof. Take $s \in S$. Then,

$$\begin{aligned} s\delta_{ij}^{1}(a_1, a_2, \cdots, a_n) &= s(0, \cdots, \underbrace{a_j}_{i^{th}}, \cdots, 0) \\ &= (s0, \cdots, sa_j, \cdots, s0) \\ &= (0, \cdots, sa_j, \cdots, 0) \\ &= \delta_{ij}^{s}(a_1, a_2, \cdots, a_n) \end{aligned}$$

□

The following properties of matrix nearrings (see, [10]) are generalized to matrix seminearrings. However, we provide the proofs for completeness.

Lemma 2.7. *(1)* $\delta_{ij}^{r} + \delta_{kl}^{s} = \begin{cases} \delta_{ij}^{r+s}, & \text{if } i = k,\ j = l; \\ \delta_{kl}^{s} + \delta_{ij}^{r}, & \text{if } i \neq k. \end{cases}$

(2) $\delta_{ij}^{r}\delta_{kl}^{s} = \begin{cases} \delta_{il}^{rs}, & \text{if } j = k; \\ \delta_{il}^{r0}, & \text{if } j \neq k; \end{cases}$

(3) $\delta_{ij}^{1}\delta_{kl}^{1} = \begin{cases} \delta_{il}^{1}, & \text{if } j = k; \\ \mathbf{0}, & \text{if } j \neq k; \end{cases}$

(4) $\delta_{ij}^{r}(\delta_{1k_1}^{r_1} + \cdots + \delta_{nk_n}^{r_n}) = \delta_{ik_j}^{rr_j}$;
where $r, s \in S$, $1 \leq i, j, k, l \leq n$.

Proof. (1) Let $(a_1, a_2, \cdots, a_n) \in S^n$. Then

$$\begin{aligned} &(\delta_{ij}^{r} + \delta_{kl}^{s})(a_1, a_2, \cdots, a_n) \\ &= \delta_{ij}^{r}(a_1, a_2, \cdots, a_n) + \delta_{ij}^{s}(a_1, a_2, \cdots, a_n) \\ &= \delta_{ij}^{r}(a_1, a_2, \cdots, a_n) + \delta_{ij}^{s}(a_1, a_2, \cdots, a_n) \\ &= (0, \cdots, \underbrace{ra_j}_{i^{th}}, \cdots, 0) + (0, \cdots, \underbrace{sa_j}_{k^{th}}, \cdots, 0) \end{aligned}$$

Case (i): If $i = k$ and $j = l$, then we get

$$\begin{aligned} (\delta_{ij}^{r} + \delta_{kl}^{s})(a_1, a_2, \cdots, a_n) &= (0, \cdots, \underbrace{ra_j + sa_j}_{i^{th}}, \cdots, 0) \\ &= (0, \cdots, \underbrace{(r+s)a_j}_{i^{th}}, \cdots, 0) \\ &= \delta_{ij}^{r+s}(a_1, a_2, \cdots, a_n) \end{aligned}$$

Case (ii): If $i \neq k$, then without loss of generality take $i < k$. Then we get

$$(\delta_{ij}^r + \delta_{kl}^s)(a_1, a_2, \cdots, a_n) = (0, \cdots, \underbrace{ra_j}_{i^{th}}, \cdots, \underbrace{sa_l}_{k^{th}}, \cdots, 0).$$

Similarly, we can verify that

$$(\delta_{kl}^s + \delta_{ij}^r)(a_1, a_2, \cdots, a_n) = (0, \cdots, \underbrace{ra_j}_{i^{th}}, \cdots, \underbrace{sa_l}_{k^{th}}, \cdots, 0).$$

Therefore, $\delta_{ij}^r + \delta_{kl}^s = \delta_{kl}^s + \delta_{ij}^r$, if $i \neq k$.

(2) Suppose $j = k$. Now

$$\begin{aligned}(\delta_{ij}^r \delta_{kl}^s)(a_1, \cdots, a_n) &= \delta_{ij}^r(\delta_{kl}^s(a_1, \cdots, a_n)) \\ &= \delta_{ij}^r(0, \cdots, \underbrace{sa_l}_{k^{th}}, \cdots, 0) \\ &= \delta_{ij}^r(0, \cdots, \underbrace{sa_l}_{j^{th}}, \cdots, 0) \text{ (since } j = k) \\ &= (0, \cdots, \underbrace{rsa_l}_{i^{th}}, \cdots, 0) \\ &= (0, \cdots, \underbrace{(rs)a_l}_{i^{th}}, \cdots, 0) \\ &= \delta_{il}^{rs}(a_1, \cdots, a_n).\end{aligned}$$

Suppose $j \neq k$. Now

$$\begin{aligned}(\delta_{ij}^r \delta_{kl}^s)(a_1, \cdots, a_n) &= \delta_{ij}^r(\delta_{kl}^s(a_1, \cdots, a_n)) \\ &= \delta_{ij}^r(0, \cdots, \underbrace{sa_l}_{k^{th}}, \cdots, 0) \\ &= (0, \cdots, \underbrace{r0}_{i^{th}}, \cdots, 0) \text{ (since } j \neq k) \\ &= \delta_{il}^{r0}(a_1, \cdots, a_n).\end{aligned}$$

(3) If $j = k$, by (2), we get $\delta_{ij}^1 \delta_{kl}^1 = \delta_{il}^1$.
Suppose $j \neq k$. Then

$$\begin{aligned}\delta_{ij}^1 \delta_{kl}^1(a_1, a_2, \cdots, a_n) &= \delta_{ij}^1(\delta_{kl}^1(a_1, \cdots, a_n)) \\ &= \delta_{ij}^1(0, \cdots, \underbrace{a_l}_{k^{th}}, \cdots, 0) \\ &= (0, \cdots, \underbrace{0}_{i^{th}}, \cdots, 0) \text{ (since } j \neq k) \\ &= \delta_{il}^0(a_1, \cdots, a_n). \\ &= \mathbf{0}(a_1, \cdots, a_n).\end{aligned}$$

(4)

$$\begin{aligned}
&\delta^r_{ij}(\delta^{r_1}_{1k_1}+\cdots+\delta^{r_n}_{nk_n})(a_1,a_2,\cdots,a_n)\\
&=\delta^r_{ij}\Big(\delta^{r_1}_{1k_1}(a_1,a_2,\cdots,a_n)+\cdots+\delta^{r_n}_{nk_n}(a_1,a_2,\cdots,a_n)\Big)\\
&=\delta^r_{ij}\Big((r_1a_{k_1},0,\cdots,0)+\cdots+(0,0,\cdots,r_na_{k_n})\Big)\\
&=\delta^r_{ij}\Big((r_1a_{k_1},r_2a_{k_2},\cdots,r_na_{k_n})\Big)\\
&=\Big(0,\cdots,\underbrace{rr_ja_{k_j}}_{i^{th}},\cdots,0\Big)\\
&=\delta^{rr_j}_{ik_j}(a_1,\cdots,a_n).
\end{aligned}$$

□

Lemma 2.8. *For any $A\in M_n(S)$ and $x,y,\cdots,z\in S$, there are $a,b,\cdots,c\in S$ such that*

$$A(\delta^x_{1k}+\delta^y_{2k}+\cdots+\delta^z_{nk})=\delta^a_{1k}+\delta^b_{2k}+\cdots+\delta^c_{nk},$$

for any $1\le k\le n$.

Proof. We prove this result by induction on $w(A)$. Suppose $w(A)=1$. Then $A=\delta^s_{ij}$ for some $1\le i,j\le n$ and $s\in S$. Now $\delta^s_{ij}(\delta^x_{1k}+\delta^y_{2k}+\cdots+\delta^z_{nk})(x_1,x_2,\cdots x_n)=\delta^r_{ij}(xx_k,yy_k,\cdots,zz_k)=(\delta^0_{1k}+\cdots+\delta^{rr_j}_{ik}+\cdots+\delta^0_{nk})(x_1,x_2,\cdots,x_n)$. Assume that the result is true when $w(A)<n$. Suppose $w(A)=n$. Then $A=B+C$ or $A=BC$ for some $B,C\in M_n(S)$ with $w(B),w(C)<n$.
Case 1: $A=B+C$.
$A(\delta^x_{1k}+\delta^y_{2k}+\cdots+\delta^z_{nk})=(B+C)(\delta^x_{1k}+\delta^y_{2k}+\cdots+\delta^z_{nk})=B(\delta^x_{1k}+\delta^y_{2k}+\cdots+\delta^z_{nk})+C(\delta^x_{1k}+\delta^y_{2k}+\cdots+\delta^z_{nk})=(\delta^{b_1}_{1k}+\delta^{b_2}_{2k}+\cdots+\delta^{b_n}_{nk})+(\delta^{c_1}_{1k}+\delta^{c_2}_{2k}+\cdots+\delta^{c_n}_{nk})=(\delta^{b_1+c_1}_{1k}+\delta^{b_2+c_2}_{2k}+\cdots+\delta^{b_n+c_n}_{nk})$.
Case 2: $A=BC$
$A(\delta^x_{1k}+\delta^y_{2k}+\cdots+\delta^z_{nk})=(BC)(\delta^x_{1k}+\delta^y_{2k}+\cdots+\delta^z_{nk})=B(C(\delta^x_{1k}+\delta^y_{2k}+\cdots+\delta^z_{nk}))=B(\delta^{c_1}_{1k}+\delta^{c_2}_{2k}+\cdots+\delta^{c_n}_{nk})$ (Since $w(C)<n$). Since $w(B)<n$, we get $A(\delta^x_{1k}+\delta^y_{2k}+\cdots+\delta^z_{nk})=\delta^{b_1}_{1k}+\delta^{b_2}_{2k}+\cdots+\delta^{b_n}_{nk}$. □

Lemma 2.9. *Let $K\in M_n(S)$, $x\in S$ and $\rho\in S^n$. Then $(K\rho)(x\underline{e})=K(\rho(x\underline{e}))$.*

Proof. We prove this by induction on $w(K)$. Suppose $w(K)=1$, then $K=\delta^a_{ij}$ for some $1\le i,j\le n$, $a\in S$. Now

$$\begin{aligned}
(K\rho)(x\underline{e})&=(\delta^a_{ij}\rho)(x(1,1,\cdots,1))\\
&=(0,\cdots,\underbrace{ax_j}_{i^{th}},\cdots,0)(x,x,\cdots,x)\\
&=(0,\cdots,\underbrace{ax_jx}_{i^{th}},\cdots,0)\\
&=\delta^a_{ij}((x_1,x_2,\cdots,x_n)x(1,1,\cdots,1)\\
&=\delta^a_{ij}(\rho x\underline{e})
\end{aligned}$$

Assume that the result is true when $w(K) < n$. Suppose $w(K) = n$. Then $K = B + C$ or $K = BC$ for some $B, C \in M_n(S)$ with $w(B), w(C) < n$.
Case 1: $K = B + C$
$(K\rho)(x\underline{e}) = ((B+C)\rho)(x\underline{e}) = (B\rho+C\rho)(x\underline{e}) = (B\rho)(x\underline{e})+(C\rho)(x\underline{e}) = B(\rho(x\underline{e}))+ C(\rho(x\underline{e}))$, (by induction) $= (B + C)(\rho(x\underline{e})) = K(\rho(x\underline{e}))$.
Case 2: $K = BC$.
$(K\rho)(x\underline{e}) = ((BC)\rho)(x\underline{e}) = B((C\rho)(x\underline{e})) = (BC)(\rho(x\underline{e})) = K(\rho(x\underline{e}))$. □

Corollary 2.10. *For any $\rho \in S^n$, there exists $X \in M_n(S)$ such that $\rho = Xe_1$.*

Theorem 2.11. *An element $r \in S$ is distributive if and only if δ^r_{ij} is distributive in $M_n(S)$.*

Proof. Suppose r is distributive in S. Then $r(s + t) = rs + rt$ for all $s, t \in S$. Let $A, B \in M_n(S)$ and $X \in S^n$.

$$\begin{aligned}
\delta^r_{ij}(A + B)(X) &= \delta^r_{ij}(AX + BX)\\
&= (i_i\delta^r\tau_j)(AX + BX)\\
&= (i_i\delta^r)\tau_j(AX + BX)\\
&= i_i\delta^r(\tau_jAX + \tau_jBX)\\
&= i_i(r(\tau_jAX + \tau_jBX))\\
&= i_i(r\tau_jAX + r\tau_jBX), \text{ (since } r \text{ is distributive in } S)\\
&= i_i\delta^r\tau_jAX + i_i\delta^r\tau_jBX\\
&= \delta^r_{ij}(AX) + \delta^r_{ij}(BX) = (\delta^r_{ij}A)X + (\delta^r_{ij}B)X = (\delta^r_{ij}A + \delta^r_{ij}B)X.
\end{aligned}$$

Conversely, suppose δ^r_{ij} is distributive over $M_n(S)$. Let $s, t \in S$. Then

$$\begin{aligned}
\delta^r_{ij}(\delta^s_{ji} + \delta^t_{ji})(1, 1, \cdots , 1) &= \delta^r_{ij}(\delta^s_{ji}(1, 1, \cdots , 1) + \delta^r_{ij}\delta^t_{ji}(1, 1, \cdots , 1))\\
&= \delta^r_{ij}((0, \cdots , \underbrace{s}_{j^{th}}, \cdots , 0) + (0, \cdots , \underbrace{t}_{j^{th}}, \cdots , 0))\\
&= (0, 0, \cdots , \underbrace{r(s + t)}_{i^{th}}, \cdots 0).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\delta^r_{ij}\delta^s_{ji} + \delta^r_{ij}\delta^t_{ji})(1, 1 \cdots , 1) &= \delta^r_{ij}\delta^s_{ji}(1, 1 \cdots , 1) + \delta^r_{ij}\delta^t_{ji}(1, 1 \cdots , 1)\\
&= (0, \cdots , \underbrace{rs}_{i^{th}}, \cdots , 0) + (0, \cdots , \underbrace{rt}_{i^{th}}, \cdots , 0)\\
&= (0, \cdots , \underbrace{rs + rt}_{i^{th}}, \cdots , 0).
\end{aligned}$$

Since δ^r_{ij} is distributive, we get $(0, 0, \cdots , \underbrace{r(s + t)}_{i^{th}}, \cdots 0) = (0, \cdots , \underbrace{rs + rt}_{i^{th}}, \cdots , 0)$.

Therefore $r(s + t) = rs + rt$. □

Theorem 2.12. *An element $r \in S$ is constant if and only if δ^r_{ij} is constant in $M_n(S)$.*

Proof. Suppose r is constant in S. Let $X = (x_1, x_2, \cdots, x_n)$. Consider $\delta_{ij}^r \mathbf{0}(X) = \delta_{ij}^r(\mathbf{0}(X)) = \delta_{ij}^r(0, \cdots, 0) = (0, \cdots, \underbrace{r0}_{i^{th}}, \cdots, 0) = (0, \cdots, \underbrace{rx_j}_{i^{th}}, \cdots, 0)$, where $x_j = \tau_j(X)$. Since r is constant, we get $\delta_{ij}^r \mathbf{0}(X) = (0, \cdots, \underbrace{r}_{i^{th}}, \cdots, 0) = \delta_{ij}^r X$. Therefore, δ_{ij}^r is constant.

Conversely, suppose δ_{ij}^r is constant in $M_n(S)$. Consider $(\delta_{ij}^r \mathbf{0})(1, 1, \cdots, 1) = \delta_{ij}^r(0, 0, \cdots, 0) = (0, \cdots, \underbrace{r0}_{i^{th}}, \cdots, 0)$.

On the other hand, $\delta_{ij}^r(1, 1, \cdots, 1) = (0, \cdots, \underbrace{r}_{i^{th}}, \cdots, 0)$. Since δ_{ij}^r is constant, we have $(\delta_{ij}^r \mathbf{0}) = \delta_{ij}^r$. That is,
$(0, \cdots, \underbrace{r0}_{i^{th}}, \cdots, 0) = (0, \cdots, \underbrace{r}_{i^{th}}, \cdots, 0)$. Therefore, $r0 = r$. □

We consider a zero symmetric matrix seminearring and $1 \in S$.

3. s-ideals in $M_n(S)$

Definition 3.1. A subset $\mathcal{Q}$ of $M_n(S)$ is a right (left) s-ideal if

(1) $A + B \in \mathcal{Q}$, and
(2) AX $(XA) \in \mathcal{Q}$,

for all $A, B \in \mathcal{Q}$, $X \in M_n(S)$.
Moreover, an s-ideal $\mathcal{A}$ of $M_n(S)$ is said to be a full s-ideal if $\mathcal{A} = K^*$ for some s-ideal K of S.

Definition 3.2. A subset $\mathcal{Q}$ of $M_n(S)$ is a right (left) invariant if $\mathcal{Q}M_n(S) \subseteq \mathcal{Q}$ $(M_n(S)\mathcal{Q} \subseteq \mathcal{Q})$.

Remark 3.3. Every right s-ideal of $M_n(S)$ is right invariant.

Lemma 3.4. *If $\mathcal{Q} \subseteq M_n(S)$ is right invariant, then $\mathcal{Q}S^n = \mathcal{Q}e_1$.*

Proof. Easy verification. □

Definition 3.5. If $K \subseteq S$, we define

$$K^* = \{A \in M_n(S) : A\rho \in K^n, \text{ for all } \rho \in S\}.$$

Proposition 3.6. *If K is a left s-ideal of S, then K^* is a two-sided s-ideal of $M_n(S)$.*

Proof. Let $A, B \in K^*$. Then $A\rho, B\rho \in K^n$ for all $\rho \in S^n$. Now, $(A + B)\rho = A\rho + B\rho \in K^n$. Therefore, $A + B \in K^*$. For any $C \in M_n(S)$, $(AC)\rho = A(C\rho) \in K^n$, since $C\rho \in S^n$. Therefore $AC \in K^*$. Also, $(CA)\rho = C(A\rho) = C\rho_1$, where $\rho_1 = A\rho \in K^n$. Now we apply induction on the weight of C. Let $w(C) = 1$, say

$C = \delta_{ij}^{s}$, $s \in S$ and $\rho_1 = (x_1, \cdots, x_n) \in K^n$. Then,

$$\begin{aligned} C\rho_1 &= \delta_{ij}^{s}(x_1, \cdots, x_n) \\ &= (0, \cdots, \underbrace{sx_j}_{i^{th}}, \cdots, 0) \\ &\in K^n, \text{ since } K \text{ is a left } s\text{-ideal.} \end{aligned}$$

Therefore, $(CA)\rho \in K^n$ for all $\rho \in S^n$, and so $CA \in K^*$. Suppose that the result in true for $w(C)<n$. Let $w(C) = n$. Then $C = P + Q$ or $C = PQ$.
Case-(i): $C = P + Q$

$$\begin{aligned} C\rho_1 &= (P + Q)\rho_1 \\ &= P\rho_1 + Q\rho_1 \\ &\in K^n + K^n \\ &= K^n. \end{aligned}$$

Therefore, $(CA)\rho \in K^n$ for all $\rho \in S^n$, and so $CA \in K^*$.
Case-(ii): $C = PQ$

$$\begin{aligned} C\rho_1 &= (PQ)\rho_1 \\ &= P(Q\rho_1) \\ &= P\rho_2, \text{ where } \rho_2 = Q\rho_1 \in K^n \\ &\in K^n. \end{aligned}$$

Hence, $(CA)\rho \in K^n$ for all $\rho \in S^n$, and so $CA \in K^*$. Therefore, K^* is a two-sided s-ideal of $M_n(S)$. □

Definition 3.7. If $\mathcal{K} \subseteq M_n(S)$, we define

$$\mathcal{K}_* = \{t \in S : t \in Im(\tau_j A) \text{ for some } A \in \mathcal{K}, 1 \le j \le n\}.$$

Proposition 3.8. *Let $\mathcal{K}$ is a two sided s-ideal of $M_n(S)$. $a \in \mathcal{K}_*$ if and only if $\delta_{11}^{a} \in \mathcal{K}$.*

Proof. Let $a \in \mathcal{K}_*$. Then there exists $A \in \mathcal{K}$, $\rho \in S^n$ and $1 \le j \le n$ such that $\tau_j(A\rho) = a$. Since $A\rho \in \mathcal{K}S^n$, by Lemma 3.4, there exists $X \in \mathcal{K}$ such that $A\rho = Xe_1$. Now,

$$\begin{aligned} Xe_1 &= X(\delta_{11}^{1} + \delta_{21}^{0} + \cdots + \delta_{n1}^{0})e_1 \\ &= (\delta_{11}^{a_1} + \cdots + \delta_{n1}^{a_n})e_1, \text{ by Lemma 2.8} \end{aligned}$$

Therefore,

$$\begin{aligned} a &= \tau_j(A\rho) \\ &= \tau_j(Xe_1) \\ &= \tau_j(\delta_{11}^{a_1} + \cdots + \delta_{n1}^{a_n})e_1 \\ &= \tau_j(\delta_{11}^{a_1}e_1 + \cdots + \delta_{n1}^{a_n}e_1) \\ &= \tau_j(a_1, \cdots, a_n) \\ &= a_j. \end{aligned}$$

Now, $\delta_{1j}^1 X(\delta_{11}^1 + \delta_{21}^0 \cdots + \delta_{n1}^0)(\alpha_1, \cdots, \alpha_n)$

$= \delta_{1j}^1(\delta_{11}^{a_1} + \delta_{21}^{a_2} + \cdots + \delta_{j1}^{a_j} + \cdots + \delta_{n1}^{a_n})(\alpha_1, \cdots, \alpha_n)$
$= \delta_{11}^{a_j}(\alpha_1, \cdots, \alpha_n)$
$= (a_j\alpha_1, 0, \cdots, 0)$
$= (a\alpha_1, 0, \cdots, 0)$
$= \delta_{11}^{a}(\alpha_1, \cdots, \alpha_n)$, for all $(\alpha_1, \cdots, \alpha_n) \in S^n$.

Therefore, $\delta_{1j}^1 X(\delta_{11}^1 + \delta_{21}^0 + \cdots + \delta_{n1}^0) = \delta_{11}^a \in \mathcal{K}$.
Conversely, suppose that $\delta_{11}^a \in \mathcal{K}$. Then $\tau_1\delta_{11}^a(1, 0, \cdots, 0) = \tau_1(a, 0, \cdots, 0) = a$. Therefore, $a \in \mathcal{K}_*$. □

Corollary 3.9. *If $\mathcal{K}$ is a two sided s-ideal of $M_n(S)$, then $s \in \mathcal{K}_*$ if and only if $\delta_{ij}^s \in \mathcal{K}$.*

Proof. By Proposition 3.8, we have $\delta_{11}^s \in \mathcal{K}$. Since $\mathcal{K}$ is a s-ideal of $M_n(S)$, we have $\delta_{ij}^s = \delta_{i1}^1\delta_{11}^s\delta_{1j}^1 \in \mathcal{K}$. □

Proposition 3.10. *If $\mathcal{K}$ is a two-sided s-ideal of $M_n(S)$, then $\mathcal{K}_*$ is a two-sided s-ideal of S.*

Proof. Suppose that $\mathcal{K}$ is a two-sided s-ideal in $M_n(S)$. To show $\mathcal{K}_*$ is a two-sided s-ideal in S. Let $a, b \in \mathcal{K}_*$. This implies $\delta_{11}^a, \delta_{11}^b \in \mathcal{K}$. Now $\delta_{11}^{a+b} = \delta_{11}^a + \delta_{11}^b \in \mathcal{K}$. Therefore, $a + b \in \mathcal{K}_*$. Let $a \in \mathcal{K}_*$ and $s \in S$. Now, $\delta_{11}^{as} = \delta_{11}^a\delta_{11}^s \in \mathcal{K}$, as K is s-ideal in $M_n(S)$. Also, $\delta_{11}^{sa} = \delta_{11}^s\delta_{11}^a \in \mathcal{K}$. Therefore, $sa \in \mathcal{K}_*$. Hence $\mathcal{K}_*$ is two-sided s-ideal of S. □

Proposition 3.11. *For two-sided s-ideal K of S and the corresponding two-sided s-ideal $\mathcal{K}$ of $M_n(S)$ the following are true.*

(i) $(\mathcal{K}_*)^* \supseteq \mathcal{K}$
(ii) $(K^*)_* = K$

Proof. (i) Suppose that $L \in \mathcal{K}$. Then $\tau_j L\rho \in \mathcal{K}_*$, for every $\rho \in S^n$ and $1 \leq j \leq n$. This implies $L\rho \in (\mathcal{K}_*)^n$, for all $\rho \in S^n$. Therefore, $L \in (\mathcal{K}_*)^*$.
(ii) $x \in (K^*)_*$ if and only if $\delta_{11}^x \in K^*$ if and only if $\delta_{11}^x\rho \in K^n$, $\forall \rho \in S^n$ if and only if $\delta_{11}^x\underline{e} \in K^n$ if and only if $(x, 0, \cdots, 0) \in K^n$ if and only if $x \in K$.

□

Proposition 3.12. *There is a bijection between the set of two-sided s-ideals of S and the set of full s-ideals of $M_n(S)$ given by $\mathcal{K} \to \mathcal{K}^*$ and $\mathcal{K} \to \mathcal{K}_*$ such that $(\mathcal{K}^*)_* = \mathcal{K}$ and $(\mathcal{K}_*)^* = \mathcal{K}$ for a s-ideal $\mathcal{K}$ of S and a s-ideal $\mathcal{K}$ of $M_n(S)$.*

Definition 3.13. $A \in M_n(S)$ is nilpotent if $A^k = \mathbf{0}$, for some $k \in \mathbb{Z}^+$; and if there is no such element in $M_n(S)$ except $\mathbf{0}$, then we call $M_n(S)$ is reduced.

Definition 3.14. An s-ideal K in a seminearring S is said to fulfill the insertion of factors property (IFP) if for every $a, b, c \in S$, $ab \in K$ implies $acb \in K$.

Theorem 3.15. *If $M_n(S)$ is reduced, then S has IFP.*

Proof. Suppose $M_n(S)$ is reduced. Let $a, b, n \in S$ such that $ab = 0$. Then, we have $\delta_{11}^{ab} = \mathbf{0}$. Now, $(\delta_{11}^{ba})^2 = \delta_{11}^{ba}\delta_{11}^{ba} = \delta_{11}^{baba} = \delta_{11}^{b0a} = \delta_{11}^{b0} = \mathbf{0}$, as S has absorbing zero. Therefore, $(\delta_{11}^{ba})^2 = \mathbf{0}$. Since $M_n(S)$ is reduced, we have that $\delta_{11}^{ba} = \mathbf{0}$. This means, $\delta_{11}^{ba}\rho = (0, 0, \cdots, 0)$, for all $\rho \in S^n$. In particular, $\delta_{11}^{ba}(1, 1, \cdots, 1) = (0, 0, \cdots, 0)$. This implies $(ba, 0, \cdots, 0) = (0, \cdots, 0)$, and so $ba = 0$. Now, $(\delta_{11}^{anb})^2 = \delta_{11}^{anb}\delta_{11}^{anb} = \delta_{11}^{anbanb} = \delta_{11}^{an0nb} = \delta_{11}^{0}$. Therefore, $(\delta_{11}^{anb})^2 = \mathbf{0}$. Since $M_n(S)$ is reduced, we have that $\delta_{11}^{anb} = \mathbf{0}$. So, $(\delta_{11}^{anb})(1, 1, \cdots, 1) = \mathbf{0}$. This is same as, $(anb, 0, \cdots, 0) = (0, 0, \cdots, 0)$. Therefore, $anb = 0$. Hence S has IFP. □

Proposition 3.16. *Let $\mathcal{K}$ be an s-ideal of $M_n(S)$. If $\mathcal{K}$ has IFP, then $\mathcal{K}_*$ has IFP in S.*

Proof. Suppose $\mathcal{K}$ has IFP. Let $a, b, c \in S$ such that $ab \in \mathcal{K}_*$. To show $acb \in \mathcal{K}_*$. Since $ab \in \mathcal{K}_*$, we have $\delta_{11}^{ab} \in \mathcal{K}$, and so $\delta_{11}^{a}\delta_{11}^{b} \in \mathcal{K}$. Since $\mathcal{K}$ has IFP, we have $\delta_{11}^{a}A\delta_{11}^{b} \in \mathcal{K}$, for all $A \in M_n(S)$. Put $A = \delta_{11}^{c}$. Then $\delta_{11}^{acb} = \delta_{11}^{a}\delta_{11}^{c}\delta_{11}^{b} \in \mathcal{K}$. Thus $\delta_{11}^{acb} \in \mathcal{K}$, and so $acb \in \mathcal{K}_*$. □

Definition 3.17. A s-ideal $\mathcal{K}$ of $M_n(S)$ is said to be prime if $\mathcal{P}\mathcal{Q} \subseteq \mathcal{K}$ implies $\mathcal{P} \subseteq \mathcal{K}$ or $\mathcal{Q} \subseteq \mathcal{K}$, for all s-ideals $\mathcal{P}, \mathcal{Q}$ of $M_n(S)$.

Proposition 3.18. *Let P be a prime s-ideal of S. Then P^* is a prime s-ideal of $M_n(S)$.*

Proof. Let P be a prime s-ideal of S. We show P^* is a prime s-ideal of $M_n(S)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ be two s-ideals of $M_n(S)$ such that $\mathcal{A}\mathcal{B} \subseteq P^*$. On a contrary, suppose that $\mathcal{A} \nsubseteq P^*$ and $\mathcal{B} \nsubseteq P^*$. Then there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $A \notin P^*$ and $B \notin P^*$. This means that there exist $\rho, \delta \in S^n$ such that $A\rho \notin P^n$ and $B\delta \notin P^n$. That is, $a = \tau_k(A\rho) \notin P$, for some $1 \leq k \leq n$. Since $A\rho \in \langle A \rangle S^n$ and $\langle A \rangle$ is a right s-ideal of $M_n(S)$, by Lemma 3.4, there exists $C \in \langle A \rangle$ such that $A\rho = Ce_1$. This implies that

$$\begin{aligned} Ce_1 &= C[\delta_{11}^{1} + \delta_{21}^{0} + \cdots + \delta_{n1}^{0}]e_1 \\ &= [\delta_{11}^{a_1} + \delta_{21}^{a_2} \cdots + \delta_{n1}^{a_n}]e_1 \\ &= (c_1, \cdots, c_n) \end{aligned}$$

Now, $\tau_k(Ce_1) = c_k = \tau_k(A\rho) = a$, implies $c_k = a \notin P$. This implies, $\delta_{11}^{c_k} \notin P^*$. Also,

$$\begin{aligned}&\delta_{1k}^{1} C[\delta_{11}^{1} + \delta_{21}^{0} + \cdots + \delta_{n1}^{0}]\\&= \delta_{1k}^{1}[\delta_{11}^{c_1} + \delta_{21}^{c_2} \cdots + \delta_{n1}^{c_n}]\\&= \delta_{11}^{c_k} \in \langle A \rangle.\end{aligned}$$

So, $\delta_{11}^{a} \in \langle A \rangle \setminus P^*$. Similarly, since $B\delta \notin P^n$, there exists $b \notin P$ such that $\delta_{11}^{b} \in \langle B \rangle \setminus I^*$. Since $a \notin P$, $b \notin P$, it follows that $\langle a \rangle \nsubseteq P$ and $\langle b \rangle \nsubseteq P$. Again, since $\langle a \rangle \langle b \rangle \nsubseteq P$, we get $c \in \langle a \rangle$ and $d \in \langle b \rangle$ such that $cd \notin P$. Therefore,

$$\delta_{11}^{cd} \notin P^* \ \cdots (1)$$

Now $c \in \langle a \rangle$. Write $X = \{a\}$. Referring to the notion of $\langle a \rangle = \bigcup_{i=0}^{\infty} X_i$, we prove $\delta_{11}^{c} \in \langle \delta_{11}^{a} \rangle$. Suppose $c \in X_m$ and $m = 0$. Then $c \in X_0 = X = \{a\}$. In this case, $\delta_{11}^{c} = \delta_{11}^{a} \in \langle \delta_{11}^{a} \rangle$. Suppose $m = 1$. Then $c = X_1 = X_0^0 \cup X_0^+ \cup X_0^*$.
If $c \in X_0^0$, then $c = a + b$. Now, $\delta_{11}^{c} = \delta_{11}^{a+b} = \delta_{11}^{a} + \delta_{11}^{b} \in \langle \delta_{11}^{a} \rangle$. If $c \in X_0^+$, then $c = as$. Now $\delta_{11}^{c} = \delta_{11}^{as} = \delta_{11}^{a}\delta_{11}^{s} \in \langle \delta_{11}^{a} \rangle$. If $c \in X_0^*$, then $c = sa$. Now $\delta_{11}^{c} = \delta_{11}^{sa} = \delta_{11}^{s}\delta_{11}^{a} \in \langle \delta_{11}^{a} \rangle$. Therefore, $c \in \langle a \rangle$. Thus, $\delta_{11}^{c} \in \delta_{11}^{a}$ for $m = 1$.
Induction hypothesis: Suppose $\delta_{11}^{c} \in \langle \delta_{11}^{a} \rangle$ for all $c \in X_{k-1}$. Suppose $c = X_k = X_{k-1} \cup X_{k-1}^{+} \cup X_{k-1}^{*}$. If $c \in X_{k-1}^{0}$, then $c = x + y$, for some $x, y \in X_{k-1}$. Now $\delta_{11}^{c} = \delta_{11}^{x+y} = \delta_{11}^{x} + \delta_{11}^{y} \in \langle \delta_{11}^{a} \rangle$. If $c \in X_{k-1}^{+}$, then $c = as$, for some $a \in X_{k-1}$. Now $\delta_{11}^{c} = \delta_{11}^{as} = \delta_{11}^{a}\delta_{11}^{s} \in \langle \delta_{11}^{a} \rangle$. If $c \in X_{k-1}^{*}$, then $c = sa$, for some $a \in X_{k-1}$. Now $\delta_{11}^{c} = \delta_{11}^{sa} = \delta_{11}^{s}\delta_{11}^{a} \in \langle \delta_{11}^{a} \rangle$. Therefore, $c \in \langle a \rangle$ implies $\delta_{11}^{c} \in \langle \delta_{11}^{a} \rangle \subseteq \langle A \rangle$. Similarly, $d \in \langle b \rangle$ implies $\delta_{11}^{d} \in \langle \delta_{11}^{b} \rangle \subseteq \langle B \rangle$. Thus, $\delta_{11}^{cd} = \delta_{11}^{c}\delta_{11}^{d} \in \langle A \rangle \langle B \rangle \subseteq \mathcal{AB} \subseteq P^*$. Therefore,

$$\delta_{11}^{cd} \in P^* \ \cdots (2)$$

Therefore, from (1) and (2), we have a contradiction. Thus, $\mathcal{A} \subseteq P^*$ or $\mathcal{B} \subseteq P^*$. □

4. Conclusion

We have defined the notion of a matrix seminearring (abbr. $M_n(S)$) over an arbitrary seminearring S. We proved various properties of matrix maps over a seminearring to obtain a one-one correspondence between the ideals of a seminearring and that of full ideals of matrix seminearring. We can extend the study to different classes of prime ideals in matrix seminearrings and corresponding radical properties.

5. Acknowledgment

The author[2*] acknowledges Manipal Institute of Technology Bengaluru, Manipal Academy of Higher Education, Manipal, and the authors[1,3,4,5*] acknowledge Manipal Institute of Technology (MIT), Manipal Academy of Higher Education, Manipal, India for their kind encouragement. The author [2*] acknowledges Indian National Science Academy (INSA), Govt. of India, for selecting to the award of visiting scientist under the award number: INSA/SP/VSP-56/2023-24/. The author[5*] acknowledges SERB, Govt. of India for the TARE project fellowship TAR/2022/000219.

References

[1] Ahsan J.: Seminear-rings characterized by their $\mathcal{S}$-ideals. I. Proc. Japan Acad. 71(Ser. A), 101-103 (1995).

[2] Booth, G.L., Groenewald, N.J.: On primeness in matrix near-rings. Arch. Math. **56**(6), 539-546 (1991).

[3] Bhavanari, S.,Kuncham, S.P.: Nearrings, fuzzy ideals, and graph theory, CRC press, (2013).

[4] Bhavanari, S.,Lokeswara Rao, M.B.V., Kuncham, S.P.: A note on primeness in near-rings and matrix near-rings. Indian J. pure appl. Math. **27**, 227-234 (1996).

[5] Bhavanari, S., Kuncham, S.P.: On finite Goldie dimension of $M_n(N)$-group N^n. Proc. of the Conf. on Nearrings and Nearfields (Springer), 301-310 (2005).

[6] Juglal, S., Groenewald, N. J., & Lee, K. S. E., Different prime R-ideals, *Algebra Colloquium*, **17** (spec 01) (2010) 887-904.

[7] Juglal, S., Groenewald, N.J.: Strongly prime near-ring modules. Arabian Journal for Science and Engineering, **36**(6) 985-995 (2011).

[8] Koppula K, Kedukodi B.S., Kuncham, S.P.: On prime strong ideals of a seminearring. Mat. Vesnik. 2020 Sep 1;72(3):243-56.

[9] Pilz, G.: Near-Rings: the theory and its applications, North Holland, **23** (1983).

[10] Meldrum, J.D.P., Van der Walt A.P.J.: Matrix near-rings. Arch. Math. **47**(4) 312-319 (1986).

[11] Meyer J.H.: Left ideals in matrix near-rings. Comm. Algebra. **17** (6), 1315-1335 (1989).

[12] Tapatee S., Kedukodi B.S., Juglal S., Harikrishnan P.K., Kuncham S.P.: Generalization of prime ideals in $M_n(N)$-group N^n. Rendiconti del Circolo Matematico di Palermo, 72(1), 449–465 (2023). https://doi.org/10.1007/s12215-021-00682-y

[13] Tapatee S., Meyer J.H., Harikrishnan P.K., Kedukodi B.S., Kuncham S.P.: Partial Order in Matrix Nearrings. Bull. Iranian Math. Soc., 48(6), 3195–3209 (2022). https://doi.org/10.1007/s41980-022-00689-w

[14] Rajani, S., Tapatee, S., Harikrishnan, P., Kedukodi, B. S., & Kuncham, S. P.: Superfluous ideals of N-groups. Rendiconti del Circolo Matematico di Palermo Series 2, 1-19 (2023). https://doi.org/10.1007/s12215-023-00888-2

[15] Tapatee S., Deepak Shetty M., Groenewald N. J., Harikrishnan P. K., Kuncham S. P.: On completely 2-absorbing ideals of N-groups. J. Discrete Math. Sci. Cryptogr. **24**(2) 541-56 (2021).

Kuncham S.P.: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education,576104, India
Email address: syamprasad.k@manipal.edu

Tapatee S.: Department of Mathematics, Manipal Institute of Technology Bengaluru, Manipal Academy of Higher Education,560064, India
Email address: sahoo.tapatee@manipal.edu

Rajani S.: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education,576104, India
Email address: rajanisalvankar@gmail.com

Kedukodi B.S.: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education,576104, India
Email address: babushrisrinivas.k@manipal.edu

Harikrishnan P.K.: Department of Mathematics, Manipal Institute of Technology Bengaluru, Manipal Academy of Higher Education,576104, India
Email address: pk.harikrishnan@manipal.edu