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ROLE OF ENVIRONMENTAL DEGRADATION ON THE SPREAD OF BACTERIAL DISEASES IN A HUMAN HABITAT: THE EFFECT OF SANITATION

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Abstract. The major factors for the transmission of various infectious diseases such as tuberculosis, typhoid fever, etc. are poor and improper sanitation, lack of hygiene and unclean drinking water. To understand the dynamics of the spread of infectious diseases, we propose a nonlinear mathematical model to study the spread and control of bacterial diseases due to environmental degradation by applying suitable sanitation effort. In the modeling process, the density of the bacteria population is assumed to be governed by a logistic model and is dependent on environmental factors conducive to the growth of the bacterial population. A suitable sanitation effort is applied to mitigate the bacterial population present in the environment. Numerical simulations are also performed to support the analytical findings. The analysis of the model reveals that by increasing the rate of sanitation effort, the bacteria population present in the environment declines which ultimately decreases the infective population.

1. Introduction

Sanitation refers to proper management of human excreta, solid and animal waste. It aims to protect human health by providing a clean environment that reduces the transmission of diseases. Inadequate sanitation is an important factor for the spread of infectious diseases such as Cholera, Typhoid, etc. worldwide. It also contributes to malnutrition, as many people consume food irrigated by wastewater. WHO [29] estimated that inadequate drinking water and poor sanitation facilities cause 502000 and 280000 diarrhea deaths . Improved sanitation is crucial for public health. Access to safe sanitation systems including household discharges, schools, and workplaces is very important in reducing the infectious diseases, improving nutritional outcomes, enhancing safety and educational projects, for women and girls and contributing to overall well-being. Since 1990, the number of people who are gaining the facilities of improved sanitation has raised from 54% to 68% but still, billions of people do not have proper toilets or latrines. In some places, people still defecate in open. The countries where this open defecation is widespread have a large number of deaths of children aged below 5 years as well as

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malnutrition and poverty are at a high level stated by WHO [30]. The UN General Assembly [31] has recognized access to sanitation and safe and clean drinking water as a human right in 2010. They have also asked for international efforts to help other countries in providing them safe, clean drinking water and sanitation.

Bacterial diseases such as Tuberculosis, Typhoid, Meningitis, etc., spread directly through direct transfer of bacteria from one person to another, as an individual with the bacterium touches or coughs or sneezes on someone who is not infected, and it also spread indirectly by touching some objects which are touched by someone ill with a cold or flu. After coming in contact with that object, if a person touches his/her eyes, nose or mouth without washing his/her hands may become infected.

The dynamics of the spread of the bacterial disease has been studied by many mathematician in the past, for instance Anderson and May [1], Gonzalez-Guzman [10], Hethcote [11], Kiss et al. [12], Ugwa et al. [28] and Nadjafikhah and Shagholi [18], considering the direct transmission of disease from one person to another. But the harmful bacteria present in the environment plays a vital role in spreading bacterial diseases, which was later taken into account by many researchers (Naresh and Pandey [19], Shukla et al. [25], Agarwal and Verma [3] and Nthiiri et al. [23]). For example, Naresh and Pandey [19] proposed a nonlinear model to study the cumulative effect of ecological factors in the habitat on the spread of tuberculosis. Their study reveals that an increase in conducive ecological factors and bacteria in the habitat increases the spread of TB. The poor environmental condition is one of the crucial factors in increasing the bacteria population. Due to an increase in pollution, household discharges and many other activities of humans, degradation of the environment increases. The harmful bacteria grow with a rapid rate in this degraded environment resulting in the spread of infectious diseases at a high level (Ghosh et al. [8], Ghosh et al. [9], Naresh and Pandey [20], Mwasa and Tchuenche [17], Naresh and Pandey [21] and Nitu and Sharma [22]). In particular, Ghosh et al. [8] proposed an SIS model to analyze the spread of bacterial diseases. They have extended their model for a socially structured population assuming that the bacteria population only affects the poor class who live in a degraded environment and does not survive in the clean environment of the rich class. They concluded that the increase in the degradation of the environment increases the spread of infectious diseases.

The first measure taken to control the spread of infectious diseases was vaccination. It is the most effective method to prevent the spread of infectious disease (Cui et al. [5], Li and Cui [14], Zhou and Cui [32], and Baafi et al. [4]). But in most cases, it is not much effective and the people who are vaccinated can also get infected. In such cases, the media plays a vital role in controlling the spread of disease by making people aware of the disease. Awareness through media not only influences the people behavior but also makes the government take effective measures on the spread of disease. Keeping, this aspect in view, many researchers like, Cui et al. [5], Misra et al. [15], Samanta et al. [26], Sharma and Misra [27], Dubey et al. [7], Agaba et al. [2], Misra et al. [16] proposed a mathematical model to control the spread of infectious diseases. In particular, Dubey et al. [7] depicted by their nonlinear model that the awareness and treatment both are important to eliminate the disease. Their analysis shows that if awareness is high and proper treatment is available, the infection can be eliminated. Also a nonlinear model was analyzed by Rai et al. [24] considering the importance of sanitation and the awareness programs. Their study reveals that the epidemic can be reduced by the sanitation and awareness programs and thus the spread of infection can be controlled.

To control the spread of infectious diseases several measures such as vaccination, treatment and awareness program among people are taken, but as discussed above proper sanitation plays an important role in curtailing the disease. If our environment is clean, the growth of harmful bacteria will decrease, which consequently decreases the spread of bacterial diseases and hence the infective population. Thus, keeping this aspect, we propose a nonlinear mathematical model to control the spread of bacterial diseases due to environmental degradation by applying proper sanitation effort. In the modeling process, sanitation effort is modeled logistically and assumed to be applied in direct proportion to the environmental degradation.

2. Mathematical Model

The formulation of our mathematical model to control the spread of bacterial diseases due to environmental degradation by applying suitable sanitation effort is done in this section. We assume that the diseases not only spread by direct contact of infectives with susceptibles but also by indirect contact of bacteria present in the environment, and the bacteria population density is controlled by applying proper sanitation effort.

Let us consider a region, at any time 't', in which the total human population $N(t)$ is divided into two subclasses, susceptible population $X(t)$ and infective population $Y(t)$. We assume that the diseases spread through bacteria with density $B(t)$ due to the increase in the cumulative density of environmental degradation $E_m(t)$. Further, we assume that this increase in the spread of bacterial diseases can be controlled by keeping the environment clean by applying suitable sanitation effort $F_s(t)$. The nonlinear mathematical model is proposed as follows,

$$\begin{aligned}
\frac{dX}{dt} &= A - \beta XY - \lambda BX - dX + \nu Y \\
\frac{dY}{dt} &= \beta XY + \lambda BX - dY - \alpha Y - \nu Y \\
\frac{dB}{dt} &= s(E_m)\left(1 - \frac{B}{L(E_m)}\right) B + s_1 Y - s_0 B \\
\frac{dE_m}{dt} &= Q_0 - \theta_0 E_m + \theta_1 (A - dN) - \theta_2 E_m F_s \\
\frac{dF_s}{dt} &= \phi E_m F_s - \phi_0 F_s^2 - \phi_1 E_m F_s + \phi_s F_s
\end{aligned} \tag{2.1}$$

where, $\phi_s = (\psi_1 - \psi_2) > 0$ and $X(0) > 0$, $Y(0) \geq 0$, $B(0) > 0$, $E_m(0) \geq 0$ and $F_s(0) > 0$.

Let us consider that the human population increases, either by birth or immigration at a constant rate A. The parameters β and λ represent the transmission rate of bacterial diseases through direct and indirect contact of susceptibles with infectives and the bacteria present in the environment respectively. The natural mortality rate of the human population is represented by a constant rate d. The parameter ν and α denote the recovery rate and disease-induced death rate of the human population respectively. The intrinsic growth rate and the carrying capacity of bacteria population density are represented by $s(E_m)$ and $L(E_m)$ respectively, which are assumed to be dependent on cumulative density of environmental degradation E_m, s_1 is the rate of release of bacteria from infective population and s_0 is the decay of bacteria population naturally or due to some control measures applied. Since we have assumed that the intrinsic growth rate and the carrying capacity of bacteria population density are dependent on the cumulative density of environmental degradation, we have,

$$s(0) = s \quad \text{and} \quad s'(E_m) \geq 0 \quad \& \quad L(0) = l \quad \text{and} \quad L'(E_m) \geq 0 \tag{2.2}$$

where, s and l are the value of $s(E_m)$ and $L(E_m)$ at E_m=0 respectively, and $s'(E_m)$ and $L'(E_m)$ denotes the derivative of the function with respect to its argument. The cumulative density of environmental degradation conducive to the growth of the bacteria population is assumed to increase at a constant rate Q_0. The parameter θ_1 is the rate of increment of degradation of the environment due to human population-related factors, θ_2 and θ_0 are the rate coefficients of depletion of environmental degradation due to sanitation effort applied and some other natural factors respectively. It is assumed in the model that the sanitation effort is applied in the direct proportion to the environmental degradation (ϕ), $\frac{\phi_s}{\phi_0}$ is its carrying capacity and ϕ_1 is the depletion rate coefficient of sanitation effort due to environmental degradation. The parameter ψ_1 and ψ_2 represent the rate of sanitation effort applied and decrease in sanitation effort due to some other factors respectively.

Since $N(t) = X(t) + Y(t)$, the above model system (2.1) can be rewritten as follows,

$$\begin{aligned}
\frac{dY}{dt} &= \beta(N - Y)Y + \lambda B(N - Y) - (d + \alpha + \nu)Y \\
\frac{dN}{dt} &= A - dN - \alpha Y \\
\frac{dB}{dt} &= s(E_m)\left(1 - \frac{B}{L(E_m)}\right)B + s_1 Y - s_0 B \\
\frac{dE_m}{dt} &= Q_0 - \theta_0 E_m + \theta_1(A - dN) - \theta_2 E_m F_s \\
\frac{dF_s}{dt} &= \phi E_m F_s - \phi_0 F_s^2 - \phi_1 E_m F_s + \phi_s F_s
\end{aligned} \tag{2.3}$$

3. Region of attraction

The region of attraction for the solution of the model system (2.3) is given as follows:

$$\Omega = \{(Y(t), N(t), B(t), E_m(t), F_s(t)) \in R_+^5 : 0 \le Y < N \le \frac{A}{d}, 0 \le B \le B_m,$$
$$0 \le E_m \le (E_m)_m, 0 < F_s \le (F_s)_m\},$$

where,

$$B_m = \frac{L(E_m)_m}{2s(E_m)_m}\left[(s(E_m)_m - s_0) + \sqrt{(s(E_m)_m - s_0)^2 + \frac{4s(E_m)_m}{L(E_m)_m}\frac{s_1 A}{d}}\right],$$
$$(E_m)_m = \frac{Q_0}{\theta_0} \quad \text{and} \quad (F_s)_m = \frac{(\phi - \phi_1)(E_m)_m + \phi_s}{\phi_0}$$

which is positively invariant and all solutions stay in Ω.

Proof. From system (2.3), we have,

$$\frac{dN}{dt} = A - dN - \alpha Y$$
$$\implies \frac{dN}{dt} \le A - dN$$
$$\implies \lim_{t\to\infty} supN(t) \le \frac{A}{d}$$

From the third equation of model system (2.3), and using the fact that $Y(t)\le \frac{A}{d}$ for large $t > 0$, we have

$$\frac{dB}{dt} \le s((E_m)_m)\left(1 - \frac{B}{L((E_m)_m)}\right)B + s_1\frac{A}{d} - s_0 B$$

From the theory of differential inequality [13], we obtain

$$\lim_{t\to\infty} supB(t) \le \frac{L(E_m)_m}{2s(E_m)_m}\left[(s(E_m)_m - s_0) + \sqrt{(s(E_m)_m - s_0)^2 + \frac{4s(E_m)_m}{L(E_m)_m}\frac{s_1 A}{d}}\right]$$
$$= B_m(\text{say}).$$

This implies that $0 \le B(t) \le B_m$ for large $t > 0$.
Further, from the fourth equation of model system (2.3), we obtain

$$\frac{dE_m(t)}{dt} \le Q_0 - \theta_0 E_m$$

From the theory of differential inequality, we have

$$\lim_{t\to\infty} supE_m(t) \le \frac{Q_0}{\theta_0} = (E_m)_m(\text{say}).$$

From the fifth equation of model system (2.3), we have

$$\frac{dF_s}{dt} \le (\phi - \phi_1)(E_m)_m F_s - \phi_0 F_s^2 + \phi_s F_s$$

From the theory of differential inequality, we have

$$\lim_{t\to\infty} sup F_s(t) \le \frac{(\phi-\phi_1)(E_m)_m+\phi_s}{\phi_0} = (F_s)_m(\text{say}).$$

□

4. Equilibrium analysis

Here, in this section, we analyze the qualitative behavior of the model system (2.3) and we obtain four feasible equilibrium points of the model system (2.3) by equating the rate of change of all dynamical variables to zero. Four non-negative equilibria are:

1. E_0 $(0, \frac{A}{d}, 0, \frac{Q_0}{\theta_0}, 0)$. This is disease-free equilibrium.
2. E_1 $(0, \overline{N}, 0, \overline{E}_m, \overline{F}_s)$. Here, $\overline{N}=\frac{A}{d}$
$\overline{E}_m = \frac{-(\phi_0\theta_0+\theta_2\phi_s)+\sqrt{(\phi_0\theta_0+\theta_2\phi_s)^2+4Q_0\phi_0\theta_2(\phi-\phi_1)}}{2\theta_2(\phi-\phi_1)}$ and
$\overline{F}_s = \frac{(\phi-\phi_1)\overline{E}_m+\phi_s}{\phi_0}$
3. E_2 $(\overline{\overline{Y}}, \overline{\overline{N}}, \overline{\overline{B}}, \overline{\overline{E}}_m, 0)$. This is equilibrium without sanitation effort.
4. E_3 $(Y^*, N^*, B^*, E_m^*, F_s^*)$. This is endemic equilibrium.

4.1. Existence of equilibrium E_2 $(\overline{\overline{Y}}, \overline{\overline{N}}, \overline{\overline{B}}, \overline{\overline{E}}_m, 0)$. The values of $\overline{\overline{Y}}$, $\overline{\overline{N}}$, $\overline{\overline{B}}$ and $\overline{\overline{E}}_m$ are obtained by solving the following set of algebraic equations,

$$\beta(N-Y)Y+\lambda B(N-Y)-(d+\alpha+\nu)Y=0 \tag{4.1}$$

$$A-dN-\alpha Y=0 \tag{4.2}$$

$$s(E_m)\left(1-\frac{B}{L(E_m)}\right)B+s_1Y-s_0B=0 \tag{4.3}$$

$$Q_0-\theta_0E_m+\theta_1(A-dN)=0 \tag{4.4}$$

From equation (4.2) we have,

$$N=\frac{A-\alpha Y}{d} \tag{4.5}$$

From equation (4.4) and (4.2) we get,

$$E_m=\frac{Q_0+\theta_1\alpha Y}{\theta_0} \tag{4.6}$$

Now, using equation (4.5) in equation (4.1) we get,

$$(\alpha+d)\beta Y^2-(\beta A-d(d+\alpha+\nu))Y-\lambda AB+(\alpha+d)\lambda BY=0 \tag{4.7}$$

From equation (4.3) we get,

$$Y=\frac{1}{s_1}\left[\frac{s(E_m)}{L(E_m)}B^2-(s(E_m)-s_0)B\right] \tag{4.8}$$

Now we show the existence of $\overline{\overline{Y}}$ and $\overline{\overline{B}}$ from equations (4.7) and (4.8), and the corresponding values of $\overline{\overline{N}}$ and $\overline{\overline{E}}_m$ can be obtained from equations (4.5) and (4.6).
From equation (4.7), we have
(i) For $B = 0$,

$$Y = 0 \quad \text{and} \quad Y = \frac{\beta A - d(d+\alpha+\nu)}{\beta(\alpha+d)} = \tilde{Y}(\text{say}),$$

which is positive, if $\beta A > d(d+\alpha+\nu)$ and negative otherwise.
(ii) At (0, 0), the slope of equation (4.7) is given by,

$$\frac{dY}{dB} = -\frac{\lambda A}{\beta A - d(d+\alpha+\nu)}$$

which is positive or negative depending upon $\tilde{Y}$ being negative or positive, respectively.
(iii) At $(0,\, \tilde{Y})$, the slope of equation (4.7) is given by,

$$\frac{dY}{dB} = \frac{\lambda d(d+\alpha+\nu)}{\beta(\beta A - d(d+\alpha+\nu))}$$

which is positive or negative depending upon $\tilde{Y}$ being positive or negative, respectively.
From equation (4.8), we observe the following points,
(i) When $Y = 0$,

$$B = 0 \quad \text{and} \quad B = \frac{L(E_m)}{s(E_m)}\left(s(E_m) - s_0\right) = \tilde{B} \quad (\text{say})$$

(ii) At (0, 0), the slope of equation (4.8) is given by,

$$\frac{dY}{dB} = -\frac{1}{s_1}\left(s(E_m) - s_0\right) < 0$$

(iii) At $(\tilde{B},\, 0)$, the slope of equation (4.8) is given by,

$$\frac{dY}{dB} = \frac{L^2(E_m)\theta_0(s(E_m)-s_0)}{s_1\theta_0 L^2(E_m) + \theta_1\alpha\tilde{B}\begin{bmatrix} \frac{L(E_m)}{s(E_m)}s_0(L(E_m)s'(E_m) - s(E_m)L'(E_m)) \\ +s(E_m)L(E_m)L'(E_m)\end{bmatrix}} > 0$$

provided $\frac{s'(E_m)}{s(E_m)} > \frac{L'(E_m)}{L(E_m)}$

Thus, after plotting Y and B corresponding to equations (4.7) and (4.8) in Figure 1 and 2, we see that there are two intersecting points (0, 0) and $(\overline{\overline{B}},\, \overline{\overline{Y}})$. After finding $\overline{\overline{Y}}$ and $\overline{\overline{B}}$, we can calculate $\overline{\overline{N}}$ and $\overline{\overline{E}}_m$ using equations (4.5) and (4.6).

4.2. Existence of equilibrium E_3 $(Y^*,\, N^*,\, B^*,\, E_m^*,\, F_s^*)$. We prove the existence of equilibrium E_3 by setting right hand side of equations in the model system (2.3) to zero and by solving the resulting algebraic equations, as given below,

$$\beta(N-Y)Y + \lambda B(N-Y) - (d+\alpha+\nu)Y = 0 \tag{4.9}$$

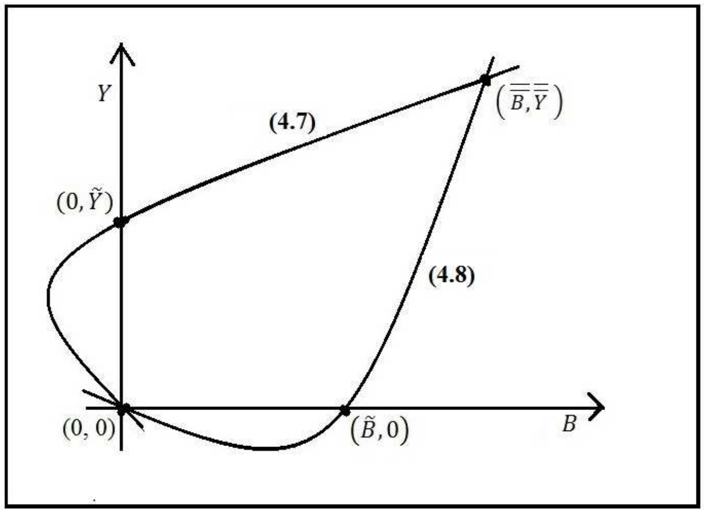

FIGURE 1. Existence of the equilibrium E_2 when $\tilde{Y} > 0$

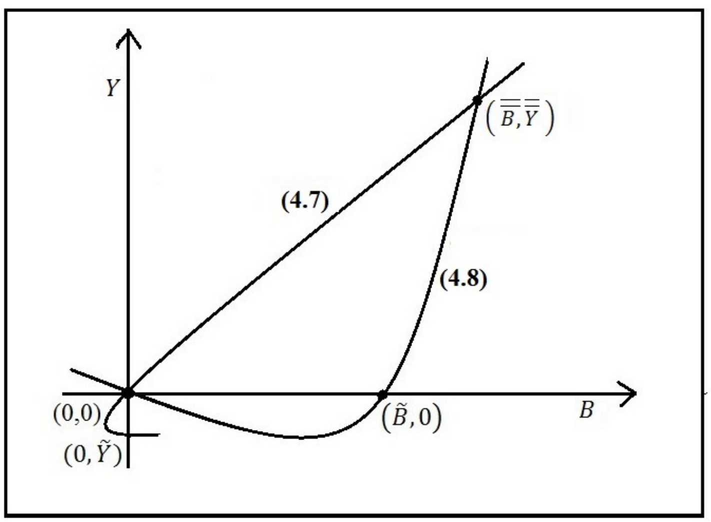

FIGURE 2. Existence of the equilibrium E_2 when $\tilde{Y} < 0$

$$A - dN - \alpha Y = 0 \tag{4.10}$$

$$s(E_m)\left(1 - \frac{B}{L(E_m)}\right)B + s_1 Y - s_0 B = 0 \tag{4.11}$$

$$Q_0 - \theta_0 E_m + \theta_1 (A - dN) - \theta_2 E_m F_s = 0 \tag{4.12}$$

$$\phi E_m F_s - \phi_0 F_s^2 - \phi_1 E_m F_s + \phi_s F_s = 0 \tag{4.13}$$

From equation (4.10) and (4.13) we have,

$$N = \frac{A - \alpha Y}{d} \tag{4.14}$$

$$F_s = \frac{(\phi - \phi_1)E_m + \phi_s}{\phi_0} \tag{4.15}$$

Now using equations (4.10) and (4.15) in equation (4.12) we have,

$$E_m = \frac{-(\theta_0\phi_0 + \theta_2\phi_s) + \sqrt{(\theta_0\phi_0 + \theta_2\phi_s)^2 + 4\theta_2\phi_0(\phi - \phi_1)(Q_0 + \theta_1\alpha Y)}}{2\theta_2(\phi - \phi_1)} = f(y) \tag{4.16}$$

Using the value of N from equation (4.14) in equation (4.9) we get,

$$(\alpha + d)\beta Y^2 - (\beta A - d(d + \alpha + \nu))Y - \lambda AB + (\alpha + d)\lambda BY = 0 \tag{4.17}$$

From equation (4.11) we get,

$$Y = \frac{1}{s_1}\left[\frac{s(E_m)}{L(E_m)}B^2 - (s(E_m) - s_0)B\right] \tag{4.18}$$

From equation (4.17), we have
(i) For $B = 0$,

$$Y = 0 \quad \text{and} \quad Y = \frac{\beta A - d(d + \alpha + \nu)}{\beta(\alpha + d)} = Y_1 \quad \text{(say)},$$

which is positive, if $\beta A > d(d + \alpha + \nu)$ and negative otherwise.
(ii) At (0, 0), the slope of equation (4.17) is given by,

$$\frac{dY}{dB} = -\frac{\lambda A}{\beta A - d(d + \alpha + \nu)}$$

which is positive or negative depending upon Y_1 being negative or positive, respectively.
(iii) At (0, Y_1), the slope of equation (4.17) is given by,

$$\frac{dY}{dB} = \frac{\lambda d(d + \alpha + \nu)}{\beta(\beta A - d(d + \alpha + \nu))}$$

which is positive or negative depending upon Y_1 being positive or negative, respectively.
From eq. (4.18), we observe the following points,
(i) When $Y = 0$,

$$B = 0 \quad \text{and} \quad B = \frac{L(E_m)\,(s(E_m) - s_0)}{s(E_m)} = B_1 \quad \text{(say)}$$

(ii) At (0, 0), the slope of equation (4.18) is given by,

$$\frac{dY}{dB} = -\frac{1}{s_1}\,(s(E_m) - s_0) < 0$$

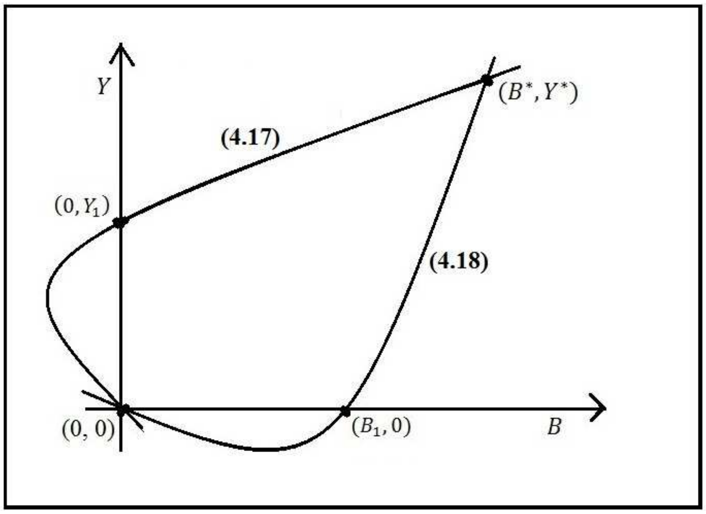

FIGURE 3. Existence of endemic equilibrium E_3 when $Y_1 > 0$

(iii) At $(B_1, 0)$, the slope of equation (4.18) is given by,

$$\frac{dY}{dB} = \frac{L^2(E_m)(s(E_m) - s_0)}{s_1 L^2(E_m) + B_1 f'(y)\left[\begin{array}{r}\frac{L(E_m)}{s(E_m)} s_0(L(E_m)s'(E_m) - s(E_m)L'(E_m)) \\ +s(E_m)L(E_m)L'(E_m)\end{array}\right]} > 0$$

provided $\frac{s'(E_m)}{s(E_m)} > \frac{L'(E_m)}{L(E_m)}$

Thus, after plotting Y and B corresponding to equation (4.17) and (4.18) in Figure 3 and 4, we see that there are two intersecting points (0, 0) and (B^*, Y^*). After finding Y^* and B^* , we can calculate N^*, E_m^* and F_s^* using equation (4.14), (4.16) and (4.15) respectively.

5. Stability Analysis

The stability behavior of equilibrium points E_0, E_1, E_2 and E_3 is presented here. The stability behavior of E_0, E_1 and E_2 is analyzed by computing variational matrix and that of E_3 is analyzed by Lyapunov method.

Theorem 5.1. *The equilibria E_0, E_1 and E_2 are unstable and the endemic equilibrium E_3 is locally asymptotically stable provided the following conditions are satisfied,*

$$(\lambda(N^* - Y^*) + s_1)^2 < \left(\beta Y^* + \frac{\lambda B^* N^*}{Y^*}\right)\left(\frac{s_1 Y^*}{B^*} + \frac{s(E_m^*)}{L(E_m^*)} B^*\right) \tag{5.1}$$

$$\theta_1^2(\phi - \phi_1) d F_s^* < \frac{2}{3}\frac{(\beta Y^* + \lambda B^*)}{\alpha}(\theta_0 + \theta_2 F_s^*)\theta_2 E_m^* \tag{5.2}$$

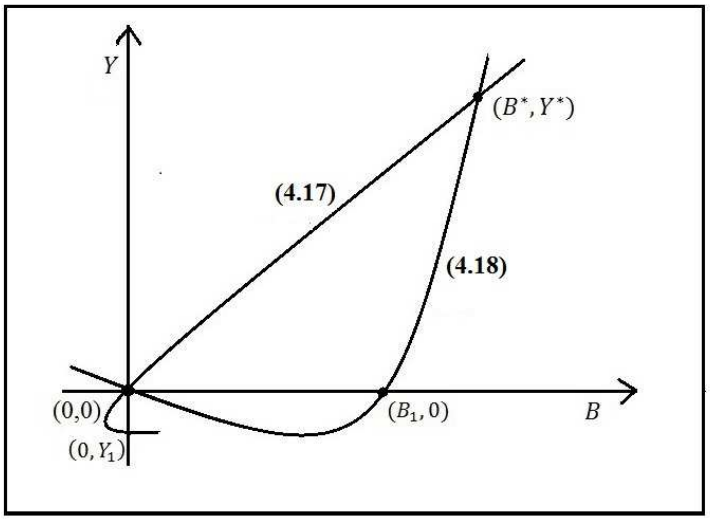

FIGURE 4. Existence of endemic equilibrium E_3 when $Y_1 < 0$

$$\theta_2 E_m^* \left(s'(E_m^*)B^* - \frac{(L(E_m^*)s'(E_m^*) - s(E_m^*)L'(E_m^*))}{L^2(E_m^*)} B^{*2} \right)^2 < \frac{2}{3}(\phi - \phi_1)F_s^*(\theta_0 + \theta_2 F_s^*) * \left(\frac{s_1 Y^*}{B^*} + \frac{s(E_m^*)}{L(E_m^*)} B^* \right) \tag{5.3}$$

Proof. The variational matrix M_0 of model system (2.3) corresponding to the equilibrium point $E_0(0, \frac{A}{d}, 0, \frac{Q_0}{\theta_0}, 0)$ is given by,

$$M_0 = \begin{bmatrix} \frac{\beta A}{d} - (d+\alpha+\nu) & 0 & \frac{\lambda A}{d} & 0 & 0 \\ -\alpha & -d & 0 & 0 & 0 \\ s_1 & 0 & s(\frac{Q_0}{\theta_0}) - s_0 & 0 & 0 \\ 0 & -\theta_1 d & 0 & -\theta_0 & \frac{-\theta_2 Q_0}{\theta_0} \\ 0 & 0 & 0 & 0 & (\phi - \phi_1)\frac{Q_0}{\theta_0} + \phi_s \end{bmatrix}$$

The characteristic equation corresponding to the above matrix is given by,

$$(d+\mu)(\theta_0+\mu)((\phi-\phi_1)\frac{Q_0}{\theta_0} + \phi_s - \mu)(\mu^2 - h_1\mu - h_2) = 0$$

where,
$h_1 = (\beta A - (d+\alpha+\nu)) + (s(\frac{Q_0}{\theta_0}) - s_0)$
$h_2 = (\beta A - (d+\alpha+\nu))(s(\frac{Q_0}{\theta_0}) - s_0) - \frac{\lambda A s_1}{d}$
From the above, it can be seen that one of the root of the characteristic equation is positive $((\phi-\phi_1)\frac{Q_0}{\theta_0} + \phi_s)$, therefore the equilibrium E_0 is unstable.
The variational matrix M_1 of model system (2.3) corresponding to the equilibrium point $E_1(0, \overline{N}, 0, \overline{E}_m, \overline{F}_s)$ is given by,

$$M_1 = \begin{bmatrix} \beta\overline{N} - (d+\alpha+\nu) & 0 & \lambda\overline{N} & 0 & 0 \\ -\alpha & -d & 0 & 0 & 0 \\ s_1 & 0 & s(\overline{E}_m) - s_0 & 0 & 0 \\ 0 & -\theta_1 d & 0 & -(\theta_0 + \theta_2\overline{F}_s) & -\theta_2\overline{E}_m \\ 0 & 0 & 0 & (\phi-\phi_1)\overline{F}_s & \omega \end{bmatrix}$$

where, $\omega = (\phi - \phi_1)\overline{E}_m - 2\phi_0\overline{F}_s + \phi_s$
The characteristic equation corresponding to the above matrix is given by,

$$(\mu^2 + l_1\mu + l_2)(\mu^2 - m_1\mu - m_2) = 0$$

where,
$l_1 = \theta_0 + \theta_2\overline{F}_s - (\phi - \phi_1)\overline{E}_m + 2\phi_0\overline{F}_s - \phi_s$
$l_2 = \theta_2(\phi - \phi_1)\overline{E}_m\overline{F}_s - (\theta_0 + \theta_2\overline{F}_s)((\phi - \phi_1)\overline{E}_m - 2\phi_0\overline{F}_s + \phi_s)$
$m_1 = (\beta\overline{N} - (d+\alpha+\nu)) + (s(\overline{E}_m) - s_0)$
$m_2 = \lambda\overline{N}s_1 - (\beta\overline{N} - (d+\alpha+\nu))(s(\overline{E}_m) - s_0)$
By Descarte's rule of sign, it is found that the equilibrium E_1 is unstable.
The variational matrix M_2 of model (2.3) corresponding to $E_2(\overline{\overline{Y}}, \overline{\overline{N}}, \overline{\overline{B}}, \overline{\overline{E}}_m, 0)$ is given by,

$$M_2 = \begin{bmatrix} \upsilon & \beta\overline{\overline{Y}} + \lambda\overline{\overline{B}} & \lambda(\overline{\overline{N}} - \overline{\overline{Y}}) & 0 & 0 \\ -\alpha & -d & 0 & 0 & 0 \\ s_1 & 0 & \xi & s'(\overline{\overline{E}}_m)\overline{\overline{B}} - \overline{\overline{B}}^2 p & 0 \\ 0 & -\theta_1 d & 0 & \theta_0 & -\theta_2\overline{\overline{E}}_m \\ 0 & 0 & 0 & 0 & (\phi-\phi_1)\overline{\overline{E}}_m + \phi_s \end{bmatrix}$$

where, $\upsilon = \beta\overline{\overline{N}} - 2\beta\overline{\overline{Y}} - \lambda\overline{\overline{B}} - (d+\alpha+\nu)$, $\xi = s(\overline{\overline{E}}_m) - \frac{2s(\overline{\overline{E}}_m)}{L(\overline{\overline{E}}_m)}\overline{\overline{B}} - s_0$ and

$$p = \left(\frac{L(\overline{\overline{E}}_m)s'(\overline{\overline{E}}_m) - s(\overline{\overline{E}}_m)L'(\overline{\overline{E}}_m)}{L^2(\overline{\overline{E}}_m)}\right)$$

Since, one of the root of the characteristic equation corresponding to the above matrix $((\phi - \phi_1)\overline{\overline{E}}_m + \phi_s) > 0$ therefore, the equilibrium E_2 is unstable.
To compute the local stability of endemic equilibrium E_3, we linearize the model system (2.3) using small perturbations y, n, b, e_m and f_s about E_3, defined as

$$Y = y + Y^*, N = n + N^*, B = b + B^*, E_m = e_m + E_m^* \text{ and } F_s = f_s + F_s^*.$$

Consider the following positive definite function:

$$U_1 = \frac{1}{2}(m_0y^2 + m_1n^2 + m_2b^2 + m_3e_m^2 + m_4f_s^2),$$

where, m_i (i = 0, 1, 2, 3, 4) are positive constants to be chosen appropriately. Differentiating above equation w.r.t t and using linearized system of model (2.3) corresponding to E_3, we get,

$$\frac{dU_1}{dt} = -m_0\left(\frac{\lambda B^* N^*}{Y^*} + \beta Y^*\right) y^2 - m_1 dn^2 - m_2\left(\frac{s_1 Y^*}{B^*} + \frac{s(E_m^*)}{L(E_m^*)} B^*\right) b^2$$
$$- m_3(\theta_0 + \theta_2 F_s^*) e_m^2 - m_4 \phi_0 F_s^* f_s^2$$
$$+ [m_0(\beta Y^* + \lambda B^*) - m_1 \alpha] ny + [m_0 \lambda (N^* - Y^*) + m_2 s_1] by$$
$$+ m_2\left(s'(E_m^*) B^* - \frac{L(E_m^*) s'(E_m^*) - L'(E_m^*) s(E_m^*)}{L^2(E_m^*)} B^{*2}\right) be_m$$
$$- m_3 \theta_1 dne_m + (-m_3 \theta_2 E_m^* + m_4(\phi - \phi_1) F_s^*) f_s e_m$$

After choosing $m_0 = 1$, $m_1 = \dfrac{\beta Y^* + \lambda B^*}{\alpha}$, $m_2 = 1$, $m_3 = \dfrac{(\phi - \phi_1) F_s^*}{\theta_2 E_m^*}$ and $m_4 = 1$. we get $\dfrac{dU_1}{dt}$ to be negative definite showing that U_1 is a Lyapunov function and hence E_3 is locally asymptotically stable provided the conditions (5.1)-(5.3) are satisfied. □

Theorem 5.2. *The endemic equilibrium E_3 is nonlinearly asymptotically stable in the region Ω provided the following conditions are satisfied.*

$$4\alpha\left(\beta + \frac{\lambda B_m}{Y^*}\right) < \beta d \tag{5.4}$$

$$\left(\frac{\lambda(N^* - Y^*)}{Y^*} + \frac{s_1}{B^*}\right)^2 < \frac{\beta s(E_m^*)}{L(E_m^*)} \tag{5.5}$$

$$\left(\left(1 - \frac{B_m}{L(E_m^*)}\right) p + s((E_m)_m) B_m \frac{q}{L_0^2}\right)^2 < \frac{2}{3} \frac{(\phi - \phi_1)}{\theta_2 (E_m)_m} \frac{s(E_m^*)}{L(E_m^*)} (\theta_0 + \theta_2 F_s^*) \tag{5.6}$$

$$(\phi - \phi_1)\theta_1^2 \alpha d < \frac{2}{3}\left(\beta + \frac{\lambda B_m}{Y^*}\right)(\theta_0 + \theta_2 F_s^*)\theta_2 (E_m)_m \tag{5.7}$$

$$\theta_2 (E_m)_m (\phi - \phi_1) < \frac{1}{3}\phi_0(\theta_0 + \theta_2 F_s^*) \tag{5.8}$$

Proof. Consider the following positive definite function, corresponding to the model system (3) about E_3,

$$U_2 = k_0\left(Y - Y^* - Y^* ln\frac{Y}{Y^*}\right) + \frac{k_1}{2}(N - N^*)^2 + k_2(B - B^* - B^* ln\frac{B}{B^*}) +$$
$$\frac{k_3}{2}(E_m - E_m^*)^2 + k_4(F_s - F_s^* - F_s^* ln\frac{F_s}{F_s^*})$$

where, k_i (i = 0, 1, 2, 3, 4) are positive constants to be chosen appropriately. Differentiating the above equation w.r.t t and using system (3), we get,

$$\begin{aligned}
\frac{dU_2}{dt} &= -k_0\left(\beta + \frac{\lambda BN}{YY^*}\right)(Y - Y^*)^2 - k_1 d(N - N^*)^2 \\
&- k_2\left(\frac{s(E_m^*)}{L(E_m^*)} + \frac{s_1 Y}{BB^*}\right)(B - B^*)^2 - k_3(\theta_0 + \theta_2 F_s)(E_m - E_m^*)^2 \\
&- k_4\phi_0(F_s - F_s^*)^2 + \left[k_0(\beta + \frac{\lambda B}{Y^*}) - k_1\alpha\right](Y - Y^*)(N - N^*) \\
&+ \left[k_0\lambda\left(\frac{N^* - Y^*}{Y^*}\right) + k_2\frac{s_1}{B^*}\right](Y - Y^*)(B - B^*) \\
&+ k_2\left[\left(1 - \frac{B}{L(E_m^*)}\right)f(E_m) + Bs(E_m)g(E_m)\right](E_m - E_m^*)(B - B^*) \\
&- k_3\theta_1 d(N - N^*)(E_m - E_m^*) \\
&+ \left[-k_3\theta_2 E_m^* + k_4(\phi - \phi_1)\right](E_m - E_m^*)(F_s - F_s^*)
\end{aligned}$$

where $f(E_m)$ and $g(E_m)$ are defined as follows,

$$f(E_m) = \begin{cases} \dfrac{s(E_m) - s(E_m^*)}{E_m - E_m^*}, & E_m \neq E_m^* \\ \dfrac{ds}{dE_m}, & E_m = E_m^* \end{cases}$$

$$g(E_m) = \begin{cases} \dfrac{L(E_m) - L(E_m^*)}{L(E_m)L(E_m^*)(E_m - E_m^*)}, & E_m \neq E_m^* \\ \dfrac{1}{L_0^2}\dfrac{dL}{dE_m}, & E_m = E_m^* \end{cases}$$

By considering the assumptions of the theorem and the mean value theorem, we have,

$$|f(E_m)| \leq p, |g(E_m)| \leq \frac{q}{L_0^2}$$

After choosing $k_0 = 1$, $k_1 = \frac{1}{\alpha}\left(\beta + \frac{\lambda B_m}{Y^*}\right)$, $k_2 = 1$, $k_3 = \frac{(\phi - \phi_1)}{\theta_2 (E_m)_m}$ and $k_4 = 1$ we get $\frac{dU_2}{dt}$ to be negative definite showing that U_2 is a Lyapunov function and hence E^* is nonlinearly asymptotically stable provided the conditions (5.4)-(5.8) are satisfied. □

6. Numerical Simulation

In this section, the numerical simulation of the model system (2.3) using MATLAB is performed. In our model, it is considered that the intrinsic growth rate ($s(E_m)$) and the carrying capacity ($L(E_m)$) of bacteria population density are the functions of cumulative density of environmental degradation E_m. Thus, for numerical simulation ($s(E_m)$) and ($L(E_m)$) are assumed to be linear function of E_m i.e., $s(E_m) = s + aE_m$ and $L(E_m) = l + bE_m$ satisfying condition (2.2).

The following set of parameter values are used in numerical simulation.

$A = 100$, $\beta = 0.002$, $\lambda = 0.000005$, $\nu = 0.02$, $d = 0.15$, $\alpha = 0.2$, $s = 0.85$, $s_0 = 0.3$, $s_1 = 0.0001$, $Q_0 = 25$, $\theta_1 = 0.001$, θ_2=0.0004, θ_0=0.1, a=0.001, l=10000, b=0.01, $\phi = 0.5$, $\phi_0 = 0.26$, $\phi_1 = 0.004$ $\psi_1 = 0.3$, $\psi_2 = 0.003$.

The equilibrium values of endemic equilibrium E_3 for the above set of parameter values are obtain as,

$Y^* = 212.408$, $N^* = 383.456$, $B^* = 6930.038$, $E_m^* = 126.921$ and $F_s^* = 243.269$

The eigenvalues corresponding to the Jacobian matrix of endemic equilibrium E_3 are:

-63.1528, -0.3209 $\pm$ 0.2538i , -0.2879 and -0.6792

It is noted here that three eigenvalues are negative and two eigenvalues have a negative real part, therefore, for the above set of parameter values the endemic equilibrium E_3 is locally asymptotically stable. The results of the model analysis are displayed graphically in Figs. 5-12. In Fig. 5, four different values of the total human population (N), infective population (Y) and bacteria population density (B) are considered. It is seen from the figure that all trajectories starting from different initial values approach to the equilibrium point. This shows that the endemic equilibrium E_3 is nonlinearly asymptotically stable. The initial starts of all trajectories to reach the equilibrium point are given below :

(1)	$Y(0) = 100$	$N(0) = 300$	$B(0) = 7000$	$E_m = 120$	$F_s(0) = 250$
(2)	$Y(0) = 300$	$N(0) = 600$	$B(0) = 5500$	$E_m = 120$	$F_s(0) = 250$
(3)	$Y(0) = 300$	$N(0) = 350$	$B(0) = 5000$	$E_m = 120$	$F_s(0) = 250$
(4)	$Y(0) = 100$	$N(0) = 550$	$B(0) = 4000$	$E_m = 120$	$F_s(0) = 250$

The variation of density of bacteria population and the infective human population with time is shown in Fig. 6 and 7 respectively for different values of s_1, the rate of release of bacteria from infective human population. It is seen that the density of bacteria population increases with increase in the rate of release of bacteria from infective human population (Fig.6). This increase in the density of bacterial population in the environment further increases the infective human population (Fig.7). Moreover, the increase in infective human population is more visible if the contact rate of susceptibles with bacteria population present in the environment is slightly enhanced.

Fig. 8 shows the variation of infective human population with time for distinct values of λ, the transmission rate of bacterial diseases due to indirect contact of susceptibles with bacteria present in the environment. It is seen from this figure that with increase in the contact rate of susceptibles with bacteria population, the infective human population increases. This increase in infective population is due to increased cumulative density of environmental degradation which provides conducive atmosphere for the growth of bacterial population. Thus, the spread of bacterial diseases can be controlled if some suitable sanitation strategies are applied to keep the environment clean thereby reducing the density of bacteria population.

In Figs.9 and 10, the variation of bacteria population density and infective human population is presented with time for different values of θ_2, the rate of depletion of environmental degradation due to applied sanitation effort. It is found that the density of bacteria population declines with increase in the depletion rate coefficient of environmental degradation due to sanitation effort applied (Fig.9).

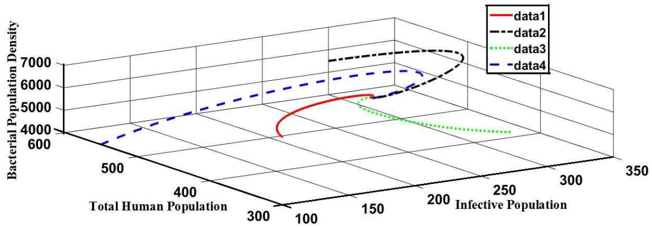

FIGURE 5. Variation of total human population with infective population and bacteria population

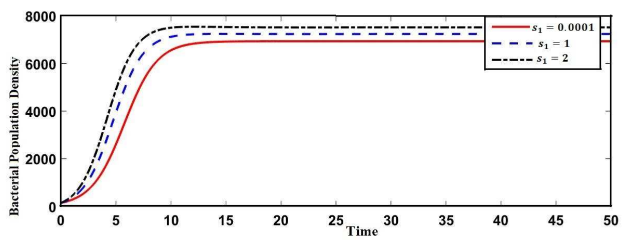

FIGURE 6. Variation of bacteria population with time for distinct values of s_1

This reduction in the density of bacteria population, as a result of sanitation effort applied, ultimately decreases the infective human population (Fig.10).

The effect of sanitation effort is explicitly shown in Fig. 11 and 12 on bacteria population density and infective human population with time, respectively, for different values of ϕ, the rate of sanitation effort applied which is assumed to be in direct proportion to density of environmental degradation. It is seen from Fig.11 that the bacteria population density decreases with increase in the rate of sanitation effort applied. This decreased bacteria population density due to increased sanitation effort ultimately reduces the infective human population (Fig.12). Since the degraded environmental conditions provide conducive breeding ground for the growth of bacteria population density, a suitable sanitation effort is to be applied to keep the environment clean and to curtail the bacterial population density. Thus, the role of sanitation effort can be of vital importance to keep the spread of bacterial diseases under control.

7. Conclusion

In this paper, we have proposed and analyzed a nonlinear mathematical model to study the role of sanitation in a human habitat to control the spread of bacterial diseases caused due to environmental degradation. In the modeling process, both direct and indirect transmission of the disease is considered and the growth rate of bacteria population density is modeled logistically, its intrinsic growth rate and carrying capacity are assumed to be dependent on conducive environmental

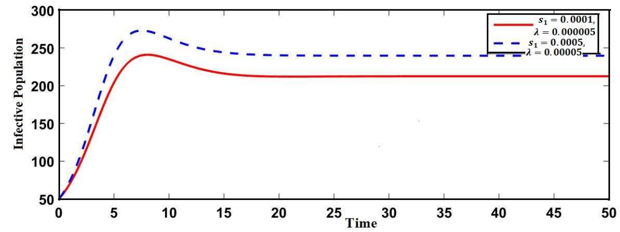

FIGURE 7. Variation of infective population with time for distinct values of s_1 and λ

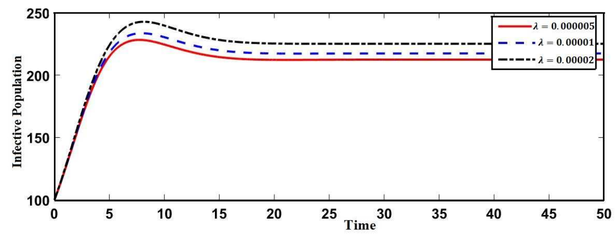

FIGURE 8. Variation of infective population with time for distinct values of λ

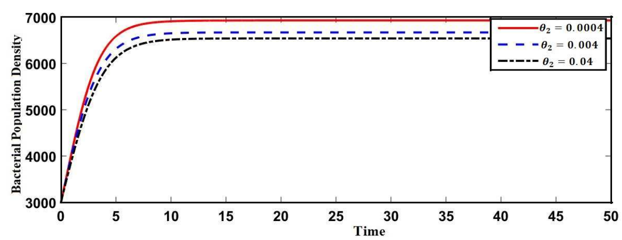

FIGURE 9. Variation of bacteria population with time for distinct values of θ_2

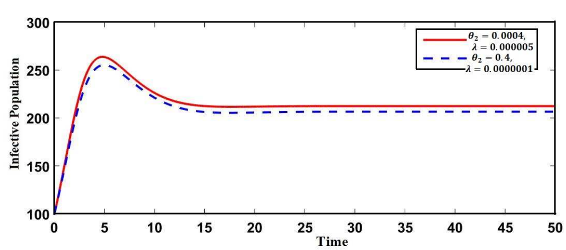

FIGURE 10. Variation of infective population with time for distinct values of θ_2 and λ

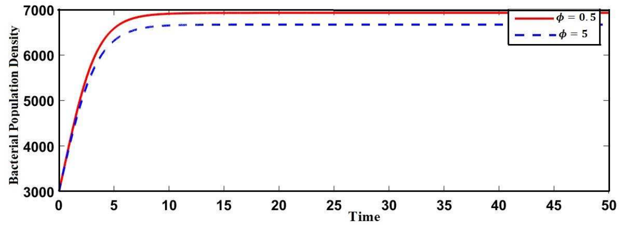

FIGURE 11. Variation of bacteria population with time for distinct values of ϕ

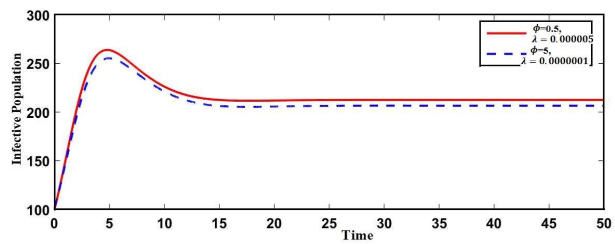

FIGURE 12. Variation of infective population with time for distinct values of ϕ and λ

degradation. The growth rate of bacteria population density is also assumed to be directly proportional to the infective population. The cumulative density of environmental degradation depends upon human population-related factors. To decline the growth of the bacterial population present in the environment, sanitation effort is applied which is modeled logistically. The proposed model has been analyzed qualitatively using stability theory. The model exhibits four non-negative equilibria whose stability is studied. The endemic equilibrium is found to be locally and nonlinearly asymptotically stable under certain conditions. The model has also been studied numerically. It has been found that if the transmission rate of bacterial diseases through indirect contact of susceptibles with bacteria present in the degraded environment increases, the infective human population increases. However, if a suitable sanitation effort is applied to keep the environment clean, the density of bacteria population declines leading to control the spread of bacterial diseases.

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