Stochastic Modeling and Applications
Vol.26 No. 1 (January-June, 2022)
ISSN: 0972-3641

Received: 06th December 2021 ***Revised: 13th January 2022*** ***Selected: 20th January 2022***

$2b$-COLORABILITY OF SOME WELL KNOWN GRAPHS

S.Saraswathi[1], M. Poobalaranjani[2]

ABSTRACT

A 2-distance coloring of an undirected Graph is a proper coloring of G in which any two vertices at distance less than or equal to 2 receive different colors. A b-coloring of G is a proper coloring of G in which each color class has at least one vertex that has a neighbor in all other color classes. A 2-distance b-coloring of G is a b-coloring as well as a 2-distance coloring and the 2-distance b-chromatic number $\chi_{2b}(G)$ of G is the greatest integer k such that there exists a 2-distance b-coloring of G with k colors. If G has a 2-distance b-coloring, it is referred to as a 2-distance b-colorable graph. In this study, two sufficient conditions are obtained for certain classes of graphs to be $2b$-colorable. Additionally, certain well-known graphs are recognized as $2b$-colorable graphs and not $2b$-colorable graphs.

Keywords: 2-distance coloring, b-coloring, 2-distance b-coloring, 2-distance b-chromatic number, 2-distance b-colorable graph.

1. Introduction

All graphs covered in this article are finite, undirected, and simple. The reader may look to Harary [1] for definitions of any words not stated here. For a graph G, a proper k-coloring is a method of assigning k colors to its vertices in such a way that no two adjacent vertices have the same color. The chromatic number $\chi(G)$ is the smallest integer k with the condition that G is properly colored using k colors. F. Kramer and H. Kramer [3,4] pioneered the notion of distance coloring in 1969. A 2-distance coloring of G is an assignment of colors to its vertices in such a way that any two vertices at a maximum distance 2, get different colors. The 2 distance chromatic number $\chi_2(G)$ of G is the lowest number k for which G has a 2-distance coloring with k colors. A survey paper on distance coloring containing all the main results on distance coloring was published by them [5] in 2008.

Another type of an interesting vertex coloring is b-coloring. It was introduced in 1999 by Irving and Manlove [2]. A b-coloring of G is a proper coloring of G in which each color class has at least one vertex that has a neighbor in every other color class. This vertex is called a color dominating vertex (or simply a cdv). The b-chromatic number $\chi_b(G)$ of G is the maximum k such that G has a b-coloring with k colors. For a graph G, the m-degree $m(G)$ is defined as the maximum i such that G has at least i vertices of degree at least $i-1$. In 2019, we [6] presented a new coloring technique that was a hybrid of 2-distance coloring and b-coloring. Formally, a 2 distance b coloring (or $2b$-coloring) is a 2-distance coloring and also a b-coloring. Consequently, the 2-distance b-chromatic number ($2b$-number) $\chi_{2b}(G)$ is defined as the largest integer k such that G has 2-distance b-coloring with k colors.

In [6], it was proved that not all graphs support 2-distance b-coloring. Hence, a new graph named $2b$-colorable graph which was described as a graph that permits a $2b$-coloring.

A natural question is "Under what condition a graph becomes a $2b$-colorable graph?" or "If a graph is $2b$-colorable, then what are the conditions are to be satisfied?". So suppose, G is a $2b$-colorable graph with k colors. Since, $\chi_b(G) \leq \Delta + 1 \leq \chi_2(G),\ k \leq \chi_b(G) \leq \Delta + 1 \leq \chi_2(G) \leq k$ must hold. These yields $\chi_b(G) = \chi_2(G) = \Delta + 1 = k.$ Thus, $2b$-coloring can exist in a graph and if exists, it requires exactly $\Delta + 1$ colors, nothing more and nothing less. In other words, a coloring which is both a 2-distance coloring and a b-coloring, then the coloring contains exactly $\Delta + 1$ colors. Hence, if a graph admits a $2b$-coloring, then its b-chromatic number attains its maximum value $\Delta + 1$, 2-distance chromatic number attains its minimum value $\Delta + 1$ and the 2-distance b-chromatic number is the common value $\Delta + 1$. Hence, if a graph G is $2b$-colorable, then $\chi_2(G) = \chi_b(G) = \chi_{2b}(G) = \Delta + 1$ is a necessary condition. In paper [6], we obtained some conditions which are sufficient for a graph to be $2b$-colorable. In this paper, two more sufficient conditions for a graph to be $2b$-colorable are obtained. In 2014, the b-chromatic number of various graphs relating to cycles, wheel related graphs Helm and Closed Helm were studied by Vaidya and Shukla [7, 8]. Their study motivated to apply $2b$-coloring on Helm and Closed Helm. In this article, with the bounds of the b-chromatic number of the above graphs, they are classified into $2b$-colorable graphs and non $2b$-colorable graphs.

2. Prior Results

In this section, some prior results based on 2-distance coloring, b-coloring and 2-distance b-coloring which are necessary for this paper are given.

Observation 2.1 : If G is a graph, then

(i) $\chi_2(G) \geq \Delta + 1$; and

(ii) if $\chi_2(G) = \Delta + 1$, then any vertex with degree Δ is a color dominating vertex.

Proposition 2.2: Let G be a graph.

(i) If G is of diameter 2, then $\chi_2(G) = |V(G)|$;

(ii) If G is a tree, then $\chi_2(G) = \Delta + 1$.

Proposition 2.3[3]: For a graph G, $\chi(G) \leq \chi_b(G) \leq m(G) \leq \Delta + 1$.

Labelling Wheel, Helm and Closed Helm graphs 2.4:

(i) The central vertex of W_n is denoted by u and the $(n-1)$-cycle of W_n is denoted by $v_1, v_2, \dots v_{n-1}, v_1$.

(ii) In H_n, the vertices of W_n are labeled as described in (i), and the pendant vertex adjacent to v_i is denoted by w_i.

(iii) Since $V(CH_n) = V(H_n)$, the vertices of CH_n are labeled as described in (ii), where w_i is adjacent to w_{i-1} and w_{i+1}.

Proposition 2.5:

(i) For $n \geq 3,\ \chi_2(P_n) = 3$;

(ii) For $m, n \geq 1,\ \chi_2(B_{m,n}) = \max\{m, n\} + 2$;

(iii) For $n \geq 3,\ \chi_2(C_n) = \begin{cases} 3, & if\ n \equiv 0 (mod\ 3) \\ 4, & if\ n \not\equiv 0 (mod\ 3) \\ & and\ n \neq 5 \\ 5, & if\ n = 5 \end{cases}$;

(iv) For $m \geq 2, n \geq 1$ or $m \geq 1, n \geq 2,\ \chi_2(K_{m,n}) = m + n$.

Proposition 2.6:

(i) For $n \geq 1$, $\chi_b(P_n) = \begin{cases} 1, & if \ n = 1 \\ 2, & if \ 2 \leq n \leq 4 \\ 3, & if \ n \geq 5 \end{cases}$;

(ii) For $n \geq 3$, $\chi_b(C_n) = \begin{cases} 2, & if \ n = 4 \\ 3, & if \ n \geq 5 \end{cases}$;

(iii) For $m \geq 1, n \geq 1$, $\chi_b(K_{m,n}) = min\{m, n\} + 1$;

(iv) For $m \geq 1, n \geq 1$, $\chi_b(B_{m,n}) = 2$;

(v) For $n \geq 4$, $\chi_b(H_n) = \begin{cases} 4, & if \ n = 4 \\ 5, & if \ n \geq 5 \end{cases}$;

(vi) For $n \geq 4$, $\chi_b(CH_n) = \begin{cases} 4, & if \ n = 4, 6 \\ 5, & if \ n = 5 \ or \ n \geq 7 \end{cases}$.

Theorem 2.7 [7]: If a graph G is $2b$-colorable, then $\chi_b(G) = \chi_2(G) = \chi_{2b}(G) = \Delta + 1$.

Corollary 2.8 [7]: If G is $2b$-colorable, then the following hold good:

(i) G has at least $\Delta + 1$ vertices of degree Δ;

(ii) $m(G) = \Delta + 1$;

(iii) Any vertex of degree Δ is a color dominating vertex.

Theorem 2.9 [7]: If G is an incomplete $2b$-colorable graph, then $diam(G) \geq 3$.

3. Two Sufficient Conditions for $2b$-colorability

In this section, two sufficient conditions for a graph to be $2b$-colorable are established.

Theorem 3.1: Suppose that G is an incomplete graph satisfying the following conditions:

(i) $\chi_b(G) = \Delta + 1$;

(ii) There are precisely $\Delta + 1$ vertices of degree Δ;

(iii) Each u in $V(G)$ is either a Δ-degree vertex or adjacent only to vertices of degree Δ.

Then G is $2b$-colorable.

Proof: Assume that c is a b-coloring of G by $(\Delta + 1)$-colors. Since $\chi_b(G) = \Delta + 1$, G has $\Delta + 1$ color classes and each containing at least one cdv. Hence, color dominating vertices are of degree Δ. Since G has exactly $\Delta + 1$ vertices of degree Δ, each vertex of degree Δ is a cdv and each color class contains exactly one cdv. Thus, each cdv has exactly one neighbor in every other color class. If c is a 2-distance coloring, then we are through. So let u, w be any two vertices of G such that $d(u, w) \leq 2$. It is enough to prove that $c(u) \neq c(w)$.

If $d(u, w) = 1$, then as c is a proper coloring, $c(u) \neq c(w)$.

If $d(u, w) = 2$, then a vertex say, x in $V(G)$ exists with the condition that$x$ is a common neighbor of u and w. Consider the following two scenarios.

Case (i): x is a color dominating vertex

Hence, x has no two neighbors in same color class. Therefore, u and w belong to different color classes. Hence, $c(u) \neq c(w)$.

Case (ii): x is not a color dominating vertex

Hence, $d(x) < \Delta$. By the given hypothesis, each neighbor of x is of degree Δ. Hence, $d(u) = d(w) = \Delta$ and hence, u and w are color dominating vertices. Then, the vertices u and w are in different color classes.

Therefore, $c(u) \neq c(w)$.

Theorem 3.2: Let G be a connected graph with a diameter at least three with exactly $\Delta + 1$ vertices of degree Δ. If each $u \in V(G)$ is either of degree Δ or of degree 1, then G is $2b$-colorable.

Proof: Since G is of diameter at least 3, G has at least three Δ-degree vertices. Let u_i, $1 \leq i \leq \Delta + 1$ be the vertices of degree Δ.

For each i, let

$$N_i^{'} = \{x \in N(u_i): d(x) = \Delta\} \text{ and } N_i^{''} = \{x \in N(u_i): d(x) = 1\}$$

Clearly $N(u_i) = N_i^{'} \cup N_i^{''} \& N_i^{'} \cap N_i^{''} = \emptyset$. Hence for each i, $|N_i^{'}| + |N_i^{''}| = \Delta$. Further $N_i^{'} \neq \emptyset$. For, if $N_i^{'} = \emptyset$, for some i, then $N(u_i) = N_i^{''}$ and hence $N[u_i]$ induces $K_{1,\Delta}$. Since G has at least three $u_i' s$, $V - N[u_i] \neq \emptyset$. Since u_i is of full degree in $\langle N(u_i) \rangle$, u_i has no neighbor in $V - N[u_i]$. If $x \in N(u_i)$, then by the choice of$u_i, x \in N_i^{''}(u_i)$. Here $d(x) = 1$ and hence x has no neighbor in $V - N[u_i]$. As no $x \in V - N[u_i]$ can be neighbor to any vertex of $N[u_i]$, G is a disconnected graph, a contradiction.

Now assign distinct $\Delta + 1$ colors to the u_i's. Hence for each i, in $\{u_i\} \cup N_i^{'}$, $|N_i^{'}| + 1$ distinct colors are used. Then for each i, assign the remaining $\Delta - |N_i^{'}| = |N_i^{''}|$ colors to the vertices of $|N_i^{''}|$. Call this coloring as c. Hence, vertices of $N_i^{''}$ also receive distinct colors. Then we observe the following.

(i) All $u_i' s$ are of distinct colors;

(ii) Each u_i has a neighbor from each of the other color classes; $i.e.$, for each i, u_i and its neighbors in $N_i^{'} \cup N_i^{''}$receive distinct $\Delta + 1$ colors;

(iii) Each u_i is a color dominating vertex and every color class includes precisely one cdv.

Claim 1: c is a proper coloring

Let $xy \in E$. If x and y are of equal degree Δ, subsequently $x = u_i$, $y = u_j$ for some i, j, $i \neq j$. By (i), $c(x) \neq c(y)$. If $d(x) = \Delta$ and $d(y) = 1$, then $x = u_i$ for some i, and $y \in N_i''$. By (ii), $c(x) \neq c(y)$. The case both $d(x) = d(y) = 1$ does not arise. Hence the claim.

Claim 2: c is a b-coloring

From (iii) and claim 1, c is a b-coloring.

Claim 3: c is a 2-distance coloring

Let $x, y \in V$ be such that $d(x, y) \leq 2$. If $d(x, y) = 1$, then by claim 1, $c(x) \neq c(y)$. So let $d(x, y) = 2$. Then they have a common neighbour say $v \in V$. Then $d(v) \geq 2$ and hence by hypothesis, $d(v) = \Delta$. Therefore $v = u_i$ for some i. Then $x, y \in N(u_i)$. By (ii), $c(x) \neq c(y)$. Hence the claim.

From claim 2 and claim 3, c is a $2b$-coloring and hence G is $2b$-colorable.

4. $2b$-colorability and Non $2b$-colorability of Some Well Known Graphs

In this section, some well-known graphs are classified into $2b$-colorable graphs and non $2b$-colorable graphs.

Proposition 4.1: Let $n \geq 3$.

(i) If $n = 3$ or 4, P_n is not $2b$-colorable;

(ii) If $n \geq 5$, P_n is $2b$-colorable and $\chi_{2b}(P_n) = 3$.

Proof:

(i) If $n = 3$ or 4 , then from proposition 2.6 (i), $\chi_b(P_n) = 2$ and from proposition 2.5 (i), $\chi_2(P_n) = 3$. Hence, $\chi_b(P_n) \neq \chi_2(P_n)$ and therefore P_n is not $2b$-colorable.

(ii) Suppose $n \geq 5$. Then $\chi_2(P_n) = \chi_b(P_n) = 3$. Hence, $2b$-coloring may exist. Define a coloring $c: V(P_n) \to \{1, 2, 3\}$ as follows:

$$c(v_i) = \begin{cases} 1, & if \ \ i \equiv 1 (mod\ 3) \\ 2, & if \ \ i \equiv 2 (mod\ 3) \\ 3, & if \ \ i \equiv 0 (mod\ 3) \end{cases} \quad \text{------ (1)}$$

Clearly, c is a proper coloring. As any P_3 receives distinct colors, c is a 2-distance coloring. It is obvious that for $2 \leq i \leq n - 1$,

(a) If $i \equiv 1 (mod\ 3)$, then v_i is a 1-cdv;

(b) if $i \equiv 2(mod\ 3)$, then v_i is a 2-cdv;

(c) if $i \equiv 0(mod\ 3)$, then v_i is a 3-cdv.

Hence, c is also a b-coloring and the result follows.

Proposition 4.2:

(i) For $n \geq 2$, $K_{1,n}$ is not $2b$-colorable;

(ii) For $m, n \geq 1$, $B_{m,n}$ is not $2b$-colorable.

Proof:

(i) Since $diam(K_{1,n}) = 2$, from theorem 2.9, $K_{1,n}$ is not $2b$-colorable.

(ii) From proposition 2.6 (iv), $\chi_b(B_{m,n}) = 2$. Further, as $B_{m,n}$ is a tree,from proposition 2.2(ii), $\chi_2(B_{m,n}) = \Delta + 1 = \max\{m, n\} + 2 \neq \chi_b(B_{m,n})$. Hence, $B_{m,n}$ is not $2b$ -colorable.

Proposition 4.3: For $n \geq 4$,

(i) If $n \equiv 0(mod\ 3)$, then C_n is $2b$-colorable and $\chi_{2b}(C_n) = 3$;

(ii) If $n \not\equiv 0(mod\ 3)$, then C_n is not $2b$-colorable.

Proof:

(i) If $n \equiv 0(mod\ 3)$, then $n \geq 6$. Hence, from propositions 2.6(ii) & 2.5(iii), $\chi_b(C_n) = \chi_2(C_n) = 3$. Hence, it is clear that if $2b$-coloring exists, then it has only 3 colors. Let us show that such a coloring exists. It can be seen that, if the edge v_1v_n is removed, then the resultant graph is P_n. Define a coloring c as given in equation (1). Then from proposition 4.1, c is a $2b$-coloring on P_n and hence, a b-coloring as well as a 2-distance coloring on P_n. This gives that c is a b-coloring on C_n. Hence, to show c is a 2-distance coloring on C_n, it is enough to show that $P_3's$ induced by v_{n-1}, v_n, v_1 and v_n, v_1, v_2 receive distinct colors. From equation (1), $c(v_1) = 1, c(v_2) = c(v_{n-1}) = 2$ and $c(v_n) = 3$. Hence,the $P_3's$ acquire different colors and hence, c is a 2-distance coloring on C_n.

(ii) Suppose $n \not\equiv 0(mod\ 3)$. If $n = 5$, then $\chi_2(C_n) = 5$ and $\chi_b(C_n) = 3$. If $n \neq 5$, then $\chi_2(C_n) = 4$ and $\chi_b(C_n) = 3$. In both cases, $\chi_b(C_n) \neq \chi_2(C_n)$ and hence, C_n is not $2b$-colorable.

Proposition 4.4: For $n \geq 4$,

(i) if $n = 5$, then H_n is $2b$-colorable;

(ii) if $n \neq 5$, then H_n is not $2b$-colorable.

Proof: First to prove the following claim.

Claim: For $n \geq 4$, $\chi_2(H_n) = \begin{cases} 5, & if\ n = 4 \\ n, & if\ n \geq 5 \end{cases}$

Proof of Claim: From the labeling of H_n given in 2.4, it is easy to observe the following.

(a) For $1 \leq i \leq n-1$, $d(u, v_i) = 1$ and $d(u, w_i) = 2$;

(b) For $1 \leq i, j \leq n-1$,

$$d(v_i, v_j) = \begin{cases} 1, & if\ |i-j| = 1\ or\ n-2 \\ 2, & if\ |i-j| > 1\ and\ \neq n-2 \end{cases};$$

$$d(w_i, w_j) \geq 3;$$

(c) For $n \geq 4$, $d(u) = n-1$, $d(v_i) = 4$, $d(w_i) = 4$. Hence,

$$\Delta(H_n) = \begin{cases} n, & if\ n = 4 \\ n-1, & if\ n \geq 5 \end{cases}$$

Take a look at the following two instances.

Case (i): $n = 4$

Then, $d(v_i, w_j) = \begin{cases} 1, & if\ i = j \\ 2, & if\ i \neq j \end{cases}$ ---------- (2)

Since in $H_n, W_n = K_4$, assign the colors 1, 2, 3 and 4 to W_n in any order. From (a) and equation (2), in a 2-distance coloring of H_n, none of the above colors can be given to any w_i. From (b), color 5 can be assigned to each w_i. Hence, this is a proper coloring. From (c), $\chi_2(H_n) \geq 5$ and hence this is a minimum 2-distance coloring. Hence, $\chi_2(H_n) = 5$.

Case (ii): $n \geq 5$

Then, $d(v_i, w_j) = \begin{cases} 1, & if \;\; i = j \\ 2, & if \;\; |i-j| = 1 \;\; or \;\; n-2 \\ 3, & if \;\; |i-j| > 1 \;\; and \;\; \neq n-2 \end{cases}$ ---------- (3)

From (i), in a 2-distance coloring of H_n, n colors are required to color W_n of H_n.

Let $c: V(H_n) \to \{1,2, \dots n\}$ be a coloring defined by,

$c(u) = n$ and,

for $1 \leq i \leq n-1$, $c(v_i) = i$ and as $c(w_i) \neq n$, $c(w_i)$ is defined as follows.

$$c(w_i) = \begin{cases} i+2, & if \;\; 1 \leq i \leq n-3 \\ 1, & if \;\; i = n-2 \\ 2, & if \;\; i = n-1 \end{cases}$$

Clearly v_i's receive $n-1$ colors. Similarly, w_i's. Such coloring of H_6 is shown in figure 1.

Clearly, c is a proper coloring. Now to show that it is a 2-distance coloring, consider the vertices v_i and w_j of distance 2 apart.

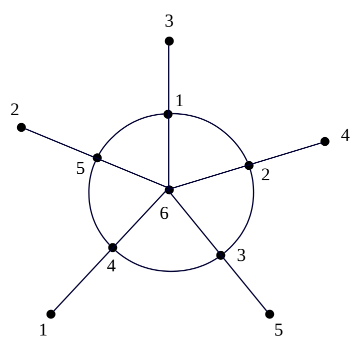

Figure 1: 2 -distance coloring of H_6

Hence, from (3), if $|i-j| = 1$, then $j = i \pm 1$, where addition is taken modulo $n-1$, the colors of w_j are tabulated in table 1.

Since $n \geq 5$, from the 3rd and the last columns, it is clear that, then $c(v_i) \neq c(w_j)$. Now suppose, $|i-j| = n-2$, then either $i = 1$ and $j = n-1$ or $i = n-1$ and $j = 1$. In the first case, $c(v_i) = 1$ and $c(w_j) = 2$, and in the second case $c(v_i) = n-1$ and $c(w_j) = 3$.

Table 1: Colors of w_j with $|i-j| = 1$ & $j = i \pm 1$

j	$j+2$	Range of i/ values of i	Range of j/ values of j	Range of $j+2$/ values of $j+2$	$c(w_j)$
$i+1$	$i+3$	$1 \leq i \leq n-4$	$2 \leq j \leq n-3$	$4 \leq j+2 \leq n-1$	$j+2$
		$i = n-3$	$j = n-2$	$j+2 = n$	1
		$i = n-2$	$j = n-1$	$j+2 = n+1$	2
		$i = n-1$	$j = n$	$j+2 = n+2$	3
$i-1$ $(\neq 0)$	$i+1$	$2 \leq i \leq n-2$	$1 \leq j \leq n-3$	$3 \leq j+2 \leq n-1$	$j+2$
		$i = 1$	$j = n-1$	$j+2 = n+1$	2

		$i = n - 1$	$j = n - 2$	$j + 2 = n$	1

Again as $n \geq 5, c(v_i) \neq c(w_j)$. Hence, c is a 2-distance coloring. Thus, $\chi_2(H_n) \leq n$.

Since $\chi_2(H_n) \geq \Delta + 1$, from (c), $\chi_2(H_n) \geq n$ and the equality holds.

Hence, the claim is proved.

Proof of (i): Suppose $n = 5$. Then the Helm graph H_n is an incomplete graph of diameter at least 3 and contains exactly $\Delta + 1$ vertices of degree Δ. Further, each $u \in V(H_n)$ is either of degree Δ or of degree 1. Hence, by theorem 3.1, H_n is a $2b$-colorable graph. From proposition 2.6 (v), $\chi_b(H_n) = 5$, and from theorem 2.7, $\chi_{2b}(H_n) = 5$.

Proof of (ii): Suppose $n \neq 5$. If $n = 4$, then $\chi_b(H_n) = 4$ and from the claim $\chi_2(H_n) = 5$.

If $n \geq 6$, then $\chi_b(H_n) = 5$ & $\chi_2(H_n) = n$. In both cases, $\chi_b(H_n) \neq \chi_2(H_n)$ and the result follows.

Proposition 4.5: Suppose $n \geq 4$.

(i) If $n = 5$, CH_n is $2b$-colorable and $\chi_{2b}(CH_n) = 5$;

(ii) For $n = 4$ or $n \geq 6$, CH_n is not $2b$-colorable.

Proof: The following claim is proved first.

Claim: For $n \geq 4, \chi_2(CH_n) = \begin{cases} 7, & if\ n = 4 \\ n, & if\ n \geq 5 \end{cases}$

Proof of the claim: Let us recall that CH_n is obtained from H_n by forming the cycle $w_1, w_2, \dots, w_{n-1}, w_1$. It is the outer $(n-1)$-cycle of CH_n.

There are two instances to analyze.

Case (i): $n = 4$

Since $diam\ (CH_n) = 2$, by the proposition 2.2 (i), $\chi_2(CH_n) = |V(CH_n)| = 7$.

Case (ii): $n \geq 5$

In CH_n, $d(w_i) = 3$.Then from (c) in the proof of the claim of proposition 4.4

$\Delta(CH_n) = n - 1 \implies \chi_2(CH_n) \geq n$ ---------- (4)

Define a coloring $c: V(CH_n) \rightarrow \{1,2, \dots n\}$ as given in case of H_n. Since, c is a 2-distance coloring in H_n, it is enough to check the adjacency of the outer cycle of CH_n. As $w_i's$ receive distinct colors, c is also 2-distance coloring in CH_n. Hence, from (4), $\chi_2(CH_n) = n$.

Proof of (i): Let $n = 5$. From proposition 2.6 (vi) and the claim, $\chi_b(CH_5) = \chi_2(CH_5) = 5$. Figure 2 shows a $2b$-coloring of CH_5 with 5 colors.

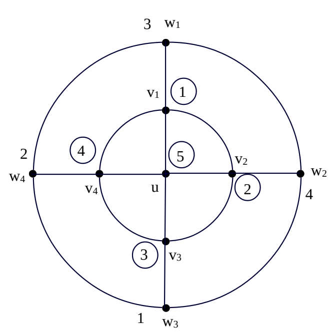

Figure 2: $2b$-coloring of CH_5 with 5 colors

Here, each v_i , $1 \leq i \leq 4$ is an i-cdv and u is an 5-cdv. Thus, CH_n is $2b$-colorable and hence, $\chi_{2b}(CH_5) = 5$.

Proof of (ii): If $n = 4$, then from the same proposition and the claim, $\chi_b(CH_n) = 4$ & $\chi_2(CH_n) = 7$. If $n \geq 6$, then $\chi_b(CH_n) = 4$ or 5 and $\chi_2(CH_n) = n \geq 6$. Hence, in both cases, $\chi_b(CH_n) \neq \chi_2(CH_n)$ and therefore, CH_n is not $2b$-colorable.

5. Conclusion

Two sufficient conditions for certain types of graphs to be $2b$-colorable are found in this article. Further, some of the well-known graphs $K_{1,n}, P_n, B_{m,n}, C_n, H_n$ & CH_n are classified into $2b$-colorable and not $2b$-colorable graphs.

References

1. Harary, F. (1988). Graph Thoery, Narosa /Addison Wesley, Indian Student Edition.
2. Irving, R. W. andManlove, D. F. (1999). The b-chromatic number of a graph, Discrete Applied Mathematics, 91, 127–141.
3. Kramer, F. and Kramer, H. (1969). Un probleme de coloration des sommets d'un graphe, C.R. Acad. Sci. Paris A 268, 46–48.
4. Kramer, F. and Kramer, H. (1969). EinFärbungs problem der Knotenpunkteeines Graphen bezüglich der Distanz *p*, Rev. Roumaine Math. Pures Appl. 14 (2), 1031–1038.
5. Kramer F. and Kramer H. (2008). A survey on Distance coloring, Discret mathematics, 308(2-3), 422-426
6. Poobalaranjani, M. and Saraswathi S. (2019). 2-Distance b-coloring of graphs, Compliance Engineering Journal, Volume 10, Issue 11, 161-165.
7. Vaidya, S. K. and Shukla, M. S. (2014). b-Chromatic number of some cycle related graphs, International Journal of Mathematics and Soft Computing, 4, 113-127.
8. Vaidya, S. K. and Shukla, M. S. (2014). The b-Chromatic Number of Helm and Closed Helm, International Journal of Mathematics and Scientific Computing, Vol. 4, No. 2, pp. 43-47.

S.SARASWATHI[1], M. POOBALARANJANI[2]:[1,2] PG & Research Department of Mathematics Seethalakshmi Ramaswami College (Autonomous), Affiliated to Bharathidasan University, Tiruchirappalli, Tamil Nadu, India.
Email: [1]sarassrc75@gmail.com, [2] mpranjani@hotmail.com