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INVERSE SPECTRAL PROBLEM FOR PT -SYMMETRIC SCHRODINGER OPERATOR ON THE GRAPH WITH LOOP

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Abstract. In the paper PT-symmetric Schrodinger operator on the lasso graph is considered. The spectral properties of this operator are investigated and related inverse problem is solved. An effective algorithm for solving the inverse problem is given and the uniqueness theorem is proved.

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1. Introduction

The main purpose of the present work is to solve the inverse problem for the PT-symmetric Hill operator on the lasso graph where the P-symmetry transformation (reflection of spatial coordinates) consists, for example, in changing the sign in front of the coordinate operator, and the T-symmetry transformation (time reversal) consists in changing the sign of the impulse (but not the coordinate), as well as replacing i on $-i$. By lasso graph, half-line attached to a loop is to be understood.

Let there be given the non-compact graph G where an edge is attached to a loop. The non-compact part of the graph is a ray $\gamma_0 = \{x| \, 0 < x < \infty\}$, compact part is the loop $\gamma_1 = \{z| \, 0 < z < 2\pi\}$ whose length we take equal to 2π and with $\gamma_2 = \{\{x = 0\} = \{z = 0\} = \{z = 2\pi\}\}$ corresponding to the attachment point. We investigate the spectral problem describing the one-dimensional scattering of a quantum particle on G. Namely, we consider the problem

$$\begin{aligned} &-Y'' + \{q(X) - \lambda^2\}Y = 0, \qquad X \in G\backslash\{\gamma_2\} \\ &Y(x = 0) = Y(z = 0) = Y(z = 2\pi), \\ &Y'(x = 0 + 0) + Y'(z = 0 + 0) - Y'(z = 2\pi - 0) = 0 \end{aligned} \tag{1.1}$$

In (1.1) differentiation with respect to the variable X is understood as differentiation with respect to x, when $X \in \gamma_0$, and as differentiation with respect to z, when $X \in \gamma_1$. Differentiation is not defined at the vertices.

We assume that the potential

$$q(X) = \begin{cases} \quad q_1(x) = \sum\limits_{n=1}^{\infty} q_{1n} e^{inx}, \quad X \in \gamma_0 \\ \quad q_2(z) = \sum\limits_{n=1}^{\infty} q_{2n} e^{inz}, \quad X \in \gamma_1 \end{cases} \tag{1.2}$$

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is defined as a complex valued function on the G with $\sum\limits_{n=1}^{\infty} |q_{kn}| < \infty, k = 1,2;$ and λ is a spectral parameter.

Then the resulting Hill operator will be as follows

$$Y''(X) + q(X)Y(X), X \in G.$$

More precisely, on the Hilbert space $L_2(G)$ with norm

$$\|f\|_{L_2(G)} = \{\|f\|^2_{L_2(\gamma_0)} + \|f\|^2_{L_2(\gamma_1)}\}^{1/2}$$

we introduce the operator L with domain

$$D(L) = \left\{ Y(X), X \in G | \begin{array}{c} Y(X) \in H^2(\gamma_0) \cup H^2(\gamma_1) \\ Y(x=0) = Y(z=0) = Y(z=2\pi) \\ Y'(x=0+0) + Y'(z=0+0) - Y'(z=2\pi-0) = 0 \end{array} \right\}$$

where $H^k(k = 1,2,..)$ are the usual Sobolev spaces.

The potentials considered in the paper have the form

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx},$$

where, in particular, for the numbers $q_n = \overline{q_n}$ the potential will be PT-symmetric, i.e. $q(x) = \overline{q(-x)}$.

Spectral analysis of operator with the potential of type (1.2) firstly was studied by M.G.Gasymov [13], where he proved the existence of the solution $f(x,\lambda)$ for the equation

$$-y''(x) + q(x)y(x) = \lambda^2 y\,(x)$$

in $L_2(-\infty,+\infty)$ of the form

$$f\,(x,\lambda) = e^{i\lambda x}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n+2\lambda} \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x}\right),$$

where the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=n+1}^{\infty} \alpha\,(\alpha-n)\,|V_{n\alpha}|; \quad \sum_{n=1}^{\infty} n\,|V_{nn}|$$

converge.

He also discussed the corresponding inverse spectral problem of finding the potential $q(x)$ for given so-called "normalizing" numbers V_{nn}, where the key role played the relation

$$\lim_{\lambda \to \frac{n}{2}} (n-2\lambda)\, f\,(x,-\lambda) = V_{nn} f\left(x, \frac{n}{2}\right). \tag{1.3}$$

As a final remark relating to the potential of the type (1.2), we mention the works K.Shin [21], R.Carlson [5,6],Guillemin and V., Uribe A [15], L.Pastur and V. Tkachenko [20] and [3,7-19,16]. More information about the potentials can be found in [13].

Let us to mention some close results.

Without a claim of completeness of investigation of inverse problems on graphs with loop here are listed the works of Akhyamov A.M, Trooshin I.Y [1], Gomilko A.M. and Pivovarchik V.N[14], Exner P. [11], Berkolaiko G.[2], Kurasov P [17], Mochizuki K. and Trooshin I.Yu. [19] .

Moreover, the potential on graphs with loop (including the potential on loop edge) can be constructed by reflection coefficients and two spectra. In order to solve the inverse problem effective algorithm is given.

Let us review briefly the contents of the paper. The Hamiltonian of the model is introduced in Section 1. Next, its spectral properties are derived in Section 2. In Section 3 we give a formulation of the inverse problem, prove the uniqueness theorem and provide a constructive procedure for the solution of the inverse problem.

2. General solution

Suppose that $f(x,\lambda), x \in \gamma_0$ is the Jost solution for equation

$$-Y'' + \{q(X) - \lambda^2\}Y = 0 \tag{2.1}$$

which satisfies the asymptotic condition $f(x,\lambda) \to e^{i\lambda x}$ $(x \to \infty)$. Then it can be constructed analogously [13], by the following theorem

Theorem 2.1. *Let $q(X)$ be in the form of (1.2). Then equation (2.1) has on γ_0 linearly independent solutions of the form*

$$f(x, \pm\lambda) = e^{i\lambda x}(1 + \sum_{n=1}^{\infty} \frac{1}{n \pm 2\lambda} \sum_{\alpha=n}^{\infty} V_{n\alpha}^{\gamma_0} e^{i\alpha x}), x \in \gamma_0, \tag{2.2}$$

where the numbers $V_{n\alpha}^{\gamma_0}$ are determined by the following recurrent relations

$$\begin{cases} \alpha(\alpha - n)V_{n\alpha}^{\gamma_0} + \sum\limits_{s=n}^{\alpha-1} q_{1\alpha-s} V_{ns}^{\gamma_0} = 0, & 1 \le n < \alpha \\ \alpha \sum\limits_{n=1}^{\alpha} V_{n\alpha}^{\gamma_0} + q_{1\alpha} = 0; \end{cases} \tag{2.3}$$

the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=n+1}^{\infty} \alpha\,(\alpha - n)\,|V_{n\alpha}^{\gamma_0}|; \quad \sum_{n=1}^{\infty} n\,|V_{nn}^{\gamma_0}|$$

are converged and fulfilled the relation

$$\lim_{\lambda \to \mp \frac{n}{2}} (n \pm 2\lambda) f(x, \pm\lambda) = V_{nn}^{\gamma_0} f(x, \mp\frac{n}{2}) \tag{2.4}$$

or that the same

$$V_{m\alpha+m}^{\gamma_0} = V_{mm}^{\gamma_0} \sum_{n=1}^{\alpha} \frac{V_{nn}^{\gamma_0}}{n+m}, \quad \alpha = 1, 2, ...; \tag{2.5}$$

The proof of the theorem is similar to one from [13] and therefore we do not give it here.

For any real $\lambda \neq 0$ the Wronskian of the functions $f(x,\lambda), f(x,-\lambda)$ is

$$W[f(x,\lambda), f(x,-\lambda)] = f(x,\lambda)f'(x,-\lambda) - f'(x,\lambda)f(x,-\lambda) = 2i\lambda.$$

This implies that these two functions form a fundamental system of solutions of equation (2.1) in $\gamma_0 = \{x|\, 0 < x < \infty\}$, and thus, if $y(x,\lambda)$ satisfies (2.1) for any real $\lambda \neq 0$ then we have some constants $C(\lambda)$, $D(\lambda)$ such that

$$y(x,\lambda) = C(\lambda)f(x,\lambda) + D(\lambda)f(x,-\lambda), x \in \gamma_0$$

Then, easy to see that, we can seek the solution to the spectral problem on $\gamma_0 = \{x|\, 0 < x < \infty\}$ in the form

$$y(x,\lambda) = f(x,-\lambda) + R_{11}(\lambda)f(x,\lambda), x \in \gamma_0$$

where $R_{11}(\lambda)$ baseing on Marchenko [20] we call as reflection coefficients.

The general solution $u(z,\lambda)$ for $z \in \gamma_1 = \{z|\, 0 < z < 2\pi\}$ of equation (2.1) we will attempt to find via its Green function.

Let $\varphi(z,\lambda), \theta(z,\lambda)$ be linear independent solutions of equation (2.1) on the loop γ_1 , satisfying the initial conditions

$$\begin{array}{l} \varphi(0,\lambda) = \theta'(0,\lambda) = 1 \\ \varphi'(0,\lambda) = \theta(0,\lambda) = 0 \end{array}.$$

Note that their Wronskian is $W[\varphi(z,\lambda), \theta(z,\lambda)] = 1$.

The Green's function on the loop $\gamma_1 = \{z|\, 0 < z < 2\pi\}$ can be constructed by means of the fundamental solutions $\varphi(z,\lambda), \theta(z,\lambda)$ taking into account the following boundary conditions

$$\begin{array}{l} G(0,t,\lambda) = G(2\pi,t,\lambda) \\ G'(0,t,\lambda) = G'(2\pi,t,\lambda) \\ G'_z(t+0,t,\lambda) - G'_z(t-0,t,\lambda) = -1 \\ \lim\limits_{z\to t+0} G(z,t,\lambda) = \lim\limits_{z\to t-0} G(z,t,\lambda). \end{array} \tag{2.6}$$

Then by virtue of (2.6), we have

$$\begin{array}{rl} G(z,t,\lambda) = & \frac{\theta(t,\lambda)+\varphi(t,\lambda)\theta(2\pi,\lambda)-\theta(t,\lambda)\varphi(2\pi,\lambda)}{\varphi(2\pi,\lambda)+\theta'(2\pi,\lambda)-2}\varphi(z,\lambda) + \\ & + \frac{\varphi(t,\lambda)\theta'(2\pi,\lambda)-\theta(t,\lambda)\varphi'(2\pi,\lambda)-\varphi(t,\lambda)}{\varphi(2\pi,\lambda)+\theta'(2\pi,\lambda)-2}\theta(z,\lambda), \qquad t \geq z \end{array}$$

$$\begin{array}{rl} G(z,t,\lambda) = & \frac{\varphi(t,\lambda)\theta(2\pi,\lambda)+\theta'(2\pi,\lambda)\varphi(t,\lambda)-\theta(,\lambda)}{\varphi(2\pi,\lambda)+\theta'(2\pi,\lambda)-2}\varphi(z,\lambda) + \\ & + \frac{\varphi(y,\lambda)-\varphi(y,\lambda)\varphi(2\pi,\lambda)-\theta(y,\lambda)\varphi'(2\pi,\lambda)}{\varphi(2\pi,\lambda)+\theta'(2\pi,\lambda)-2}\theta(z,\lambda), \qquad t \leq z. \end{array}$$

Then easy to see that the function

$$\begin{array}{rl} G(z,0,\lambda) = G(z,2\pi,\lambda) = & \frac{\theta(2\pi,\lambda)}{\varphi(2\pi,\lambda)+\theta'(2\pi,\lambda)-2}\varphi(z,\lambda) + \\ & + \frac{1-\varphi(2\pi,\lambda)}{\varphi(2\pi,\lambda)+\theta'(2\pi,\lambda)-2}\theta(z,\lambda). \end{array} \tag{2.7}$$

is a solution of equation (2.1) on the loop $\gamma_1 = \{z|\, 0 < z < 2\pi\}$ up to constant . So, we can take as a solution on the loop γ_1 the function

$$u(z,\lambda) = \alpha G(z,0,\lambda) \tag{2.8}$$

where α is any constant. So, the following theorem is proved.

Theorem 2.2. *For any real $\lambda \neq 0$ problem (1.1-1.2) on the non-compact graph G has a solution of the form*

$$Y(X,\lambda) = \begin{cases} y(x,\lambda = f(x,-\lambda) + R_{11}(\lambda) f(x,\lambda), & X \in \gamma_0 \\ u(z,\lambda) = \alpha G(z,0,\lambda) & X \in \gamma_1 \end{cases} \tag{2.9}$$

Let us find $R_{11}(\lambda)$ on such way that the solution of the form (2.9) would satisfy boundary conditions in (1.1)
From the the boundary conditions in (1.1) we have

$$\begin{aligned} & f(0,-\lambda) + R_{11}(\lambda) f(0,\lambda) = \alpha G(0,0,\lambda) = \alpha G(2\pi,0,\lambda) \\ & f'(0,-\lambda) + R_{11}(\lambda) f'(0,\lambda) + \alpha[G'_z(0+0,0,\lambda) - G'_z(0-0,0,\lambda]) = 0. \end{aligned}$$

Taking into account the boundary conditions (2.6) for the Green function on the loop, we obtain

$$\begin{aligned} & f(0,-\lambda) + R_{11}(\lambda) f(0,\lambda) = \alpha G(0,0,\lambda) \\ & f'(0,-\lambda) + R_{11}(\lambda) f'(0,\lambda) = \alpha. \end{aligned} \tag{2.10}$$

Thus

$$f(0,-\lambda) + R_{11}(\lambda) f(0,\lambda) = [f'(0,-\lambda) + R_{11}(\lambda) f'(0,\lambda)] G(0,0,\lambda).$$

So, we get the following relations that will be used in future

$$G(0,0,\lambda) = \frac{f(0,-\lambda) + R_{11}(\lambda) f(0,\lambda)}{f'(0,-\lambda) + R_{11}(\lambda) f'(0,\lambda)} \tag{2.11}$$

and

$$R_{11}(\lambda) = \frac{\alpha - f'(0,-\lambda)}{f'(0,\lambda)} = -\frac{f(0,-\lambda) - G(0,0,\lambda) f'(0,-\lambda)}{f(0,\lambda) - G(0,0,\lambda) f'(0,\lambda)}. \tag{2.12}$$

3. The Inverse Spectral Problem On Lasso Graph

If the graph has at least one loop, then the potential on the loop cannot be reconstructed using local methods: calculation of the potential requires consideration of the whole loop at once.

The main idea of the solution of the inverse problem for the considered system is its reduction two independent problems of reconstruction of the potential $q(X) = [q_1(x), q_2(z)]$, to recover $q_1(x)$ on the edge γ_0 and to recover $q_2(z)$ on the edge γ_1. Since the coefficients $R_{11}(\lambda)$ can be found by using matching conditions

$$y(0) = u(0) = u(2\pi)$$

and

$$y'(0+0) + u'(0+0) - u'(2\pi - 0) = 0$$

at the central vertex, it is natural to formulate inverse problem - recovering of the potential $q(X)$ at non-compact graph G by reflection coefficient, the set of eigenvalues of Dirichlet problems

$$\begin{aligned} &-u''(z,\lambda)+q_2(z)u(z,\lambda)=\lambda^2 u(z,\lambda), \ \ z\in[0,2\pi] \\ &u(0,\lambda)=u(2\pi,\lambda)=0 \end{aligned} \tag{3.1}$$

and the spectrum of Neumann boundary value problem

$$\begin{aligned} &-u''(z,\lambda)+q_2(z)u(z,\lambda)=\lambda^2 u(z,\lambda), \ \ z\in[0,2\pi] \\ &u'(0,\lambda)=u'(2\pi,\lambda)=0 \end{aligned} \tag{3.2}$$

Inverse problem: Given the spectral data: $\{\lambda_n\}$ - the spectrum of the Dirichlet problem (3.1), $\{\mu_n\}$- the spectrum of Neumann boundary value problem (3.2) and reflection coefficient $R_{11}(\lambda)$, construct the potential $q(X)$.

Lemma 3.1. *All numbers $V_{nn}^{\gamma_0}$ can be determined by specifying the reflection coefficients $R_{11}(\lambda)$ as*

$$\lim_{\lambda\to\frac{n}{2}}(n-2\lambda)\,R_{11}(\lambda)=-V_{nn}^{\gamma_0}$$

Proof. Indeed , from relation (2.12), we get

$$R_{11}\,(\lambda)=\frac{\alpha-f'(0,-\lambda)}{f'(0,\lambda)}$$

Then by using (2.4), we have

$$\begin{aligned} &\lim_{\lambda\to\frac{n}{2}}(n-2\lambda)R_{11}\,(\lambda)=\lim_{\lambda\to\frac{n}{2}}(n-2\lambda)\tfrac{\alpha-f'(0,-\lambda)}{f'(0,\lambda)}= \\ &=\lim_{\lambda\to\frac{n}{2}}\tfrac{(n-2\lambda)\alpha-(n-2\lambda)f'(0,-\lambda)}{f'(0,\lambda)}=-\lim_{\lambda\to\frac{n}{2}}\tfrac{(n-2\lambda)f'(0,-\lambda)}{f'(0,\lambda)}=-V_{nn}^{\gamma_0} \end{aligned}$$

.

Note that

$$V_{m,\alpha+m}^{\gamma_0}=V_{m,m}^{\gamma_0}\sum_{n=1}^{\alpha}\frac{V_{n,\alpha}^{\gamma_0}}{n+m},\quad \alpha=1,2,..$$

are fundamental equations for defining q_{1n} from $V_{nn}^{\gamma_0}$. In fact, if $V_{nn}^{\gamma_0}$ are known, then (2.3) gives recurrent formulas for defining $V_{n\alpha}^{\gamma_0}$. Thus, for the numbers $V_{nn}^{\gamma_0}$ the function $q_1(x)$ may be reconstructed uniquely and effectively. □

Theorem 3.2. *The specification of spectral data uniquely determines the potential $q(X)$.*

Proof. All numbers q_{1n} can be determined from (2.3) by using the "normalizing" numbers $V_{nn}^{\gamma_0}$ and the potential $q_1(x)$ may be reconstructed using above given algorithm uniquely and effectively on the edge γ_0.

Since specifying numbers $V_{nn}^{\gamma_0}$ makes possible to construct the function $f\,(x,\lambda)$, then knowing the reflection coefficient $R_{11}(k)$, we can find values of the spectral parameter λ that are roots of the equation

$$f'\,(0,-\lambda)+R_{11}(\lambda)f'\,(0,\lambda)=0. \tag{3.3}$$

Then from (2.10) we directly see that for these λ , $\alpha = 0$ and from (2.8) obtain that the solution to the spectral problem on the loop must satisfy the boundary conditions

$$u(0,\lambda) = u(2\pi,\lambda) = 0,$$
$$u'(0,\lambda) = u'(2\pi,\lambda) = 0.$$

Let us consider the problem of reconstruction of the potential $q_2(z)$ on the loop γ_1. As initial data, we take the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ where the first of which coincides with eigenvalues of the spectral problem (3.1) and the second determines the eigenvalue of problem (3.2). Then it is noticeable that $\{\lambda_n\}$ and $\{\mu_n\}$ coincide with the zeros of

$$\Phi_1(\lambda) = \theta(2\pi,\lambda),$$
$$\Phi_2(\mu) = \varphi'(2\pi,\mu)$$

that means that the functions $\theta(2\pi,\lambda)$ and $\varphi'(2\pi,\mu)$ can be recovered by using $\{\lambda_n\}$ - eigenvalues of spectral problem (3.1) and $\{\mu_n\}$-the eigenvalue of problem (3.2) respectively.

Let us introduce the function

$$S(\lambda) = \frac{g'(0,\lambda) + i\lambda g(0,\lambda)}{g'(0,-\lambda) + i\lambda g(0,-\lambda)}$$

which will play an important role in solving the inverse problem on the loop.
Here the function $g(z,\lambda)$ is a solution of the problem

$$-u''(z,\lambda) + q(z)u(z,\lambda) = \lambda^2 u(z,\lambda) \tag{3.4}$$

in the space $L_2[0,\infty)$ with the potential

$$q(z) = \begin{cases} q_2(z) & on\ z \in [0,2\pi] \\ 0 & on \quad z > 2\pi \end{cases}$$

with the boundary condition $u'(0) = 0$ and moreover fulfilling the condition

$$\lim_{\mathrm{Im} z\to\infty} g(z,\lambda)e^{-i\lambda z} = 1.$$

Then, from [13] follows that, $g(z,\lambda)$ can be represented as

$$g(z,\lambda) = \begin{cases} \tilde{f}(z,\lambda) & on\ z \in [0,2\pi] \\ e^{i\lambda z} & on \quad z > 2\pi, \end{cases}$$

where

$$\tilde{f}(z,\lambda) = e^{i\lambda z}\Big(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}^{\gamma_1}}{n+2\lambda} e^{i\alpha z}\Big).$$

The numbers $V_{n\alpha}^{\gamma_1}$ are determined by the following recurrent relations

$$\begin{cases} \alpha(\alpha-n)V_{n\alpha}^{\gamma_1} + \sum\limits_{s=n}^{\alpha-1} q_{2\alpha-s}V_{ns}^{\gamma_1} = 0, & 1 \le n < \alpha \\ \alpha \sum\limits_{n=1}^{\alpha} V_{n\alpha}^{\gamma_1} + q_{2\alpha} = 0, & \end{cases} \tag{3.5}$$

where the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=n+1}^{\infty} \alpha\,(\alpha-n)\,|V_{n\alpha}^{\gamma_1}|; \quad \sum_{n=1}^{\infty} n\,|V_{nn}^{\gamma_1}|$$

converge and the relation

$$\lim_{\lambda\to\mp\frac{n}{2}} (n \pm 2\lambda)\tilde{f}(z,\pm\lambda) = V_{nn}^{\gamma_1}\tilde{f}(z,\mp\frac{n}{2}) \tag{3.6}$$

is fulfilled.

□

Lemma 3.3. *For the solution $g(x,\lambda)$ of equation (19), the relation*

$$g(z,\lambda) = e^{2i\lambda\pi}[\theta'(2\pi,\lambda) - i\lambda\theta(2\pi,\lambda)]\varphi(z,\lambda) + e^{2i\lambda\pi}[i\lambda\varphi(2\pi,\lambda) - \varphi'(2\pi,\lambda)]\theta(z,\lambda)$$

is fulfilled.

From the Lemma 2 we have

$$\begin{aligned} g(0,\lambda) &= e^{2i\lambda\pi}[\theta'(2\pi,\lambda) - i\lambda\theta(2\pi,\lambda)] \\ g'(0,\lambda) &= e^{2i\lambda\pi}[i\lambda\varphi(2\pi,\lambda) - \varphi'(2\pi,\lambda)] \end{aligned}$$

or

$$\begin{aligned} g'(0,\lambda) + i\lambda g(0,\lambda) &= e^{2i\lambda\pi}[i\lambda(\theta'(2\pi,\lambda) + \varphi(2\pi,\lambda)) + \lambda^2\theta(2\pi,\lambda) - \varphi'(2\pi,\lambda)] = \\ &= e^{2i\lambda\pi}[i\lambda F(\lambda) + \lambda^2\theta(2\pi,\lambda) - \varphi'(2\pi,\lambda)] \end{aligned}$$

where $F(\lambda) = \theta'(2\pi,\lambda) + \varphi(2\pi,\lambda)$ is a Lyapunov function (Hill discriminant).

Taking into account formulas (10) and (14) we have

$$G(0,0,\lambda) = \frac{\theta(2\pi,\lambda)}{\varphi(2\pi,\lambda) + \theta'(2\pi,\lambda) - 2} = \frac{f(0,-\lambda) + R_{11}(\lambda)\,f(0,\lambda)}{f'(0,-\lambda) + R_{11}(\lambda)\;f'(0,\lambda)}$$

Lemma 3.4. *Zeros of the functions $f(0,-\lambda) + R_{11}(\lambda)\,f(0,\lambda)$ and $f'(0,-\lambda) + R_{11}(\lambda)\;f'(0,\lambda)$ do not coincide.*

Proof. Let us assume contrary. Let λ^* be a common root for both functions. Then

$$f(0,-\lambda^*) + R_{11}(\lambda^*)\,f(0,\lambda^*) = 0,$$

$$f'(0,-\lambda^*) + R_{11}(\lambda^*)\;f'(0,\lambda^*) = 0,$$

from that we have

$$R_{11}(\lambda^*) = -\frac{f(0,-\lambda^*)}{f(0,\lambda^*)} = -\frac{f'(0,-\lambda^*)}{f'(0,\lambda^*)}$$

or

$$f(0,-\lambda^*)f'(0,\lambda^*) - f'(0,-\lambda^*)f(0,\lambda^*) = 0$$

which cannot take place since these solutions are linearly independent.

The Lemma is proved.

It turns out that the roots of equation (3.3) are eigenvalues of the periodic boundary-value problem, at the same time, are roots of the dispersion relation $F(\lambda) = 2$. Therefore, the Lyapunov function $F(\lambda)$ can be recovered by the roots of the equation (3.3).

Since $\theta(2\pi, \lambda)$ and $\varphi'(2\pi, \lambda)$ can be recovered by using $\{\lambda_n\}$ - eigenvalues of spectral problem (3.1) and $\{\mu_n\}$-the eigenvalue of problem (3.2), respectively we find out that the function

$$g'(0, \lambda) + i\lambda g(0, \lambda) = e^{2i\lambda\pi}[i\lambda F(\lambda) + \lambda^2\theta(2\pi, \lambda) - \varphi'(2\pi, \lambda)]$$

can be reconstructed specifying spectral data.

Thus, indentifying the spectral data uniquely determines the function

$$S(\lambda) = \frac{g'(0, \lambda) + i\lambda g(0, \lambda)}{g'(0, -\lambda) + i\lambda g(0, -\lambda)}.$$

Then taking into account (21), we can find

$$\lim_{\lambda \to -\frac{n}{2}} (n + 2\lambda) S(\lambda) = V_{nn}^{\gamma_1}$$

By using the results obtained above, we obtain the following procedure for the solution of the inverse problem recovering the potential $q_2(z)$ uniquely and effectively on the edge γ_1:

1. Taking into account (3.6), we get

$$V_{m\alpha+m}^{\gamma_1} = V_{mm}^{\gamma_1} \sum_{n=1}^{\alpha} \frac{V_{nn}^{\gamma_1}}{n+m}, \quad \alpha = 1, 2, ...;$$

from which all the numbers $V_{n\alpha}^{\gamma_1}$ are defined.

2. From recurrent formula (3.5), we find all numbers q_{2n}.

So, the inverse problem has a unique solution, and the numbers q_{2n} are defined constructively by the spectral data on the edge γ_1.

The theorem is proved.

Using the results obtained above we arrive at the following procedure for the solution of the Inverse Problem.

Algoritm:

Let the spectral data $\{\lambda_n\}$ - the spectrum of the Dirichlet problem (3.1), $\{\mu_n\}$- the spectrum of Neumann boundary value problem (3.2) and reflection coefficient $R_{11}(\lambda)$ are given.

To construct the potential $q_1(x)$ on γ_0, one have to:

1. Use $R_{11}(\lambda)$ to find $V_{nn}^{\gamma_0}$ by Lemma 1, calculate all numbers $V_{n\alpha}^{\gamma_0}$ from (2.5).

2. Use recurrent relation (6) find all numbers $q_{1\alpha}$ to recover potential $q_1(x)$.

To construct potential $q_2(x)$ on γ_1, one have to:

3. Use the $\{\lambda_n\}$- spectrum of the Dirichlet problem given by (3.1) and the $\{\mu_n\}$ - spectrum of Neumann boundary value problem given by (3.2) to construct the functions $\theta(2\pi, \lambda)$ and $\varphi'(2\pi, \lambda)$ correspondingly .

4. Use roots of (3.3) to construct dispersion relation $F(\lambda) = \theta'(2\pi, \lambda) + \varphi(2\pi, \lambda)$ in order to construct the function

$$g'(0, \lambda) + i\lambda g(0, \lambda) = e^{2i\lambda\pi}[i\lambda F(\lambda) + \lambda^2\theta(2\pi, \lambda) - \varphi'(2\pi, \lambda)]$$

together with functions $\theta(2\pi, \lambda)$ and $\varphi'(2\pi, \lambda)$.

5. Construct

$$S\left(\lambda\right)=\frac{g'(0,\lambda)+i\lambda g(0,\lambda)}{g'(0,-\lambda)+i\lambda g(0,-\lambda)}$$

6. Use relation

$$\lim_{\lambda\to-\frac{n}{2}}(n+2\lambda)S\left(\lambda\right)=V_{nn}^{\gamma_1}$$

to find the numbers $V_{nn}^{\gamma_1}$.

7. Find all numbers $V_{n\alpha}^{\gamma_1}$ by using

$$V_{m\alpha+m}^{\gamma_1}=V_{mm}^{\gamma_1}\sum_{n=1}^{\alpha}\frac{V_{nn}^{\gamma_1}}{n+m},\quad \alpha=1,2,...;$$

and use (3.5) for finding the numbers $q_{2\alpha}$ to recover potential $q_2(x)$ on γ_1.

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