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EXTREMAL SOLUTION TO GENERALIZED DIFFERENTIAL EQUATIONS UNDER INTEGRAL BOUNDARY CONDITION

MIGUEL VIVAS-CORTEZ*, OSWALDO J. LARREAL B.,
AND JUAN E. NÁPOLES VALDÉS

Abstract. In this article, by utilizing the monotone iterative strategy coupled with the strategy of upper and lower solutions, we get the existence of extremal iteration solution to generalized differential equations under boundary conditions of type Riemann-Stieltjes, which at the same time provides the principle of comparison for these solutions. The results obtained are illustrated with appropriate examples.

♣ Note to author: Use 2000 Mathematics Subject Classification.

1. Preliminaries

The notion of derivative is in the very essence of Ordinary Calculus, and it has attracted attention of many researchers and mathematicians such as Newton, L'Hospital, Leibniz, Abel, Euler, Riemann, etc. Later, several types of fractional derivatives, what will we denote D^α, have been introduced to date Euler, Riemann–Liouville, Abel, Fourier, Caputo, Hadamard, Grunwald–Letnikov, Miller–Ross, Riesz among others, extended the derivative concept to fractional order derivative (see [10], [13] and [14]).

In recent years, Fractional and Generalized Calculus has received a lot of the attention, not only in Pure Mathematics, but in multiple fields of applied science. Between its own theoretical development and the multiplicity of applications, the field has grown rapidly in recent years, in such a way that a single definition of the fractional derivative or integral does not exist, or at least is not unanimously accepted, in [2] suggests and justifies the idea of a fairly complete classification of the known operators of the Generalized and Fractional Calculus, we can also point out that in the work the authors study this phenomenon and support the appearance of various operators, both in theoretical and practical research. Let us point out that these developments have been obtained in different contexts, and not with a single starting point, that is, they are taken as a basis, from the Riemann-Liouville integral, to that of Katugampola, through other formulations such as Weyl's, Hadamard, or Erdelyi-Kober, in this way various authors have defined different integral operators, even from different notions of generalized local derivatives, this last point of view is the one present in our work.

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We believe that it is convenient to take into account the historical route that is presented in Chapter 1 of [1] where a historic route of differential operators, whether local or global, is made, starting from Newton's classical formulation and arriving at Caputo's Definition, which serves as the basis for presenting a differential operator, with a new parameter, and providing a great variety of applications, taking into account the difference between both types of differential operators, global and local. A seminal question is addressed in 1.5.1 (p.24), where sentence "We can therefore conclude that both the Riemann-Liouville and Caputo operators are not derivatives, and then they are not fractional derivatives, but fractional operators. We agree with the result [15] that, the local fractional operator is not a fractional derivative" (p.24). For all the above, we can affirm that there is a great variety of integral operators, which have proven their usefulness in solving a great variety of applications and in successive theoretical developments.

Classical fractional differential operators have a group of known deficiencies, although local operators appeared in the 60s, it was not until 2014 that these disadvantages were overcome when Khalil et al. [9], defined and formalized the operators using the classic idea of the limit of a certain incremental quotient and obtained a derivative that was called conformable, in 2018, a new work direction was opened when what was called non-conformable was introduced (see [6] and [11]). These differential (local) operators have proven their usefulness in multiple applications, for example [2, 3, 8, 4, 5, 12, 7].

Motivated by the above works, we consider the existence of solutions for the following nonlinear conformable fractional differential equation involving integral boundary condition, using the method of upper and lower solutions and its associated monotone iterative technique

$$\begin{cases} \mathcal{N}_1^{\alpha} u(t) = f(t, u(t)),\ t \in (0,1) \\ u(0) = \displaystyle\int_0^1 u(t) d\mu(t) \end{cases} \tag{1.1}$$

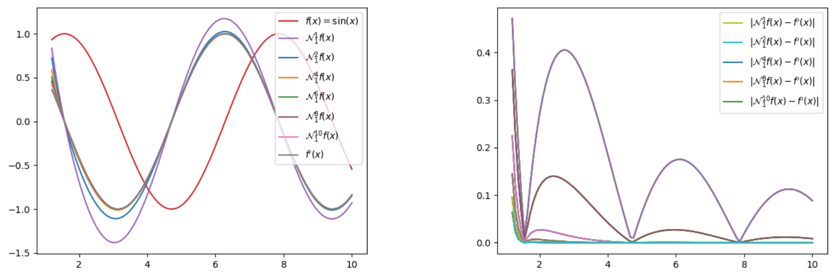
where $f \in C((0,1) \times \mathbb{R}, \mathbb{R}), \int_0^1 x(t) d\mu(t)$ denotes the Riemann-Stieltjes integral with positive Stieltjes measure of μ, and $\mathcal{N}_1^{\alpha} f(t)$ stands for the $\mathcal{N}$-derivative.

Definition 1.1. Let $f : [0, +\infty) \to \mathbb{R}$ a function. Then the $\mathcal{N}$-derivative of f of order α is defined by $\mathcal{N}_1^{\alpha} f(t) = \lim_{h \to 0} \dfrac{f\left(t + he^{t^{-\alpha}}\right) - f(t)}{h}$ for all $t > 0, \alpha \in (0,1)$. If f is $\alpha-$differentiable in some $(0, a)$, and $\lim_{t \to 0^+} \mathcal{N}_1^{(\alpha)} f(t)$ exists, then define $\mathcal{N}_1^{(\alpha)} f(0) = \lim_{t \to 0^+} \mathcal{N}_1^{(\alpha)} f(t)$

Lemma 1.2. *Let $f : [0, +\infty) \to \mathbb{R}$ be differentiable, then*

$$\mathcal{N}_1^{\alpha} f(t) = e^{t^{-\alpha}} f'(t)$$

Theorem 1.3 (Mean value theorem)**.** *Let $[a, b] \subset [0, +\infty)$, and let $f : [0, +\infty) \to \mathbb{R}$. Suppose that Let f be a continuous on $[a, b]$ and suppose that f is differentiable*



(A) Graphics $f(x) = \sin(x)$, $\mathcal{N}_1^\alpha f(x)$ for $\alpha = 1, 2, 4, 6, 8, 10$ and $f'(x)$.

(B) Graphics absolute error between $f'(x)$ and $\mathcal{N}_1^\alpha f(x)$.

FIGURE 1. Graphics of comparison between $f'(x)$ and $\mathcal{N}_1^\alpha f(x)$.

on $[a, b]$. *Then there exists a constant* $\xi \in (a, b)$, *such that*

$$e^{-\xi^{-\alpha}} \mathcal{N}_1^\alpha f(\xi) = \frac{f(b) - f(a)}{b - a}$$

Proof. By the lemma 1.2 we know that $\mathcal{N}_1^\alpha f(t) = e^{t^{-\alpha}} f'(t)$, so

$$f(t) = \int_0^t \mathcal{N}_1^\alpha f(\tau) e^{-\tau^{-\alpha}} \mathrm{d}\tau$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \mathcal{N}_1^\alpha f(\xi) e^{-\xi^{-\alpha}}$$

□

Definition 1.4. Let $f : [0, +\infty) \to \mathbb{R}$ be differentiable, the inverse of the $\mathcal{N}$-derivative is defined as

$$\mathcal{N}_{-1}^\alpha F(s) = \int_0^s e^{-\tau^{-\alpha}} F(\tau) d\tau$$

Lemma 1.5. *Let* $f : [0, +\infty) \to \mathbb{R}$ *be differentiable and* $0 < \alpha < 1$. *Then, for all* $t > 0$ *we have*

$$\mathcal{N}_{-1}^\alpha \mathcal{N}_1^\alpha f(t) = f(t) - f(0) \tag{1.2}$$

$$\mathcal{N}_1^\alpha \mathcal{N}_{-1}^\alpha f(s) = f(s) \tag{1.3}$$

1.1. Properties. Let f and g be N -differentiable at a point $t > 0$ and $\alpha \in (0, 1]$. Then

(1) $\mathcal{N}_1^\alpha (af + bg)(t) = a\mathcal{N}_1^\alpha (f)(t) + b\mathcal{N}_1^\alpha (g)(t)$
(2) $\mathcal{N}_1^\alpha (t^q) = e^{t^{-\alpha}} q t^{q-1}, q \in \mathbb{R}$.
(3) $\mathcal{N}_1^\alpha (C) = 0$, C constant.
(4) $\mathcal{N}_1^\alpha (fg)(t) = f\mathcal{N}_1^\alpha (g)(t) + g\mathcal{N}_1^\alpha (f)(t)$
(5) $\mathcal{N}_1^\alpha \left(\frac{f}{g}\right)(t) = \frac{g\mathcal{N}_1^\alpha (f)(t) - f\mathcal{N}_1^\alpha (g)(t)}{g^2(t)}$

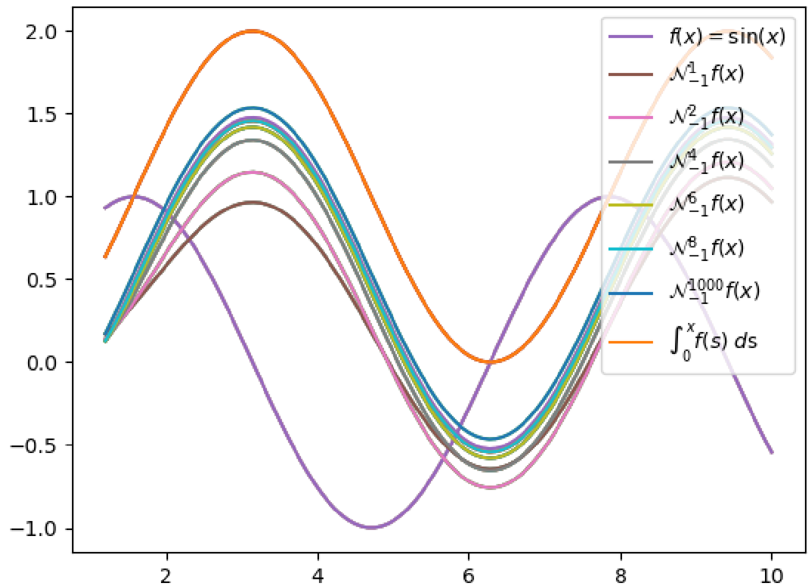


FIGURE 2. Graphics $f(x) = \sin(x)$, $\mathcal{N}^{\alpha}_{-1}f(x)$ for $\alpha = 1, 2, 4, 6, 8, 1000$ and $\int_0^x f(s)$ds.

(6) If, in addition, f is differentiable then $\mathcal{N}^{\alpha}_1(f) = e^{t^{-\alpha}} f'(t)$.
(7) Being f differentiable and $\alpha = n$ integer, we have $\mathcal{N}^n_1(f)(t) = e^{t^{-n}} f'(t)$.

Remark 1.6. with observation 6 we can obtain properties for the trigonetric, exponential and logarithmic functions, so in this way we can try to solve differential equations using this method.

Definition 1.7. A function $u \in C((0,1), \mathbb{R})$ is known as a lower solution of (1.1), if it satisfies

$$\begin{aligned} \mathcal{N}^{\alpha}_1 u(t) \leq & f(t, u(t)), t \in (0,1) \\ u(0) \leq & \int_0^1 u(t) d\mu(t) \end{aligned} \tag{1.4}$$

If inequalities (1.4) are reversed, then u is an upper solution of problem (1.1) .

Lemma 1.8. *Let* $0 < \alpha < 1$, $a \in \mathbb{R}$ *and* $A, B \in C((0,1), \mathbb{R})$*. Next, we present the following existence and uniqueness results for linear equations.*

$$\begin{cases} \mathcal{N}^{\alpha}_1 u(t) = - A(t)u(t) + B(t), & t \in (0,1) \\ u(0) = \displaystyle\int_0^1 u(t) d\mu(t) + a \end{cases} \tag{1.5}$$

If $\Gamma_\alpha = 1 - \int_0^1 e^{-\mathcal{N}_{-1}^{\alpha} A(t)} d\mu(t) \neq 0$ *then the equation has a unique solution*

$$u(t) = e^{-\mathcal{N}_{-1}^{\alpha} A(t)} \left[u(0) + \mathcal{N}_{-1}^{\alpha} \left(B(t) e^{\mathcal{N}_{-1}^{\alpha} A(t)} \right) \right] \tag{1.6}$$

Proof. Multiplying both sides of the first equation of the problem (1.5) by $e^{\mathcal{N}_{-1}^{\alpha} A(t)}$ and using Lemma 1.2, we can get

$$e^{\mathcal{N}_{-1}^{\alpha} A(t)} \mathcal{N}_1^{a} u(t) + A(t) u(t) e^{\mathcal{N}_{-1}^{\alpha} A(t)} = B(t) e^{\mathcal{N}_{-1}^{\alpha} A(t)}$$

In other words, by means of the product rule (item 4 of the properties above), equality turns to

$$\mathcal{N}_1^{\alpha} \left[e^{\mathcal{N}_{-1}^{\alpha} A(t)} u(t) \right] = B(t) e^{\mathcal{N}_{-1}^{\alpha} A(t)} \tag{1.7}$$

Applying $\mathcal{N}_{-1}^{\alpha}$ to both and (1.2) side of (1.7), we have

$$e^{\mathcal{N}_{-1}^{\alpha} A(t)} u(t) - u(0) = \mathcal{N}_{-1}^{\alpha} \left(B(t) e^{\mathcal{N}_{-1}^{\alpha} A(t)} \right) \tag{1.8}$$

$$u(t) - u(0) e^{-\mathcal{N}_{-1}^{\alpha} A(t)} = e^{-\mathcal{N}_{-1}^{\alpha} A(t)} \mathcal{N}_{-1}^{\alpha} \left(B(t) e^{\mathcal{N}_{-1}^{\alpha} A(t)} \right) \tag{1.9}$$

Then

$$u(t) = e^{-\int_0^t e^{-\tau^{-\alpha}} A(\tau) d\tau} \left(u(0) + \int_0^t e^{-\tau^{-\alpha}} B(\tau) e^{\mathcal{N}_{-1}^{\alpha} A(\tau)} d\tau \right) \tag{1.10}$$

From the equation (1.9) we obtain:

$$\int_0^1 \left[u(t) - u(0) e^{-\mathcal{N}_{-1}^{\alpha} A(t)} \right] d\mu(t) = \int_0^1 \left[e^{-\mathcal{N}_{-1}^{\alpha} A(t)} \mathcal{N}_{-1}^{\alpha} \left(B(t) e^{\mathcal{N}_{-1}^{\alpha} A(t)} \right) \right] d\mu(t)$$

$$\int_0^1 u(t) d\mu(t) - u(0) \int_0^1 e^{-\mathcal{N}_{-1}^{\alpha} A(t)} d\mu(t) = \int_0^1 \left[e^{-\mathcal{N}_{-1}^{\alpha} A(t)} \mathcal{N}_{-1}^{\alpha} \left(B(t) e^{\mathcal{N}_{-1}^{\alpha} A(t)} \right) \right] d\mu(t)$$

From the hypothesis (1.5) we obtain:

$$u(0) \left(1 - \int_0^1 e^{-\mathcal{N}_{-1}^{\alpha} A(t)} d\mu(t) \right) = a + \int_0^1 \left[e^{-\mathcal{N}_{-1}^{\alpha} A(t)} \mathcal{N}_{-1}^{\alpha} \left(B(t) e^{\mathcal{N}_{-1}^{\alpha} A(t)} \right) \right] d\mu(t)$$

If $\Gamma_\alpha = \left(1 - \int_0^1 e^{-\mathcal{N}_{-1}^{\alpha} A(t)} d\mu(t) \right) \neq 0$ then problem (1.5) has a unique solution, beside

$$u(0) = \frac{a + \int_0^1 \left[e^{-\mathcal{N}_{-1}^{\alpha} A(t)} \mathcal{N}_{-1}^{\alpha} \left(B(t) e^{\mathcal{N}_{-1}^{\alpha} A(t)} \right) \right] d\mu(t)}{1 - \int_0^1 e^{-\mathcal{N}_{-1}^{\alpha} A(t)} d\mu(t)} \tag{1.11}$$

□

Example 1.9. Consider the differential equation

$$\mathcal{N}_1^{1} u = -\left(e^{\frac{1}{t}} \sin(t) \right) u + t^2 e^{\cos(t) - 1 + \frac{1}{t}}, \quad u(0) = 1 \tag{1.12}$$

Here $A(t) = e^{\frac{1}{t}} \sin(t)$ and $B(t) = t^2 e^{\cos(t) - 1 + \frac{1}{t}}$. So by (1.6) we obtain:

$$u(t) = \left(\frac{t^3}{3} + 1\right) e^{\cos(t) - 1}$$

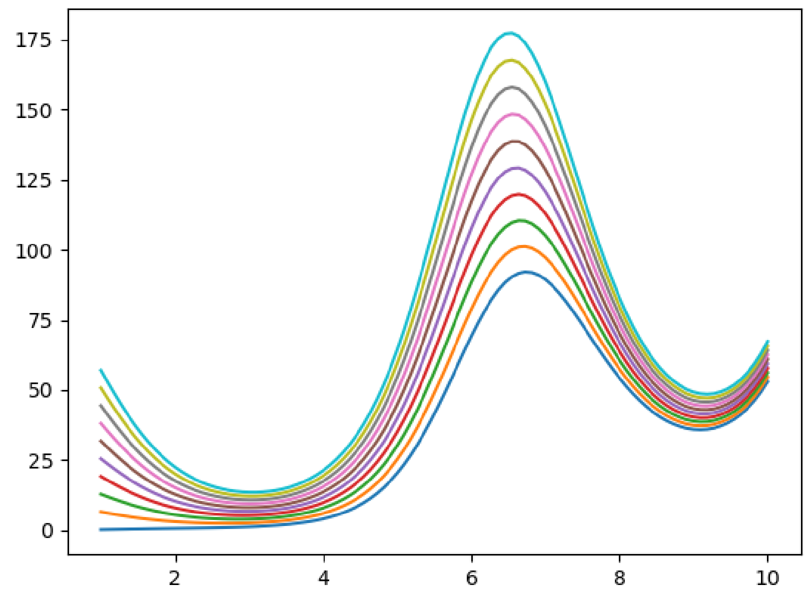


FIGURE 3. Flux of the equation (1.12).

Lemma 1.10. *Let $0 < \alpha < 1$. Suppose that $A, u \in C((0,1), \mathbb{R})$ satisfies*

$$\begin{cases} \mathcal{N}_1^{\alpha} u(t) \leq -A(t)u(t) + B(t), \ t \in (0,1) \\ u(0) \leq \displaystyle\int_0^1 u(t) d\mu(t) + a \end{cases}$$

Then $u(t) \leq 0$ on $(0,1)$ provided $\Gamma_\alpha > 0$.

Proof. Let $B(t) = \mathcal{N}_1^{\alpha} u(t) + A(t)u(t)$ and $a = u(0) - \displaystyle\int_0^1 u(t) d\mu(t)$, we know that $B(t) \leq 0, \ a \leq 0$ and

$$\begin{cases} \mathcal{N}_1^{\alpha} u(t) = -A(t)u(t) + B(t), \ t \in (0,1) \\ u(0) = \displaystyle\int_0^1 u(t) d\mu(t) + a \end{cases}$$

Using $\Gamma_\alpha > 0$, we have then by (1.10), we can conclude that

$$u(t) \leq u(0) e^{-\mathcal{N}_{-1}^{\alpha} A(t)} \leq 0$$

The proof is complete. □

2. Main Results

In this section, we prove the existence of extremal solutions for conformable fractional differential equations involving integral boundary condition. For convenience, we list some assumptions.

H1: $f : (0,1) \times \mathbb{R} \to \mathbb{R}$ is continuous.
H2: Assume that $\nu_0, \omega_0 \in E = C(0,1)$ is lower and upper solution of problem (1.1), and $\nu_0(t) \le \omega_0(t)$
H3: There exists a function $A \in E$ with $\Gamma_\alpha > 0$ which satisfies.

$$f(t,x) - f(t,x) \le A(t)(\bar{x} - x)$$

for $\nu_0(t) \le x \le x \le \omega_0(t)$

Theorem 2.1. *Assume that H1, H2 and H3 hold. Then there exist monotone iterative sequences* $\{\nu_n\}_{n=0}^{\infty}, \{\omega_n\}_{i=0}^{\infty} \subset E$ *such that*

$$\lim_{n\to\infty} \nu_n = v, \lim_{n\to\infty} \omega_n = w$$

uniformly on $(0,1)$*, and* v*,* w *are the extremal solutions of problem* (1.1) *in the sector* $[\nu_0, \omega_0] = \{g \in E : \nu_0(t) \le g(t) \le \omega_0(t),\ 0 < t < 1\}$

Proof. For all $\nu_m, \omega_n \in E$, let

$$\begin{cases} \mathcal{N}_1^\alpha \nu_{n+1}(t) = f\left(t, \nu_n(t)\right) - A(t)\left(\nu_{n+1}(t) - \nu_n(t)\right),\ t \in (0,1) \\ \mathcal{N}_1^\alpha \omega_{n+1}(t) = f\left(t, \omega_n(t)\right) - A(t)\left(\omega_{n+1}(t) - \omega_n(t)\right),\ t \in (0,1) \\ \nu_{n+1}(0) = \displaystyle\int_0^1 \nu_{n+1}(t)d\mu(t),\ \omega_{n+1}(0) = \int_0^1 \omega_{n+1}(t)d\mu(t) \end{cases} \tag{2.1}$$

Thus, the iterative sequences $\{\nu_n\}$ and $\{\omega_n\}$ can be constructed by Lemma 1.8. Firstly, we shall prove that

$$\nu_n \le \nu_{n+1} \le \omega_{n+1} \le \omega_n,\ n = 0, 1, 2, \ldots$$

Let $p = \nu_0 - \nu_1$. According to (2.1) and definition 1.7, we have

$$\begin{cases} \mathcal{N}_1^\alpha p(t) = \mathcal{N}_1^\alpha \nu_0(t) - \mathcal{N}_1^\alpha \nu_1(t) \le f\left(t, \nu_0(t)\right) - f\left(t, \nu_0(t)\right) + A(t)\left(\nu_1(t) - \nu_0(t)\right),\ t \in (0,1) \\ p(0) \le \int_0^1 \nu_0(t)d\mu(t) - \int_0^1 \nu_1(t)d\mu(t) \end{cases}$$

i.e.,

$$\begin{cases} \mathcal{N}_1^\alpha p(t) \le -A(t)p(t),\ t \in (0,1) \\ p(0) \le \displaystyle\int_0^1 p(t)d\mu(t) \end{cases}$$

Therefore, by lemma 1.10 we have $\nu_0(t) \le \nu_1(t)$. Similarly, we can prove that $\omega_1(t) \le \mathrm{w}_0(t),\ t \in (0,1)$ Now, let $r = \nu_1 - \omega_1$, according to (2.1) and (H3), we have

$$\begin{cases} \mathcal{N}_1^\alpha r(t) = f(t,\nu_0(t)) - f(t,tu_0(t)) - A(t)(\nu_1(t)-\nu_0(t)-\omega_1(t)+\omega_0(t)) \\ \qquad\qquad \leq A(t)(\omega_0(t)-\nu_0(t)) - A(t)(\nu_1(t)-\nu_0(t)-\omega_1(t)+\omega_0(t)) \\ \qquad\qquad = -A(t)r(t) \\ r(0) = \int_0^1 r(t)d\mu(t). \end{cases}$$

By lemma 1.10, we have $\nu_1(t) \leq \omega_1(t),\ t \in (0,1)$.

Secondly, we show that ν_1, ω_1 are lower and upper solutions of (1.1), respectively.

$$\begin{cases} \mathcal{N}_1^\alpha \nu_1(t) = f(t,\nu_0(t)) - A(t)(\nu_1(t)-\nu_0(t)) - f(t,\nu_1(t)) + f(t,\nu_1(t)) \\ \qquad\qquad \leq A(t)(\nu_1(t)-\nu_0(t)) - A(t)(\nu_1(t)-\nu_0(t)) + f(t,\nu_1(t)) \\ \qquad\qquad = f(t,\nu_1(t)) \\ \nu_1(0) = \int_0^1 \nu_1(t)d\mu(t) \end{cases}$$

According to (H_3) and definition 1.7, we deduce that ν_1 is a lower solution of (1.1). Similarly, ω_1 is a upper solutions of (1.1). By the above arguments and mathematical induction, it is clear that

$$\nu_0 \leq \cdots \leq \nu_n \leq \nu_{n+1} \leq \omega_{n+1} \leq \omega_n \leq \cdots \leq \omega_0,\ n = 0,1,2,\ldots \tag{2.2}$$

Thirdly, we show that $\lim_{n\to\infty} \nu_n = v, \lim_{u\to\infty} \omega_n = w$. Hence, we need to conclude that $\nu_n,\ \omega_n$ are uniformly bounded and equicontinuous on $(0,1)$. Obviously, the uniform boundedness of sequences $\nu_n,\ \omega_n$ follows from (2.2). Thus, there exists $L > 0$ such that

$$|f(t,\nu_n(t)) - A(t)(\nu_{n+1}(t)-\nu_n(t))| \leq L$$

and

$$|f(t,\omega_n(t)) - A(t)(\omega_{n+1}(t)-\omega_n(t))| \leq L$$

Using Theorem 1.3, we get

$$\begin{aligned} |\nu_n(t_1) - \nu_n(t_2)| &= \frac{1}{\alpha}|D_\alpha \nu_n(\xi)|\,|t_1^\alpha - t_2^\alpha| \\ &= \frac{1}{\alpha}|f(\xi,\nu_{n-1}(\xi)) - M(\xi)(\nu_n(\xi)-\nu_{n-1}(\xi))|\,|t_1^\alpha - t_2^\alpha| \end{aligned}$$

Therefore, $\{\nu_n\}$ are equicontinuous. Similarly, we obtain that $\{\omega_n\}$ are equicontinuous too. By Arzela-Ascoli Theorems, we conclude that $\{\nu_n\},\ \{\omega_n\}$ have subsequences $\{\nu_{n_k}\},\ \{\omega_{n_k}\}$ such that $\{\nu_{n_k}\} \to v$, and $\{\omega_{n_k}\} \to w$ when $k \to \infty$. This together with the monotonicity of sequences $\{\nu_n\}$ and $\{\omega_n\}$ implies

$$\lim_{n\to\infty} \nu_n(t) = v(t),\ \lim_{n\to\infty} \omega_n(t) = w(t)$$

uniformly on $(0,1)$. Please note that the sequence $\{\nu_n\}$ satisfies

$$\begin{cases} \nu_n(t) = e^{-\mathcal{N}_{-1}^\alpha A(t)}[\nu_{n-1}(0) + R\nu_{n-1}(t)],\ t \in (0,1) \\ \nu_n(0) = \int_0^1 \nu_n(t)d\mu(t), \quad n = 1,2,\ldots \end{cases} \tag{2.3}$$

where

$$R\nu_{n-1}(t) = \mathcal{N}_{-1}^{\alpha}\left[\left(f\left(t, \nu_{n-1}(s)\right) + A(s)\nu_{n-1}(s)\right) e^{\mathcal{N}_{-1}^{\alpha} A(t)}\right]$$

Let $n \to \infty$ in (9) . We have

$$\begin{cases} v(t) = e^{-\mathcal{N}_{-1}^{\alpha} A(t)}[v(0) + Rv(t)],\ t \in (0,1) \\ v(0) = \int_0^1 v(t) d\mu(t) \end{cases}$$

This shows that v is a solution of the nonlinear problem (1). Similarly, we obtain to is a solution of the nonlinear problem (1) too. And

$$\nu_0(t) \le v(t) \le w(t) \le \omega_0(t),\ t \in (0,1)$$

Finally, we are going to prove that v, to are minimal and maximal solutions of (1.1) in the sector $[\nu_0, \omega_0]$. In the following, we show this using induction arguments. Suppose that $g(t)$ is any solution of (1.1) in the $[\nu_0, \omega_0]$ that is

$$\nu_0(t) \le g(t) \le \omega_0(t),\ t \in (0,1)$$

Assume that $\nu_n(t) \le g(t) \le \omega_n(t)$ hold. Let $p(t) = \nu_{n+1}(t) - g(t)$, we have

$$\begin{cases} \mathcal{N}_1^{\alpha} p(t) = \mathcal{N}_1^{\alpha} \nu_{m+1}(t) - \mathcal{N}_1^{\alpha} g(t) \\ \qquad = f\left(t, \nu_n(t)\right) - A(t)\left(\nu_{n+1}(t) - \nu_n(t)\right) - f(t, g(t)) \\ \qquad \le A(t)\left(g(t) - \nu_n(t)\right) - A(t)\left(\nu_{n+1}(t) - \nu_n(t)\right) \\ \qquad = -A(t)p(t) \\ p(0) = \displaystyle\int_0^1 p(t) d\mu(t) \end{cases}$$

Then, by lemma 1.10, we have $\nu_{n+1}(t) \le g(t),\ t \in (0,1)$. By similar method, we can show that $g(t) \le \omega_{n+1}(t),\ t \in (0,1)$. Therefore,

$$\nu_n \le g \le \omega_n, \quad n = 1, 2, \ldots$$

By taking $n \to \infty$ in the above inequalities, we get that $v \le g \le w$. That is v, w are extremal solutions of problem (1.1) in $[\nu_0, \omega_0]$. Thus, the proof is finished.

□

3. Conclusions

In this article, using the monotone iterative technique we investigate the existence results for extremal solutions of some generalized differential equations. A fundamental detail is that almost all the results derived in the paper are more-or-less straightforward extensions of well-known results from the theory of the first-order ordinary differential equations, since the generalized derivative is a version of the first-order derivative.

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Escuela de Ciencias Físicas y Matemáticas, Facultad de Ciencias Exactas y Naturales,Pontificia Universidad Católica del Ecuador, Av. 12 de octubre 1076, Apartado: 17-01-2184, Quito, Ecuador
E-mail address: mjvivas@puce.edu.ec

Departamento de Matemáticas y Estadísticas, Instituto de Ciencias Básicas, Universidad Técnica de Manabí, 130105, Portoviejo, Ecuador.
E-mail address: oswaldo.larreal@utm.edu.ec

Juan E. Nápoles Valdés:, UNNE, FaCENA, Ave. Libertad 5450, Corrientes 3400, Argentina
E-mail address: jnapoles@exa.unne.edu.ar