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INTERPOLATING MARTINGALE MEASURES AND INTERPOLATING DEFLATORS OF ONE-STEP PROCESSES ON COUNTABLE PROBABILITY SPACES

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Abstract. In this paper we present generalizations of two results on the existence of interpolating martingale measures for one-step process defined on a countable probability space. The initial random variable of such a process is a constant a and the final one takes finite or countable many different values b_k, $b_k \neq a$. The first theorem generalizes the main result of the work of I.V. Pavlov (GSA, 2018, N2) and the second one — the results of I.V. Pavlov, I.V. Tsvetkova, V.V. Shamraeva (Theory Probab. Appl., 2017, N1). The meaning of these generalizations is as follows. The main point is that in previous papers the existence of interpolational martingale measures was proved with rational b_k, and in the present work a partial transition from rational b_k to arbitrary ones was made. We also automatically obtain the existence of interpolation deflators introduced in this paper.

1. Introduction

Let us consider on a countable set Ω a static real-valued stochastic process (s. p.) $Z = (Z_0, Z_1)$, where $Z_0 = a = \text{const}$, and among the values of the random variable (r. v.) Z_1 there may be coinciding ones. All the different values of Z_1 are numbered and denoted by b_k $(1 \leq k < r+1)$ (the number $r > 1$ can be either finite or infinite). Let us denote $B_k = \{\omega\colon Z_1(\omega) = b_k\}$. We say that b_k is of the order m_k, $1 \leq m_k \leq \infty$, if r. v. Z_1 takes this value m_k times. It means that for any k $B_k = \bigcup\limits_{i=1}^{m_k} \omega_k^i$, where $\omega_k^i \in \Omega$ and $Z_1(\omega_k^i) := b_k^i = b_k$. Denote by $\mathcal{F}_0$ the trivial σ-field $\{\Omega, \varnothing\}$, and by $\mathcal{F}_1$ the set of all subsets of Ω. It is clear that s. p. Z is adapted to the one-step filtration $\mathbf{F} = (\mathcal{F}_0, \mathcal{F}_1)$.

Denote by $\mathcal{P}$ the set of all non degenerate probability measures on $\mathcal{F}_1$, i.e. $P \in \mathcal{P}$ iff $p_k^i := P(\omega_k^i) > 0$ for all $1 \leq k < r+1$, $1 \leq i < m_k+1$. Denote $p_k = \sum\limits_{i=1}^{m_k} p_k^i$ $(1 \leq k < r+1)$ and introduce the set $\mathcal{P}(Z, \mathbf{F})$ of all non degenerate martingale

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measures (m. m.) P of the process Z. It is obvious that any $P = (p_k^i) \in \mathcal{P}(Z, \mathbf{F})$ satisfies the system:

$$\begin{cases} \sum\limits_{k=1}^{r} p_k = 1 \\ \sum\limits_{k=1}^{r} |b_k| p_k < \infty \\ \sum\limits_{k=1}^{r} b_k p_k = a \\ p_k > 0, \ 1 \le k < r+1. \end{cases} \tag{1.1}$$

Conversely, if $(p_k)_{k=1}^r$ is a solution of the system (1.1), then writing arbitrarily for any k a representation $p_k = \sum\limits_{i=1}^{m_k} p_k^i$ $(1 \le k < r+1)$, where each term of the sum is strictly positive, we obtain a measure $P = (p_k^i) \in \mathcal{P}(Z, \mathbf{F})$. Since $r > 1$, then it is easy to see that the resolvability of the system (1.1) is equivalent to the fulfilment of the condition

$$\inf_k b_k < a < \sup_k b_k. \tag{1.2}$$

We assume that this condition is satisfied.

Consider a special family of m.m.

Definition 1.1. We say that a probability measure $P = (p_k^i) \in \mathcal{P}(Z, \mathbf{F})$ satisfies noncoincidence barycenter condition (in the weak sence) if $\forall l$, $1 \le l < r+1$, and for all subsets $J \subset \{(k,i), 1 \le k < r+1, 1 \le i < m_k+1\}$ with finite complementation J^c (in the set of all duble indexes $\{(k,i), 1 \le k < r+1, 1 \le i < m_k+1\}$) the following inequalities hold:

$$b_l \ne \frac{\sum\limits_J b_k p_k^i}{\sum\limits_J p_k^i}. \tag{1.3}$$

The set of all martingale measures satisfying (1.3) is denoted by $\mathcal{SP}(Z, \mathbf{F})$.

It is easy to see that if $\mathcal{SP}(Z, \mathbf{F}) \ne \emptyset$, then $b_k \ne a, 1 < k < r+1$. We suppose here that the last condition is always fulfilled.

The set $\mathcal{SP}(Z, \mathbf{F})$ plays an important role in the theory of Haar interpolations of financial markets. In arbitrage-free incomplete markets using measures $P \in \mathcal{SP}(Z, \mathbf{F})$ we obtain more fair prices of various contingent claims. That is why the question if whether $\mathcal{SP}(Z, \mathbf{F})$ is not empty is significant.

In this paper, we investigate the set $\mathcal{SP}(Z, \mathbf{F})$ first in the case $r = \infty$ (as in the work [1]), and then in the case $r < \infty$ (as in the work [2]). After that we introduce the notion of interpolating deflator and discuss possibilities of its application in the specified fild.

2. Set $\mathcal{SP}(Z, \mathbf{F})$ in the case $r = \infty$

A non-zero sequence $\mathbf{r} = (r_1, r_2, \dots)$ will be called finitary if its components are rational and among them only a finite number are non-zero. For a sequence of real numbers $\mathbf{d} = (d_1, d_2, \dots)$ we denote by $\mathcal{L}(\mathbf{d})$ a set of numbers of the form $\sum r_i d_i$, where $\mathbf{r}$ runs through all finitary sequences.

The following theorem generalizes the main result from [1].

Theorem 2.1. *Suppose that $r = \infty$, number a is irrational, and the sequence $\mathbf{b}$ contains an infinite number of different rational terms. If $a \notin \mathcal{L}(\mathbf{b})$, then the set $\mathcal{SP}(Z, \mathbf{F})$ is not empty.*

Proof. First we prove that the system (1.1) has a rational solution. From the condition (1.2) it follows that the system (1.1) has a solution (not necessarily rational). Introduce the countable set of indexes $K = \{k : b_k \, is \, rational\}$ and let $L = K^c$, where K^c is the complementation of K in $\mathbb{N} := \{1, 2, \dots\}$. Represent the system (1.1) in the form:

$$\begin{cases} \sum\limits_{k\in K} \frac{p_k}{1-\sum_{n\in L} p_n} = 1 \\ \sum\limits_{k=1}^{\infty} |b_k| p_k < \infty \\ \sum\limits_{k\in K} \frac{b_k p_k}{1-\sum_{n\in L} p_n} = \frac{a}{1-\sum_{n\in L} p_n} - \sum\limits_{k\in L} \frac{b_k p_k}{1-\sum_{n\in L} p_n} \\ p_k > 0, \ k \in \mathbb{N}. \end{cases} \tag{2.1}$$

Fix some solution $(p_1^{(o)}, p_2^{(o)}, \dots)$ of (2.1) and denote:

$$A = \frac{a}{1 - \sum_{n\in L} p_n^{(o)}} - \sum_{k\in L} \frac{b_k p_k^{(o)}}{1 - \sum_{n\in L} p_n^{(o)}}. \tag{2.2}$$

Now consider the system:

$$\begin{cases} \sum\limits_{k\in K} q_k = 1 \\ \sum\limits_{k\in K} |b_k| q_k < \infty \\ \sum\limits_{k\in K} b_k q_k = A \\ q_k > 0, \ k \in K. \end{cases} \tag{2.3}$$

System (2.3) has the solution $q_k = q_k^{(o)} = \frac{p_k^{(o)}}{1-\sum_{n\in L} p_n^{(o)}}$, $k \in K$. Hence

$$\inf_{k\in K} b_k < A < \sup_{k\in K} b_k. \tag{2.4}$$

Let us replace in (2.2) the numbers $p_n^{(o)}$, $n \in L$, with very close rational positive numbers $p_n^{(r)}$, $n \in L$, so that the inequalities (2.4) and $1 - \sum_{n\in L} p_n^{(r)} > 0$ hold. We get the number

$$A_0 = \frac{a}{1 - \sum_{n\in L} p_n^{(r)}} - \sum_{k\in L} \frac{b_k p_k^{(r)}}{1 - \sum_{n\in L} p_n^{(r)}},$$

satisfying the inequality

$$\inf_{k\in K} b_k < A_0 < \sup_{k\in K} b_k.$$

From the results of [1] it follows that the system with rational coefficiets

$$\begin{cases} \sum\limits_{k\in K} q_k = 1 \\ \sum\limits_{k\in K} |b_k| q_k < \infty \\ \sum\limits_{k\in K} b_k q_k = A_0 \\ q_k > 0,\ k \in K. \end{cases}$$

admits rational solutions $q_k^{(r)}$, $k \in K$. Denote $p_k^{(r)} := q_k^{(r)} \left(1 - \sum_{n\in L} p_n^{(r)}\right)$, $k \in K$. Now it is obvious that the vector $\left(p_1^{(r)}, p_2^{(r)}, \dots\right)$ is a rational solution of the system (1.1).

For any k let us decompose $p_k^{(r)} = \sum\limits_{i=1}^{m_k} p_k^i$ $(1 \le k < r+1)$, where each term of the sum is strictly positive and rational. We obtain a probability measure $P = (p_k^i)$ defined on $\mathcal{F}_1$. Let us prove that under conditions of theorem 2.1 P is included in $\mathcal{SP}(Z, \mathbf{F})$.

Using martingale and normalization properties write the inequality (1.3) in the form:

$$b_l \ne \frac{a - \sum\limits_{J^c} b_k p_k^i}{1 - \sum\limits_{J^c} p_k^i}.$$

Since the set J^c is finite, the last inequality is equivalent to the condition $a \notin \mathcal{L}(\mathbf{b})$ in the formulation of the theorem. Hence $P \in \mathcal{SP}(Z, \mathbf{F})$. Q.E.D. □

Proposition 2.2. *Let the number a and all the terms of the sequence* $\mathbf{b}$ *be rational. If a martingale measure $P = (p_1, p_2, \dots)$ is such that $\mathcal{L}(\mathbf{P})$ consists only of irrational numbers, then $P \in \mathcal{SP}(Z, \mathbf{F})$.*

Proof. The proof is trivial. □

Example 2.3. Let $(d_1, d_2, \dots)$ be an arbitrary positive sequence such that $\mathcal{L}(\mathbf{d})$ consists only of irrational numbers (for example, if a number t is transcendental, then the sequence $(t, t^2, t^3 \dots)$ satisfies this property). Find the sequence $(c_1, c_2, \dots)$ of positive rational numbers such that $\sum c_i d_i = 1$. Put $p_i = c_i d_i$. Let a be an arbitrary rational number. Find a sequence of rational numbers $\mathbf{b}$ such that $\sum b_i p_i = a$. Then $P = (p_1, p_2, \dots)$ belongs to $\mathcal{SP}(Z, \mathbf{F})$.

3. Set $\mathcal{SP}(Z, \mathbf{F})$ in the case $r < \infty$

Without loss of generality, we can assume that $0 < b_1 < b_2 < ... < b_r$.

Theorem 3.1. *Let $3 < r < \infty$, $m_1 < \infty, \dots, m_{r-2} < \infty$, $m_{r-1} = m_r = \infty$, $b_1 < a < b_2$. Then $\mathcal{SP}(Z, \mathbf{F}) \ne \infty$.*

Proof. 1) Transform first inequalities (1.3) from definition 1.1. Represent the finite set J^c in the form:

$$J^c = \bigcup_{k=1}^{r} J_k, \tag{3.1}$$

where $J_k \subset \{(k,i), 1 \le i < m_k + 1\}$. It is easy to see that a measure $P \in \mathcal{P}(Z, \mathbf{F})$ satisfies the definition 1.1 iff for any finite subsets $J_k \subset \{(k,i), 1 \le i < m_k + 1\}$ the following inequalities are true:

$$\begin{cases} (b_l - b_1)\left(p_1 - \sum_{J_1} p_j\right) + \cdots + (b_l - b_{l-1})\left(p_{l-1} - \sum_{J_{l-1}} p_j\right) + \\ +(b_l - b_{l+1})\left(p_{l+1} - \sum_{J_{l+1}} p_j\right) + \cdots + (b_l - b_r)\left(p_r - \sum_{J_r} p_j\right) \neq 0, \\ l = 2, \ldots, r-1 \end{cases} \tag{3.2}$$

(for $l = 1$ and $l = r$ the inequalities satisfy automatically).

2) Let $(p_1, p_2, \ldots, p_r)$ be a solution of the system (1.1). If we express p_1 and p_2 through the $p_3, \ldots, p_r$, then we see that $p_1 > \frac{b_2 - a}{b_2 - b_1}$ and $p_2 \uparrow \frac{a - b_1}{b_2 - b_1}$ as $p_3 \downarrow 0, \ldots, p_r \downarrow 0$.

3) Consider the case $m_1 = 1, \ldots, m_{r-2} = 1$.

Let $l = 2$. If $J_1 = \{p_1\}$, then (3.2) is fulfilled automatically since the left part of (3.2) is more than 0. If $J_1 = \emptyset$, we will ensure inequality:

$$(b_2 - b_1)p_1 > (b_3 - b_2)p_3 + (b_4 - b_2)p_4 + \cdots + (b_r - b_2)p_r. \tag{3.3}$$

Let $l = 3$. Arguing in a similar way, we will require the inequalities:

$$\begin{cases} (b_3 - b_1)p_1 > (b_4 - b_3)p_4 + \cdots + (b_r - b_3)p_r \\ (b_3 - b_2)p_2 > (b_4 - b_3)p_4 + \cdots + (b_r - b_3)p_r. \end{cases} \tag{3.4}$$

For $l = 3$:

$$\begin{cases} (b_4 - b_1)p_1 > (b_5 - b_4)p_5 + \cdots + (b_r - b_4)p_r \\ (b_4 - b_2)p_2 > (b_5 - b_4)p_5 + \cdots + (b_r - b_4)p_r \\ (b_4 - b_3)p_3 > (b_5 - b_4)p_5 + \cdots + (b_r - b_4)p_r. \end{cases} \tag{3.5}$$

....

For $l = r - 1$:

$$\begin{cases} (b_{r-1} - b_1)p_1 > (b_r - b_{r-1})p_r \\ (b_{r-1} - b_2)p_2 > (b_r - b_{r-1})p_r \\ \cdots\cdots\cdots\cdots \\ (b_{r-1} - b_{r-3})p_1 > (b_r - b_{r-1})p_r \\ (b_{r-1} - b_{r-2})p_1 > (b_r - b_{r-1})p_r. \end{cases} \tag{3.6}$$

Now decompose arbitrarily $b_{r-1} = \sum_{i=1}^{\infty} b_{r-1}^i$, $b_r = \sum_{i=1}^{\infty} b_r^i$, where each term in the sums is strictly positive. It is easy to see that if the components of constructed measure $P = (p_k^i)$ satisfy the inequalities (3.3)–(3.6), then P belongs to $\mathcal{SP}(Z, \mathbf{F})$. Show that P really satisfy (3.3)–(3.6).

Denote $c := \min_{1\leq k<j\leq r}(b_j - b_k)$, $C := \max_{1\leq k<j\leq r}(b_j - b_k)$. Let $\tilde{p}_3^1, \ldots, \tilde{p}_r^1$ be positive numbers satisfying the inequality $c\frac{b_2-a}{b_2-b_1} > C\sum_{k=3}^{r}\tilde{p}_k^1$ and such that the unique solution $x_1 = \tilde{p}_1^1$, $x_2 = \tilde{p}_2^1$ of the system

$$\begin{cases} x_1 + x_2 = 1 - \sum\limits_{k=3}^{r} \tilde{p}_k^1 \\ b_1x_1 + b_2x_2 = a - \sum\limits_{k=3}^{r} b_k\tilde{p}_k^1 \end{cases}$$

is positive. From 2) it follows that $(\tilde{p}_1^1, \tilde{p}_2^1, \ldots, \tilde{p}_r^1)$ satisfies the first inequalities of (3.3)–(3.6).

Let $\tilde{p}_4^2, \ldots, \tilde{p}_r^2$ be positive numbers satisfying the inequality $c\tilde{p}_2^1 > C\sum_{k=4}^{r}\tilde{p}_k^2$ and such that $\tilde{p}_k^2 < \tilde{p}_k^1$, $k = 4, \ldots, r$. Put $\tilde{p}_3^2 = \tilde{p}_3^1$. As in the previous case we find $\tilde{p}_1^2 > \frac{b_2-a}{b_2-b_1}$ and $\tilde{p}_2^2 > \tilde{p}_2^1$. Now $(\tilde{p}_1^2, \tilde{p}_2^2, \ldots, \tilde{p}_r^2)$ satisfies the first and second inequalities of (3.3)–(3.6).

In the same way, ensuring the fulfillment of inequalitiy $c\tilde{p}_3^2 > C\sum_{k=5}^{r}\tilde{p}_k^3$ and up to $c\tilde{p}_{r-2}^{r-3} > C\tilde{p}_r^{r-2}$ we obtain a vector $(\tilde{p}_1^{r-2}, \tilde{p}_2^{r-2}, \ldots, \tilde{p}_r^{r-2})$ satisfying all the inequalities of (3.3)–(3.6),

4) At last consider the case $m_1 < \infty, \ldots, m_{r-2}\infty$. Let us denote $p_k^i = \frac{p_k}{m_k}$, $k = 1, \ldots, r-2$, $i = 1, \ldots, m_k$. Changing properly the constants c and C introduced in 3) and using the same arguments as in 3), we obtaine a measure $P \in \mathcal{SP}(Z, \mathbf{F})$. Q.E.D. □

Corollary 3.2. *Let* $3 < r < \infty$, $m_1 = m_2 = \infty$, $m_3 < \infty, \ldots, m_r < \infty$, $b_{r-1} < a < b_r$. *Then* $\mathcal{SP}(Z, \mathbf{F}) \neq \emptyset$.

Proof. The proof follows from.the obvious equality $\mathcal{SP}(\alpha Z + \beta, \mathbf{F}) = \mathcal{SP}(\mathbf{Z}, \mathbf{F})$, where $\alpha \neq 0$ and β are constants. □

Remark 3.3. If $r = 3$, then $\mathcal{SP}(Z, \mathbf{F}) \neq \emptyset$ only under conditions (1.2) and $b_k \neq a, 1 < k \leq 3$ (c.f. [2]).

4. Interpolating martingale deflators

It is well known that on non-arbitrage financial (B,S)-markets with a fixed physical probability Q, there is a one-to-one correspondence between martingale measures (m.m.) equivalent to Q and martingale (relative to Q) deflators (m.d.). If (B,S)-market is defined on no more than a countable probabilistic space $(\Omega, \mathcal{F}, Q)$, then for construction of hedge portfolios it turned out to be useful to consider the most fair m.m., which we called interpolating m.m. [1]-[2]. We also call the corresponding deflators interpolating ones. Let us clarify this definition in our one-step model.

Let the physical measure Q on $\mathcal{F}_1$ is strictly positive on all subsets of Ω (except for $\emptyset$). Let $Z = (Z_k, \mathcal{F}_k, Q)_{k=0}^1$ be a discounted stock price, and $H = (H_k, \mathcal{F}_k, Q)_{k=0}^1$ be a strictly positive m.d. with $H_0 = 1$. We fix a family of interpolating filtrations $\mathbf{F} = \{\mathbf{F}^\alpha\}$ indexed by a parameter α, where $\mathbf{F}^\alpha = (\mathcal{F}_n^\alpha)_{n=0}^\infty$ and for each index α the equalities $\mathcal{F}_0^\alpha = \mathcal{F}_0$, $\mathcal{F}_\infty^\alpha = \mathcal{F}_1$ hold. Consider the following martingale interpolations of the deflator H: $H_n^\alpha = E^Q[H_1 \mid \mathcal{F}_n^\alpha]$, $n = 0, 1, 2, \ldots$. On the other hand, let P be a $\mathbf{F}$ -interpolating m.m. of the process Z, that is, for each α the process $Z_n^\alpha = E^P[Z_1 \mid \mathcal{F}_n^\alpha]$, $n = 0, 1, 2, \dots$, admits a unique m.m. (namely, only the measure P). If in the model under consideration the measure P corresponds to H (i.e. $dP = H_1 dQ$), then the generalized Bayes formula implies that $H_n^\alpha Z_n^\alpha = E^Q[H_1 Z_1 \mid \mathcal{F}_n^\alpha]$ for any α, i.e. H_n^α is a m.d. of the process Z_n^α. Since P is the unique m.m. of the process Z_n^α, then H_n^α is the unique m.d. of this process. From the above reasoning the following proposition follows.

Proposition 4.1. *M.d. H of the process Z, corresponding to the m.m. P, is an interpolating m.d. if and only if it satisfies the following uniqueness property: $\forall\alpha$ the process H_n^α is the unique martingale deflator of the process Z_n^α.*

Corollary 4.2. *If $\mathbf{F}$ is equal to the set of all interpolating special Haar filtrations (c.f. section 5), then for a fixed physical measure Q there is a one-to-one correspondence between the set of all interpolating martingale deflators and $\mathcal{SP}(Z, \mathbf{F})$.*

5. Application to interpolation of financial markets

The importance of measures from $\mathcal{SP}(Z, \mathbf{F})$ consists in the following. Transform $P = \{p_k^i,\ 2 < k < r + 1, 1 \le i < m_k + 1\}$ to a sequence $(q_1, q_2, \dots)$. For any permutation $\{k_1, \dots, k_n, \dots\}$ of $\{1, 2, \dots, n, \dots\}$ introduce interpolating special Haar filtration (ISHF) of $\mathbf{F} = (\mathcal{F}_\mathbf{0}, \mathcal{F}_\mathbf{1})$ in the manner:

$\mathcal{H}_0 = \mathcal{F}_0$,
$\mathcal{H}_1 = \sigma\{B_{k_1}\}$,
. . . .
$\mathcal{H}_n = \sigma\{B_{k_1}, B_{k_2}, \dots, B_{k_n}\}$,
. . . .
$\mathcal{H}_\infty = \mathcal{F}_1$.

Let $P \in \mathcal{P}(Z, \mathbf{F})$ and consider $Y_n := E^P[Z_1 | \mathcal{H}_n]$. Then the process $Y = (Y_n, \mathcal{H}_n)_{n=0}^\infty$ is called a special martingale Haar interpolation of Z. We say that $P \in \mathcal{P}(Z, \mathbf{F})$ satisfies uniqueness property if for $\mathbf{F}$ and <u>every</u> ISHF $\mathbf{H} = (\mathcal{H}_\mathbf{n})_\mathbf{n=0}^\infty$ $|\mathcal{P}(Y, \mathbf{H})| = \mathbf{1}$ (Y is a martingale <u>only</u> with respect to the initial measure P). It is easy to see that $P \in \mathcal{P}(Z, \mathbf{F})$ satisfies uniqueness property if and only if $P \in \mathcal{SP}(Z, \mathbf{F})$. This fact can be used in the transformation of arbitrage-free financial (B,S)-markets to complete ones (c.f. [3]– [13]).

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