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BAYESIAN ANALYSIS OF TWO PARAMETER WEIBULL DISTRIBUTION USING DIFFERENT LOSS FUNCTIONS

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Abstract. In this paper, we study the parameter estimation of Weibull distribution using Bayesian approach. Here, we use both informative and non-informative priors. We obtain the estimators and their posterior risks under various asymmetric and symmetric loss functions. Since, Bayes estimators are not having a closed form under these loss functions, we use an approximation technique introduced by Lindley to found out the Bayes estimates. A comparative study is also done between the proposed estimators using Monte Carlo simulation study based on the associated posterior risk. We also examine the effect of various loss functions using different priors.

1. Introduction

Weibull distribution, introduced by Waloddi Weibull (1951), is the most commonly used life time distribution. It is one among the important distributions used in reliability and risk assessment because of its wide variety of shapes in accordance with altering parameters. This distribution has also widely employed in many fields of renewable energy, geothermal energy, medical, biological, environmental and earth sciences (Rinne, 2008).
The probability density function for the two parameter Weibull distribution with scale and shape parameters α and δ respectively both positive is given by

$$f\left(x \mid \alpha, \delta\right) = \left(\frac{\delta}{\alpha}\right)\left(\frac{x}{\alpha}\right)^{\delta-1} exp\left[-\left(\frac{x}{\alpha}\right)^{\delta}\right],\ x > 0. \tag{1.1}$$

Figure 1 and Figure 2 respectively shows the graph of Weibull probability density functions for various choices of α and δ.

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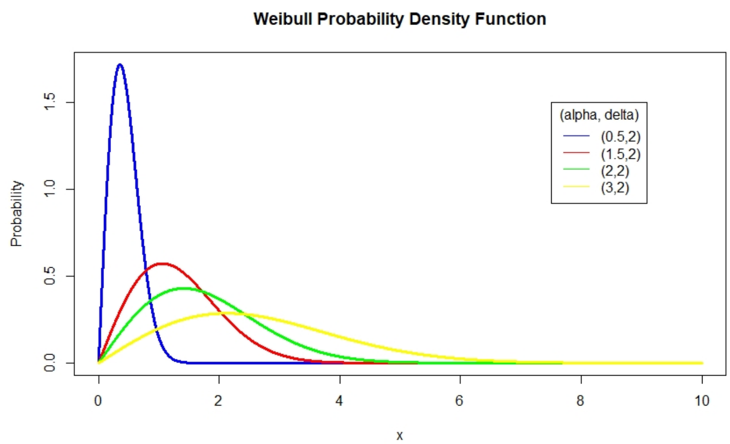

FIGURE 1. plot of Weibull distribution for selected values of α and fixed δ

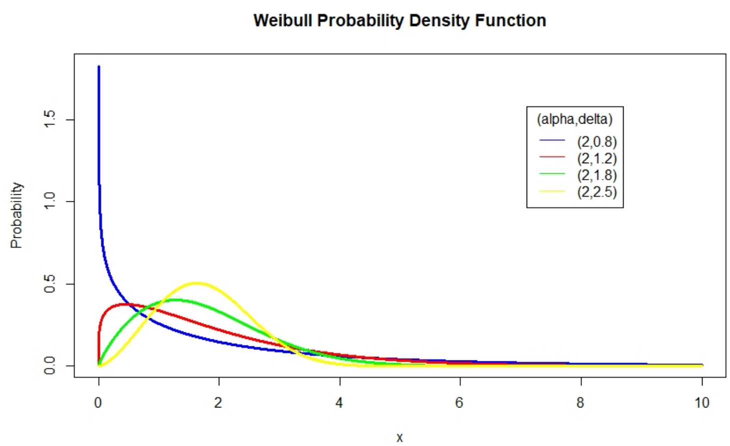

FIGURE 2. plot of Weibull distribution for selected values of δ and fixed α

Depending on values of the shape parameter δ, the two parameter Weibull distribution may have increasing or decreasing hazard function. Because of its skewness, this distribution can be considered as an initial choice when modeling monotone hazard rates. This distribution is quite useful to model extreme value data also.

Various studies have been done to obtain the best estimation method for Weibull distribution. Though there is an enormous number of literature available on the estimation of Weibull distribution using the frequentist approach, researchers has been given much attention recently, to the Bayesian inference of the Weibull parameters. Banerjee and Kundu (2008) have been considered the Gamma prior for censored data from two parameter Weibull distribution. Using Jeffrey and extended Jeffrey prior Ahmed et al. (2010) compared the Bayesian estimation with

MLE for Weibull distribution. Pandey et al. (2011) studied the scale parameter of Weibull distribution under LINEX loss function with a comparison to the maximum likelihood estimator. Huang and Wu (2011) discussed the Bayesian method for the parameters of Weibull model with progressive censoring using a bivariate prior distribution under squared error loss and compared it to the method of maximum likelihood. Guure et al. (2012) worked on two parameter Weibull distribution and derived the Bayes estimators under three loss functions using extension of Jeffrey's prior. A Bayesian study of two parameter Weibull distribution under various loss functions and different priors by Aslam et al. (2014). Babacan and Kaya (2019) conducted a simulation study of the Bayes estimator for parameters in Weibull distribution under Gamma priors and squared error loss function. They concluded that since the Bayesian estimator of the parameters cannot be found out in explicit form, they used the approximation technique by Lindley to obtain the approximate estimators. Yanuar et al. (2019) provided Bayesian estimate of scale parameter of Weibull distribution using inverse gamma and Jeffrey's prior. Yilmaz et al. (2020) conducted a comparative study on Bayesian and classical estimation methods of the parameters for Weibull distribution. In classical method, he estimated the parameters by using nine different methods and in Bayesian he computed the Bayesian estimators using two methods viz. Lindley approximation method and Tierney Kadane approximation method under different symmetric and asymmetric loss functions. He also suggested Lindley approximation method for computation of Bayesian estimators.
Bayes decision becomes a Bayes estimator when the decision minimizes risk function. The best decision is that with minimum posterior risk. The decision-theoretical viewpoint consider additional information related to the possible consequences of the decisions. Hence, the choice of suitable loss function is very crucial in decision theory. Therefore, to obtain Bayes estimators we consider different loss functions. Again, in Bayesian approach the major problem for a specific model depends upon the selection of prior distribution and loss functions. Hence, we use different priors, both informative and non-informative to compute the estimators and corresponding posterior risks under different loss functions.

The remaining sections of the paper are arranged as follows. In section 2, we discuss the maximum likelihood method of estimation for estimating the Weibull parameters. Posterior distribution using both informative and non-informative priors are derived in Section 3. In section 4, we use Lindley's approximation technique to obtain Bayes estimators under various loss functions (symmetric and asymmetric) using different priors. Elicitation of hyperparameters are explained in Section 5. In section 6, some simulation studies are conducted to illustrate the performance of different estimators based on posterior risk. The effect of different loss functions based on informative and non-informative priors are also studied here. Some conclusions are made in section 7.

2. Maximum Likelihood Estimation

Let $x_1, x_2..., x_n$ denotes a random sample of size n drawn from Weibull distribution with probability density function (1.1).

Then the likelihood function is

$$\ell(x;\alpha,\delta)=\prod_{i=1}^{n}\left(\frac{\delta}{\alpha}\right)\left(\frac{x_i}{\alpha}\right)^{(\delta-1)}e^{-}\left(\frac{x_i}{\alpha}\right)^{\delta} \tag{2.1}$$

so that the log-likelihood equation can be defined as

$$L=nlog(\delta)-n\delta log(\alpha)+(\delta-1)\sum_{i=1}^{n}log(x_i)-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}. \tag{2.2}$$

Under differentiation of equation (2.2) with respect to α and δ and equating to zero, equation (2.2) becomes

$$\frac{\partial L}{\partial\alpha}=-n\left(\frac{\delta}{\alpha}\right)+\left(\frac{\delta}{\alpha}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}=0 \text{ and} \tag{2.3}$$

$$\frac{\partial L}{\partial\delta}=\frac{n}{\delta}-nlog(\alpha)+\sum_{i=1}^{n}log(x_i)-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}log\left(\frac{x_i}{\alpha}\right)=0. \tag{2.4}$$

This system of equations are infeasible to obtain their explicit solutions. In this situation, the ML estimates of α and δ are computed by using "MASS" package in RStudio. Nelder-Mead optimization method is employed for solving the equations analytically.

3. Priors and Posterior Distributions

Recently, Bayesian approach became more popular for analyzing failure time data. If the prior knowledge of the parameter is in hand it is better to make use of an informative prior. Otherwise, make use of a prior which is non-informative. Posterior distribution is the combination of sample information within the likelihood function and the probabilistic information about the parameters available as prior distribution. Here, we consider Weibull model as the sampling distribution, for deriving posterior distribution it is mingled with both informative and non-informative priors.

3.1. Posterior Distribution using Non-Informative Prior

Here, we use Jeffreys' (1946) prior based on Fisher's information given by

$$\pi(\gamma)\propto\sqrt{I(\gamma).}$$

Sinha and Sloan (1988) deals with the non-informative prior for the two parameter Weibull distribution given by

$$\pi_1(\alpha,\delta)\propto\left(\frac{1}{\alpha\delta}\right). \tag{3.1}$$

Then the joint posterior distribution using (2.1) and (3.1) is given by

$$\pi_1(\alpha,\delta\mid x)=\frac{1}{K_1}\left(\frac{1}{\alpha\delta}\right)\left(\frac{\delta^n}{\alpha^{n\delta}}\right)\prod_{i=1}^{n}x_i^{\delta-1}e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}};\ \alpha,\delta>0 \tag{3.2}$$

where
$$K_1 = \int_0^\infty \int_0^\infty \left(\frac{1}{\alpha\delta}\right)\left(\frac{\delta^n}{\alpha^{n\delta}}\right)\prod_{i=1}^{n} x_i^{\delta-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}} d\alpha d\delta.$$

3.2. Posterior Distribution using Informative Priors

Here, we consider two informative priors viz. Gamma Prior and Gumbel type-II Prior for both α and δ.

3.2.1. Posterior Distribution using Gamma Prior

Here, we consider independent gamma priors for both α and δ of Weibull distribution. That is $\alpha \sim gamma(a_2, b_2)$ and $\delta \sim gamma(a_1, b_1)$.

The joint prior distribution of α and δ is given by

$$\pi_2(\alpha,\delta) \propto \delta^{a_1-1}e^{-b_1\delta}\alpha^{a_2-1}e^{-b_2\alpha};\ \alpha,\delta,a_1,b_1,a_2,b_2>0 \text{ and thereby} \tag{3.3}$$

the joint posterior distribution of α and δ is

$$\pi_2(\alpha,\delta \mid x) = \frac{1}{K_2}\delta^{a_1-1}e^{-b_1\delta}\alpha^{a_2-1}e^{-b_2\alpha}\left(\frac{\delta^n}{\alpha^{n\delta}}\right)\prod_{i=1}^{n} x_i^{\delta-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}} \tag{3.4}$$

where $K_2 = \int_0^\infty \int_0^\infty \delta^{a_1-1}e^{-b_1\delta}\alpha^{a_2-1}e^{-b_2\alpha}\left(\frac{\delta^n}{\alpha^{n\delta}}\right)\prod_{i=1}^{n} x_i^{\delta-1}e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}} d\alpha d\delta.$

3.2.2. Posterior Distribution using Gumbel type-II Prior

We propose here that δ and α follow independent Gumbel type II priors having probability density functions

$$\pi(\delta) = \frac{a_3}{\delta^2}e^{-\left(\frac{a_3}{\delta}\right)};\ \delta, a_3 > 0 \text{ and}$$
$$\pi(\alpha) = \frac{b_3}{\alpha^2}e^{-\left(\frac{b_3}{\alpha}\right)};\ \alpha, b_3 > 0$$

respectively.

Then the joint prior distribution is given by

$$\pi_3(\alpha,\delta) \propto \frac{e^{\left(\frac{-a_3}{\delta}\right)}}{\delta^2}\frac{e^{\left(\frac{-b_3}{\alpha}\right)}}{\alpha^2};\ \alpha,\delta,a_3,b_3>0 \text{ and} \tag{3.5}$$

the joint posterior distribution of the parameters α and δ is

$$\pi_3(\alpha,\delta \mid x) = \frac{1}{K_3}\frac{e^{\left(\frac{-a_3}{\delta}\right)}}{\delta^2}\frac{e^{\left(\frac{-b_3}{\alpha}\right)}}{\alpha^2}\left(\frac{\delta^n}{\alpha^{n\delta}}\right)\prod_{i=1}^{n} x_i^{\delta-1} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}} \tag{3.6}$$

where $K_3 = \int_0^\infty \int_0^\infty \frac{e^{\left(\frac{-a_3}{\delta}\right)}}{\delta^2}\frac{e^{\left(\frac{-b_3}{\alpha}\right)}}{\alpha^2}\left(\frac{\delta^n}{\alpha^{n\delta}}\right)\prod_{i=1}^{n} x_i^{\delta-1}e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}} d\alpha d\delta.$

4. Bayes Estimators and Posterior Risks under Different Loss Functions

In this section, we considered different loss functions (asymmetric and symmetric) to obtain Bayes estimators and their respective posterior risks for the unknown parameters of Weibull distribution. The SELF (squared error loss function), PLF (precautionary loss function) and LINEX (linear exponential loss function) are the different loss functions used to obtain Bayes estimators and their respective posterior risks. For more details, see, Legendre (1805), Norstrom (1996) and Soliman (2000).

Bayes estimators and their posterior risks are explained in Table 1 for the unknown parameter γ under mentioned loss functions.

Table 1. Bayes estimators and posterior risks under different loss functions

Loss Function	Bayes Estimator (BE)	Posterior Risk (PR)
$SELF = (d-\gamma)^2$	$E(\gamma\|x)$	$E(\gamma^2\|x) - [E(\gamma\|x)]^2$
$PLF = \frac{(d-\gamma)^2}{d}$	$\sqrt{E(\gamma^2\|x)}$	$2[\sqrt{E(\gamma^2\|x)} - E(\gamma\|x)]$
$LINEX = e^{m(d-\gamma)} - m(d-\gamma) - 1$	$\frac{-1}{m} log[E(e^{-m\gamma\|x})]$	$log[E(e^{-m\gamma\|x})] + mE(\gamma\|x)$

4.1. Lindley Approximation

The joint posterior distribution of α and δ in the equations (3.2), (3.4) and (3.6) takes a ratio form of two integrals which cannot be solved analytically.
The posterior expectation can be expressed as

$$I(x) = E[u(\alpha,\delta)|x] = \frac{\int u(\alpha,\delta) exp[L(\alpha,\delta) + \rho(\alpha,\delta)]d(\alpha,\delta)}{\int exp[L(\alpha,\delta) + \rho(\alpha,\delta)]d(\alpha,\delta)}$$

where $u(\alpha,\delta)$ is a function of α and δ only, $L(\alpha,\delta)$ is the log-likelihood and $\rho(\alpha,\delta)$ is the log of joint prior of α and δ.

According to Lindley (1980), if the sample size n is sufficiently large, the above equation can be approximately evaluated as:

$$\begin{aligned} I(x) = u(\hat{\alpha},\hat{\delta}) &+ 0.5[(\hat{u}_{\alpha\alpha} + 2\hat{u}_{\alpha}\hat{\rho}_{\alpha})\hat{\sigma}_{\alpha\alpha} + (\hat{u}_{\delta\alpha} + 2\hat{u}_{\delta}\hat{\rho}_{\alpha})\hat{\sigma}_{\delta\alpha} + (\hat{u}_{\alpha\delta} + 2\hat{u}_{\alpha}\hat{\rho}_{\delta})\hat{\sigma}_{\alpha\delta} \\ &+ (\hat{u}_{\delta\delta} + 2\hat{u}_{\delta}\hat{\rho}_{\delta})\hat{\sigma}_{\delta\delta}] + 0.5[(\hat{u}_{\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{u}_{\delta}\hat{\sigma}_{\alpha\delta})(\hat{L}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{L}_{\alpha\delta\alpha}\hat{\sigma}_{\alpha\delta} + \hat{L}_{\delta\alpha\alpha}\hat{\sigma}_{\delta\alpha} \\ &+ \hat{L}_{\delta\delta\alpha}\hat{\sigma}_{\delta\delta}) + (\hat{u}_{\alpha}\hat{\sigma}_{\delta\alpha} + \hat{u}_{\delta}\hat{\sigma}_{\delta\delta})(\hat{L}_{\delta\alpha\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{L}_{\alpha\delta\alpha}\hat{\sigma}_{\alpha\delta} + \hat{L}_{\delta\alpha\delta}\hat{\sigma}_{\delta\alpha} + \hat{L}_{\delta\delta\delta}\hat{\sigma}_{\delta\delta})] \end{aligned}$$

where $\hat{\alpha}$ and $\hat{\delta}$ are the MLE's of α and δ respectively.

$\hat{u}_{\alpha} = \frac{\partial u(\hat{\alpha},\hat{\delta})}{\partial \alpha}$; $\hat{u}_{\delta} = \frac{\partial u(\hat{\alpha},\hat{\delta})}{\partial \delta}$; $\hat{u}_{\alpha\alpha} = \frac{\partial^2 u(\hat{\alpha},\hat{\delta})}{\partial \alpha^2}$; $\hat{u}_{\delta\delta} = \frac{\partial^2 u(\hat{\alpha},\hat{\delta})}{\partial \delta^2}$; $\hat{u}_{\alpha\delta} = \frac{\partial^2 u(\hat{\alpha},\hat{\delta})}{\partial \alpha \partial \delta}$
$\hat{L}_{\alpha\alpha\alpha} = \frac{\partial^3 L(\hat{\alpha},\hat{\delta})}{\partial \alpha^3}$; $\hat{L}_{\delta\delta\delta} = \frac{\partial^3 L(\hat{\alpha},\hat{\delta})}{\partial \delta^3}$; $\hat{L}_{\alpha\delta} = \frac{\partial^2 L(\hat{\alpha},\hat{\delta})}{\partial \alpha \partial \delta}$ and so on.

Since, α and δ are independent $\hat{\sigma}_{\alpha\delta} = 0$.

With the above defined expressions, the values of the estimates for Weibull distribution are as follows.

(4.1)

$$E[u(\alpha,\delta)|x] = u(\hat{\alpha},\hat{\delta}) + 0.5[(\hat{u}_{\alpha\alpha}\hat{\sigma}_{\alpha\alpha}) + (\hat{u}_{\delta\delta}\hat{\sigma}_{\delta\delta})] + \hat{u}_{\alpha}\hat{\rho}_{\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{u}_{\delta}\hat{\rho}_{\delta}\hat{\sigma}_{\delta\delta} + 0.5[(\hat{u}_{\alpha}\hat{\sigma}^2_{\alpha\alpha}L_{\alpha\alpha\alpha}) + (\hat{u}_{\delta}\hat{\sigma}^2_{\delta\delta}L_{\delta\delta\delta})]$$

where $L(\alpha,\delta)$ is the log-likelihood equation (2.2). Also

$$L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2} = n\left(\frac{\delta}{\alpha^2}\right) - \left(\frac{\delta}{\alpha}\right)^2 \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta} - \frac{\delta}{\alpha^2}\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta},$$

$$\hat{\sigma}_{\alpha\alpha} = -\frac{1}{L_{\alpha\alpha}}, \tag{4.2}$$

$$\begin{aligned} L_{\alpha\alpha\alpha} = \frac{\partial^3 L}{\partial\alpha^3} = -2n\left(\frac{\delta}{\alpha^3}\right) + \left(\frac{\delta^3}{\alpha^3}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta} + 2\left(\frac{\delta^2}{\alpha^3}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta} \\ + \left(\frac{\delta^2}{\alpha^3}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta} + 2\left(\frac{\delta}{\alpha^3}\right)\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}, \end{aligned} \tag{4.3}$$

$$L_{\delta\delta} = \frac{\partial^2 L}{\partial\delta^2} = \frac{-n}{\delta^2} - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}\left(log\left(\frac{x_i}{\alpha}\right)\right)^2,$$

$$\hat{\sigma}_{\delta\delta} = -\frac{1}{L_{\delta\delta}} \text{ and} \tag{4.4}$$

$$L_{\delta\delta\delta} = \frac{\partial^3 L}{\partial\delta^3} = \frac{2n}{\delta^3} - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\delta}\left(log\left(\frac{x_i}{\alpha}\right)\right)^3. \tag{4.5}$$

The log joint prior density of Jeffrey's prior is

$$\rho(\alpha,\delta) = -log(\alpha) - log(\delta),$$

$$\rho_{\alpha} = \frac{\partial\rho(\alpha,\delta)}{\partial\alpha} = \frac{-1}{\alpha} \text{ and} \tag{4.6}$$

$$\rho_{\delta} = \frac{\partial\rho(\alpha,\delta)}{\partial\delta} = \frac{-1}{\delta}. \tag{4.7}$$

The log joint prior density of Gamma prior is as follows:

$$\rho(\alpha,\delta) = (a_1-1)log(\delta) - b_1\delta + (a_2-1)log(\alpha) - b_2\alpha,$$

$$\rho_{\alpha} = \frac{\partial\rho(\alpha,\delta)}{\partial\alpha} = \frac{a_2-1}{\alpha} - b_2 \text{ and} \tag{4.8}$$

$$\rho_{\delta} = \frac{\partial\rho(\alpha,\delta)}{\partial\delta} = \frac{a_1-1}{\delta} - b_1. \tag{4.9}$$

Again, the log of the joint prior density of Gumbel type-II prior is

$$\rho(\alpha,\delta) = \frac{-a_3}{\delta} - log(\delta^2) - \frac{-b_3}{\alpha} - log(\alpha^2),$$

$$\rho_\alpha = \frac{\partial\rho(\alpha,\delta)}{\partial\alpha} = \frac{b_3 - 2\alpha}{\alpha^2} \text{ and} \tag{4.10}$$

$$\rho_\delta = \frac{\partial\rho(\alpha,\delta)}{\partial\delta} = \frac{a_3 - 2\delta}{\delta^2}. \tag{4.11}$$

Table 2 shows the $u(\alpha)$ and $u(\delta)$ functions under the mentioned loss functions.

TABLE 2

Loss Function	$u(\alpha)$	u_α	$u_{\alpha\alpha}$	$u(\delta)$	u_δ	$u_{\delta\delta}$
SELF	α	1	0	δ	1	0
PLF	α^2	2α	2	δ^2	2δ	2
LINEX	$e^{-m\alpha}$	$-me^{-m\alpha}$	$m^2e^{-m\alpha}$	$e^{-m\delta}$	$-me^{-m\delta}$	$m^2e^{-m\delta}$

We derive the estimates of α and δ under the mentioned loss functions using Jeffrey's, Gamma and Gumbel type-II prior as follows.

4.1.1. Lindley's Approximation of α and δ using SELF

The approximate Bayes estimates of δ using Jeffrey's, Gamma and Gumbel type-II priors under SELF are obtained respectively as follows.

$$\hat{\delta}_{JS} = \hat{\delta} + \left(\frac{-1}{\hat{\delta}}\right)\sigma_{\delta\delta} + 0.5(L_{\delta\delta\delta}\sigma^2_{\delta\delta}), \tag{4.12}$$

$$\hat{\delta}_{GS} = \hat{\delta} + \left(\frac{a_1 - 1}{\hat{\delta}} - b_1\right)\sigma_{\delta\delta} + 0.5(L_{\delta\delta\delta}\sigma^2_{\delta\delta}) \text{ and} \tag{4.13}$$

$$\hat{\delta}_{GUS} = \hat{\delta} + \left(\frac{a_3 - 2\hat{\delta}}{\hat{\delta}^2}\right)\sigma_{\delta\delta} + 0.5(L_{\delta\delta\delta}\sigma^2_{\delta\delta}) \text{ respectively.} \tag{4.14}$$

The posterior risks of δ using Jeffrey's, Gamma and Gumbel type-II priors under SELF are obtained and they are

$$R(\hat{\delta}_{JS}) = \hat{\delta}^2 - \sigma_{\delta\delta} + (\hat{\delta}L_{\delta\delta\delta}\sigma^2_{\delta\delta}) - (\hat{\delta}_{JS})^2, \tag{4.15}$$

(4.16)

$$R(\hat{\delta}_{GS}) = \hat{\delta}^2 + 0.5(2\sigma_{\hat{\delta}\delta}) + 2\hat{\delta}\left(\frac{a_1 - 1}{\hat{\delta}} - b_1\right)\sigma_{\delta\delta} + 0.5(2\hat{\delta}L_{\delta\delta\delta}\sigma^2_{\delta\delta}) - (\hat{\delta}_{GS})^2 \text{ and}$$

$$R(\hat{\delta}_{GUS}) = \hat{\delta}^2 + 0.5(2\sigma_{\delta\delta}) + 2\hat{\delta}\left(\frac{a_3 - 2\hat{\delta}}{\hat{\delta}^2}\right)\sigma_{\delta\delta} + 0.5(2\hat{\delta}L_{\delta\delta\delta}\sigma^2_{\delta\delta}) - (\hat{\delta}_{GUS})^2. \tag{4.17}$$

The approximate Bayes estimates of α using Jeffrey's, Gamma and Gumbel type-II priors under SELF are in the following form.

$$\hat{\alpha}_{JS} = \hat{\alpha} + \left(\frac{-1}{\hat{\alpha}}\right)\sigma_{\alpha\alpha} + 0.5(L_{\alpha\alpha\alpha}\sigma^2_{\alpha\alpha}), \tag{4.18}$$

$$\hat{\alpha}_{GS} = \hat{\alpha} + \left(\frac{a_2 - 1}{\hat{\alpha}} - b_2\right)\sigma_{\alpha\alpha} + 0.5(L_{\alpha\alpha\alpha}\sigma^2_{\alpha\alpha}) \text{ and} \tag{4.19}$$

$$\hat{\alpha}_{GUS} = \hat{\alpha} + \left(\frac{b_3 - 2\hat{\alpha}}{\hat{\alpha}^2}\right)\sigma_{\alpha\alpha} + 0.5(L_{\alpha\alpha\alpha}\sigma^2_{\alpha\alpha}). \tag{4.20}$$

Also, the posterior risks of α using Jeffrey's, Gamma and Gumbel type-II priors under SELF are obtained respectively as follows.

$$R(\hat{\alpha}_{JS}) = \hat{\alpha}^2 + \sigma_{\alpha\alpha} - 2\sigma_{\alpha\alpha} + \hat{\alpha}L_{\alpha\alpha\alpha}\sigma^2_{\alpha\alpha} - (\hat{\alpha}_{JS})^2, \tag{4.21}$$

(4.22)

$$R(\hat{\alpha}_{GS}) = \hat{\alpha}^2 + 0.5(2\sigma_{\alpha\alpha}) + 2\hat{\alpha}\left(\frac{a_2 - 1}{\hat{\alpha}} - b_2\right)\sigma_{\alpha\alpha} + 0.5(2\hat{\alpha}L_{\alpha\alpha\alpha}\sigma_{\alpha}\alpha^2) - (\hat{\alpha}_{GS})^2 \text{ and}$$

(4.23)

$$R(\hat{\alpha}_{GUS}) = \hat{\alpha}^2 + 0.5(2\sigma_{\alpha\alpha}) + 2\hat{\alpha}\left(\frac{b_3 - 2\hat{\alpha}}{\hat{\alpha}^2}\right)\sigma_{\alpha\alpha} + 0.5(2\hat{\alpha}L_{\alpha\alpha\alpha}\sigma^2_{\alpha\alpha}) - (\hat{\alpha}_{GUS})^2.$$

4.1.2. Lindley's Approximation of α and δ using PLF

The approximate Bayes estimates of δ using Jeffrey's, Gamma and Gumbel type-II priors under PLF are obtained ad they are

$$\hat{\delta}_{JP} = \left(\hat{\delta}^2 + \sigma_{\delta\delta} - 2\sigma_{\delta\delta} + \hat{\delta}L_{\delta\delta\delta}\sigma^2_{\delta\delta}\right)^{\frac{1}{2}}. \tag{4.24}$$

$$\hat{\delta}_{GP} = \left(\hat{\delta}^2 + 0.5(2\sigma_{\delta\delta}) + 2\hat{\delta}\left(\frac{a_1 - 1}{\hat{\delta}} - b_1\right)\sigma_{\delta\delta} + 0.5(2\hat{\delta}L_{\delta\delta\delta}\sigma^2_{\delta\delta})\right)^{\frac{1}{2}}. \tag{4.25}$$

(4.26)

$$\hat{\delta}_{GUP} = \left(\hat{\delta}^2 + 0.5(2\sigma_{\delta\delta}) + 2\hat{\delta}\left(\frac{a_3 - 2\hat{\delta}^2}{\hat{\delta}}\right)\sigma_{\delta\delta} + 0.5(2\hat{\delta}L_{\delta\delta\delta}\sigma^2_{\delta\delta})\right)^{\frac{1}{2}} \text{ respectively.}$$

The corresponding posterior risks under PLF are attained from the equations (4.12)-(4.14) and (4.24)-(4.26).

By using the similar procedure, the approximate Bayes estimates and posterior risks of α using Jeffrey's, Gamma and Gumbel type-II priors under PLF are obtained.

4.1.3. Lindley's Approximation of α and δ using LINEX

The approximate Bayes estimates of δ using Jeffrey's, Gamma and Gumbel type-II priors under LINEX are obtained respectively as follows.

(4.27)
$$\hat{\delta}_{JL} = \frac{-1}{m} log[e^{-m\hat{\delta}} + 0.5\left(m^2 e^{-m\hat{\delta}}\sigma_{\delta\delta}\right) + \left(\frac{1}{\hat{\delta}}\right) a e^{-m\hat{\delta}}\sigma_{\delta\delta} - 0.5\left(m e^{-m\hat{\delta}} L_{\delta\delta\delta}\sigma^2_{\delta\delta}\right)],$$

(4.28)
$$\hat{\delta}_{GL} = \frac{-1}{m} log\left[e^{-m\hat{\delta}} + 0.5\left(m^2 e^{-m\hat{\delta}}\sigma_{\delta\delta}\right) - m e^{-m\hat{\delta}}\left(\frac{a_1 - 1}{\hat{\delta}} - b_1\right)\sigma_{\delta\delta} - 0.5\left(m e^{-m\hat{\delta}} L_{\delta\delta\delta}\sigma^2_{\delta\delta}\right)\right] \text{ and}$$

(4.29)
$$\hat{\delta}_{GUL} = \frac{-1}{m} log\left[e^{-m\hat{\delta}} + 0.5\left(m^2 e^{-m\hat{\delta}}\sigma_{\delta\delta}\right) - m e^{-m\hat{\delta}}\left(\frac{a_3 - 2\hat{\delta}}{\hat{\delta^2}}\right)\sigma_{\delta\delta} - 0.5\left(m e^{-m\hat{\delta}} L_{\delta\delta\delta}\sigma^2_{\delta\delta}\right)\right].$$

Again, the approximate Bayes estimates of α using Jeffrey's, Gamma and Gumbel type-II priors under LINEX are obtained respectively as follows.

(4.30)
$$\hat{\alpha}_{JL} = \frac{-1}{m} log[e^{-m\hat{\alpha}} + 0.5\left(m^2 e^{-m\hat{\alpha}}\sigma_{\alpha\alpha}\right) + \left(\frac{1}{\hat{\alpha}}\right) m e^{-m\hat{\alpha}}\sigma_{\alpha\alpha} - 0.5\left(m e^{-m\hat{\alpha}} L_{\alpha\alpha\alpha}\sigma^2_{\alpha\alpha}\right)],$$

(4.31)
$$\hat{\alpha}_{GL} = \frac{-1}{m} log\left[e^{-m\hat{\alpha}} + 0.5\left(m^2 e^{-m\hat{\alpha}}\sigma_{\alpha\alpha}\right) - m e^{-m\hat{\alpha}}\left(\frac{a_2 - 1}{\hat{\alpha}} - b_2\right)\sigma_{\alpha\alpha} - 0.5\left(m e^{-m\hat{\alpha}} L_{\alpha\alpha\alpha}\sigma^2_{\alpha\alpha}\right)\right] \text{ and}$$

(4.32)
$$\hat{\alpha}_{GUL} = \frac{-1}{m} log\left[e^{-m\hat{\alpha}} + 0.5\left(m^2 e^{-m\hat{\alpha}}\sigma_{\alpha\alpha}\right) - m e^{-m\hat{\alpha}}\left(\frac{b_3 - 2\hat{\alpha}}{\hat{\alpha^2}}\right)\sigma_{\alpha\alpha} - 0.5\left(m e^{-m\hat{\alpha}} L_{\alpha\alpha\alpha}\sigma^2_{\alpha\alpha}\right)\right].$$

We follow the similar procedure for attaining the posterior risks of α and δ under LINEX loss function.

5. Elicitation of Hyperparameters

Here, elicitation of hyperparameters is performed by using the method suggested by Aslam (2003). For that we require the derivation of prior predictive distribution and which is obtained as

$$\pi(z) = \int_{\gamma} f(z|\gamma)\pi(\gamma)d\gamma.$$

Then using gamma prior the prior predictive distribution is obtained as,

$$\pi(z) = \frac{{b_1}^{a_1}{b_2}^{a_2}}{\Gamma(a_1)\Gamma(a_2)} \int_0^{\infty}\int_0^{\infty} \frac{\delta^{a_1}}{\alpha^{\delta - a_2 - 1}} z^{\delta-1} e^{-((\frac{x}{\alpha})^{\delta} - b_1\delta - b_2\alpha)} d\alpha d\delta, \ z > 0. \tag{5.1}$$

We have to consider four different intervals for the elicitation of the four hyperparameters. Here, consider equation (5.1), the expert probabilities given are 0.01, 0.02, 0.1 and 0.2 for the intervals (0.2, 1.2), (1.2, 2.2), (2.2, 3.2) and (3.2, 4.2) respectively so that

$\int_{0.2}^{1.2} p(z)dz = 0.01, \quad \int_{1.2}^{2.2} p(z)dz = 0.02,$

$\int_{2.2}^{3.2} p(z)dz = 0.1, \quad \int_{3.2}^{4.2} p(z)dz = 0.2.$

The above equations are solved simultaneously by using the Mathematica software. Thus, the elicited values of the hyperparameters a_1, b_1, a_2 and b_2 obtained by applying this methodology are 1.68, 3.01, 3.38 and 0.53 respectively.
Again, using Gumbel type-II prior we obtain the prior predictive distribution as

$$\pi(z) = b_3 a_3 \int_0^\infty \int_0^\infty \frac{1}{\alpha^3 \delta} \left(\frac{z}{\alpha}\right)^{\delta - 1} e^{-((\frac{x}{\alpha})^\delta + \frac{b_3}{\alpha} + \frac{a_3}{\delta})} d\alpha d\delta. \tag{5.2}$$

In this case, the expert probabilities are assumed to be 0.1 for both the intervals (0.5, 1) and (1, 1.5). Using the procedure that have been discussed above, the values of the hyperparameters a_3 and b_3 are found to be 4.51 and 2.17 respectively.

6. Simulation Study

Here, we carried out a study to assess the performance of Bayesian estimators obtained in Section 4 through Monte Carlo simulation with respect to posterior risks for different sample sizes and values of parameters. All programs were written using the R software. Here, we choose samples of sizes, n=10, 50 and 100. The parameter values are α=0.5, 1.5 and δ=0.8, 1.2. Also the values for the loss parameter are $m = \pm 0.6$ and ± 1.6. This was iterated 5000 times.

TABLE 3. Maximum Likelihood Estimates of α and δ

n	α	δ	$\hat{\alpha}_{ML}$	$\hat{\delta}_{ML}$
10	0.5	0.8	0.3554	0.7632
		1.2	0.3982	1.1449
	1.5	0.8	1.0662	0.7632
		1.2	1.1948	1.1449
50	0.5	0.8	0.4435	0.7959
		1.2	0.4616	1.1939
	1.5	0.8	1.3307	0.7959
		1.2	1.3847	1.1939
100	0.5	0.8	0.4995	0.816
		1.2	0.4997	1.2241
	1.5	0.8	1.4981	0.816
		1.2	1.499	1.2241

It is clear from the Table 3 that the maximum likelihood estimates of δ is same for both $\alpha = 0.5$ and 1.5. The Bayes estimators and corresponding posterior risks of α and δ are reported in Tables (4-9).

TABLE 4. Bayes estimators for δ (n=10) and their posterior risks (in parenthesis)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
0.5	0.8	SELF	0.73832(0.02965)	0.71384(0.02783)	0.93253(0.00159)
		PLF	0.75814(0.03963)	0.73308(0.03847)	0.93338(0.00171)
		LINEX(m=0.6)	0.72958(0.00524)	0.70576(0.00484)	0.93168(0.00050)
		LINEX(m=1.6)	0.71594(0.03581)	0.69364(0.03231)	0.92832(0.00672)
		LINEX(m=-0.6)	0.74733(0.0054)	0.72243(0.00515)	0.93269(9.78e-05)
		LINEX(m=-1.6)	0.76254(0.03874)	0.73754(0.03791)	0.93181(-0.00114)
	1.2	SELF	1.10758(0.06671)	1.00251(0.04783)	1.2819(0.04933)
		PLF	1.13730(0.05943)	1.02609(0.04716)	1.30101(0.03820)
		LINEX(m=0.6)	1.08813(0.01167)	0.98955(0.00777)	1.26576(0.00969)
		LINEX(m=1.6)	1.05920(0.07734)	0.97375(0.04601)	1.23341(0.07760)
		LINEX(m=-0.6)	1.12793(0.01220)	1.01825(0.00944)	1.29535(0.00805)
		LINEX(m=-1.6)	1.16184(0.08680)	1.05017(0.07625)	1.31195(0.04805)

TABLE 5. Bayes estimators for δ (n=50) and their posterior risks (in parenthesis)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
0.5	0.8	SELF	0.7912(0.00731)	0.78468(0.00721)	0.83420(0.00587)
		PLF	0.79589(0.00922)	0.78926(0.00916)	0.83771(0.00702)
		LINEX(m=0.6)	0.78909(0.00131)	0.78253(0.00128)	0.82239(0.00108)
		LINEX(m=1.6)	0.78550(0.00925)	0.77904(0.00902)	0.82920(0.00799)
		LINEX(m=-0.6)	0.79348(0.00132)	0.78686(0.00130)	0.83591(0.00103)
		LINEX(m=-1.6)	0.79715(0.00939)	0.79053(0.00935)	0.83858(0.00700)
	1.2	SELF	1.18697(0.01646)	1.16050(0.01539)	1.22528(0.01552)
		PLF	1.19389(0.01383)	1.16712(0.01323)	1.23160(0.01264)
		LINEX(m=0.6)	1.18206(0.00294)	1.15598(0.00271)	1.22053(0.00284)
		LINEX(m=1.6)	1.17408(0.02063)	1.14895(0.01848)	1.21234(0.02070)
		LINEX(m=-0.6)	1.19193(0.00297)	1.16521(0.00282)	1.22984(0.00273)
		LINEX(m=-1.6)	1.20015(0.02108)	1.17338(0.02060)	1.23697(0.01870)

TABLE 6. Bayes estimators for δ (n=100) and their posterior risks (in parenthesis)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
0.5	0.8	SELF	0.81315(0.00351)	0.80980(0.00348)	0.83263(0.00324)
		PLF	0.81531(0.00431)	0.81195(0.00429)	0.83458(0.00389)
		LINEX(m=0.6)	0.81210(0.00063)	0.80870(0.00062)	0.83165(0.00058)
		LINEX(m=1.6)	0.81036(0.00446)	0.80704(0.00440)	0.82997(0.00426)
		LINEX(m=-0.6)	0.81420(0.00063)	0.81085(0.00062)	0.83360(0.00057)
		LINEX(m=-1.6)	0.81597(0.00450)	0.81261(0.00449)	0.83515(0.00403)
	1.2	SELF	1.21982(0.00790)	1.20684(0.00762)	1.23715(0.00775)
		PLF	1.22306(0.00674)	1.21000(0.00631)	1.24028(0.00626)
		LINEX(m=0.6)	1.21746(0.00141)	1.20458(0.00135)	1.23481(0.00140)
		LINEX(m=1.6)	1.21357(0.01000)	1.20094(0.00944)	1.23085(0.01008)
		LINEX(m=-0.6)	1.22220(0.00142)	1.20916(0.00138)	1.23946(0.00137)
		LINEX(m=-1.6)	1.22616(0.01013)	1.21309(0.00999)	1.24320(0.00967)

Table 7. Bayes estimators for α (n=10, $\alpha = 0.5$ and 1.5) and their posterior risks (in parenthesis)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
0.5	0.8	SELF	0.40920(0.01879)	0.60398(-0.0401)	0.72079(-0.11182)
		PLF	0.43156(0.04470)	0.56981(-0.06833)	0.63853(-0.16452)
		LINEX(m=0.6)	0.40338(0.00349)	0.61697(-0.00779)	0.75956(-0.02325)
		LINEX(m=1.6)	0.39296(0.02599)	0.64414(-0.06425)	0.86409(-0.22926)
		LINEX(m=-0.6)	0.41464(0.00326)	0.59275(-0.00673)	0.69110(-0.01781)
		LINEX(m=-1.6)	0.42279(0.02173)	0.57697(-0.0432)	0.65397(-0.10691)
	1.2	SELF	0.43077(0.01103)	0.52703(-0.00250)	0.56593(-0.0020)
		PLF	0.44339(0.02524)	0.52274(-0.00857)	0.55158(-0.02870)
		LINEX(m=0.6)	0.42740(0.00194)	0.52835(-0.00082)	0.57095(-0.00017)
		LINEX(m=1.6)	0.42152(0.01479)	0.53040(-0.00539)	0.58041(-0.02317)
		LINEX(m=-0.6)	0.43401(0.00190)	0.52566(-0.00075)	0.56131(-0.00270)
		LINEX(m=-1.6)	0.43908(0.01329)	0.52330(-0.00595)	0.55436(-0.01850)
1.5	0.8	SELF	1.22762(-0.16114)	1.74298(-0.16283)	1.41716(0.07202)
		PLF	1.29468(0.06411)	1.66587(0.05037)	1.442349(0.06420)
		LINEX(m=0.6)	1.17237(0.03314)	1.83872(0.04158)	1.38770(0.01767)
		LINEX(m=1.6)	1.07148(0.24982)	2.18476(0.19804)	1.29963(0.18803)
		LINEX(m=-0.6)	1.27280(0.04712)	1.67533(0.04058)	1.43237(0.03131)
		LINEX(m=-1.6)	1.32299(0.15259)	1.59559(0.13581)	1.43751(0.12583)
	1.2	SELF	1.29259(0.09937)	1.54303(0.07560)	1.36701(0.07927)
		PLF	1.33047(0.07576)	1.53903(0.00799)	1.39570(0.05739)
		LINEX(m=0.6)	1.26121(0.01882)	1.54462(0.01721)	1.34051(0.01589)
		LINEX(m=1.6)	1.20553(0.13929)	1.53281(0.11634)	1.28623(0.12924)
		LINEX(m=-0.6)	1.32041(0.01869)	1.53785(0.01719)	1.38804(0.01620)
		LINEX(m=-1.6)	1.35680(0.10274)	1.52519(0.11214)	1.41174(0.09890)

Table 8. Bayes estimators for α (n=50, $\alpha = 0.5$ and 1.5) and their posterior risks (in parenthesis)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
0.5	0.8	SELF	0.4562(0.0060)	0.4425(0.0058)	0.4622(0.0062)
		PLF	0.4626(0.0131)	0.4494(0.0139)	0.4685(0.0125)
		LINEX(m=0.6)	0.4542(0.0010)	0.4406(0.0011)	0.4604(0.0010)
		LINEX(m=1.6)	0.4511(0.0078)	0.4375(0.0079)	0.4574(0.0077)
		LINEX(m=-0.6)	0.4578(0.0010)	0.4443(0.0011)	0.4640(0.0010)
		LINEX(m=-1.6)	0.4607(0.0075)	0.4474(0.0079)	0.4668(0.0072)
	1.2	SELF	0.4687(0.0029)	0.4620(0.0029)	0.4712(0.0028)
		PLF	0.4718(0.0062)	0.4652(0.0064)	0.4742(0.0061)
		LINEX(m=0.6)	0.4678(0.0005)	0.4611(0.0005)	0.4703(0.0005)
		LINEX(m=1.6)	0.4663(0.0037)	0.4596(0.0038)	0.4688(0.0037)
		LINEX(m=-0.6)	0.4695(0.0005)	0.4629(0.0005)	0.4720(0.0005)
		LINEX(m=-1.6)	0.4710(0.0037)	0.4644(0.00381)	0.4734(0.0036)
1.5	0.8	SELF	1.3684(0.0544)	1.2017(0.0392)	1.3466(0.0556)
		PLF	1.3882(0.0395)	1.2179(0.0324)	1.3671(0.0410)
		LINEX(m=0.6)	1.3517(0.0099)	1.1909(0.0064)	1.3298(0.0100)
		LINEX(m=1.6)	1.3237(0.0071)	1.1774(0.0388)	1.3025(0.0705)
		LINEX(m=-0.6)	0.5074(0.0006)	0.4987(0.0006)	1.3630(0.0098)
		LINEX(m=-1.6)	1.4081(0.0635)	1.2401(0.0615)	1.3885(0.06714)
	1.2	SELF	1.4060(0.0264)	1.3253(0.0233)	1.3955(0.0267)
		PLF	1.4153(0.0187)	1.3341(0.0175)	1.4051(0.0191)
		LINEX(m=0.6)	1.3979(0.0048)	1.318(0.0040)	1.3874(0.0048)
		LINEX(m=1.6)	1.3844(0.0344)	1.3086(0.0267)	1.3741(0.0342)
		LINEX(m=-0.6)	1.4138(0.0046)	1.3326(0.0043)	1.4035(0.0047)
		LINEX(m=-1.6)	1.4261(0.0321)	1.3457(0.0325)	1.4162(0.0331)

Table 9. Bayes estimators for α (n=100, α = 0.5 and 1.5) and their posterior risks (in parenthesis)

α	δ	Loss Function	Jeffreys	Gamma	Gumbel
0.5	0.8	SELF	0.50631(0.00287)	0.52968(0.00283)	0.53139(0.00272)
		PLF	0.50995(0.00620)	0.53235(0.00534)	0.53396(0.00512)
		LINEX(m=0.6)	0.50519(0.00061)	0.52881(0.00052)	0.53056(0.00050)
		LINEX(m=1.6)	0.50332(0.00404)	0.52729(0.00382)	0.52909(0.00036)
		LINEX(m=-0.6)	0.5074(0.00070)	0.4987(0.00060)	0.5095(0.00050)
		LINEX(m=-1.6)	0.50923(0.00467)	0.53182(0.00343)	0.53345(0.00329)
	1.2	SELF	0.50340(0.00165)	0.51379(0.00146)	0.51455(0.00144)
		PLF	0.50504(0.00327)	0.51522(0.00285)	0.51595(0.00280)
		LINEX(m=0.6)	0.50291(0.00028)	0.51335(0.00025)	0.51411(0.00025)
		LINEX(m=1.6)	0.50208(0.00212)	0.51259(0.00192)	0.51337(0.00189)
		LINEX(m=-0.6)	0.50390(0.00029)	0.51423(0.00026)	0.51498(0.00026)
		LINEX(m=-1.6)	0.50472(0.00210)	0.51494(0.00183)	0.51568(0.00180)
1.5	0.8	SELF	1.51851(0.03327)	1.57666(0.0298)	1.52859(0.03050)
		PLF	1.52942(0.02183)	1.55436(0.01950)	1.58534(0.01740)
		LINEX(m=0.6)	1.50843(0.00589)	1.54511(0.00561)	1.56800(0.00559)
		LINEX(m=1.6)	1.49159(0.04306)	1.52833(0.04197)	1.5283(0.04310)
		LINEX(m=-0.6)	1.52834(0.00604)	1.53821(0.00591)	1.58449(0.0058)
		LINEX(m=-1.6)	1.54375(0.04038)	1.56640(0.0390)	1.55306(0.03493)
	1.2	SELF	1.51012(0.01487)	1.53598(0.01362)	1.52614(0.01425)
		PLF	1.51503(0.00983)	1.53081(0.00932)	1.54041(0.00895)
		LINEX(m=0.6)	1.51455(0.00263)	1.52180(0.00260)	1.53180(0.00250)
		LINEX(m=1.6)	1.49812(0.01919)	1.52450(0.01837)	1.51433(0.01889)
		LINEX(m=-0.6)	1.50563(0.00265)	1.51721(0.00275)	1.50261(0.00274)
		LINEX(m=-1.6)	1.52170(0.01852)	1.52600(0.01721)	1.53696(0.01689)

Figures 3 and 4 show the graph of posterior risks of $\hat{\alpha}$ and $\hat{\delta}$ under LINEX loss function.

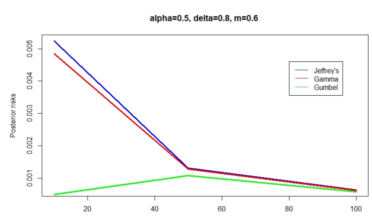

Figure 3. The posterior risks of $\hat{\delta}$ under LINEX loss function

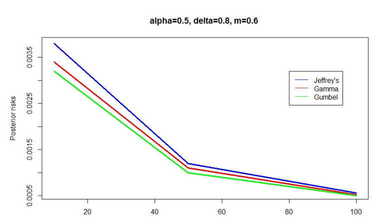

Figure 4. The posterior risks of $\hat{\alpha}$ under LINEX loss function

Here, we compared the performance of the estimators based on their posterior risks. In comparison of Jeffrey's (non-informative prior) with Gamma and Gumbel (informative priors), informative priors gives the better estimates of δ for all the loss functions. When comparing informative priors, we can see that the better estimates of δ and α are obtained when using the Gumbel type-II prior because the posterior risks are smaller for the loss functions considered here except SELF and LINEX with loss parameter 1.6. But for loss functions SELF and LINEX with loss parameter 1.6, the better estimates of δ and α are obtained through gamma prior. One can easily observe that the LINEX loss function with loss parameter 0.6 is a better choice of loss function for the estimation of α and δ. Since, corresponding posterior risks obtained are smaller for all the priors under consideration. Also, Tables (4-9) depicts that the increase of sample size results in the decrease of posterior risks and the increase in parameter values result in the increase of the posterior risks.

7. Conclusion

In this paper, we studied the Bayesian analysis of Weibull distribution under different loss functions using priors both non-informative and informative priors . Here, we employed Lindley's approximation technique for the computation of Bayes estimators and the maximum likeklihood estimators are obtained using MASS() package in RStudio. We conducted Monte Carlo simulation study to compare the different estimators based on posterior risks. We studied the effects of loss functions (both symmetric and asymmetric) by using Jeffrey's, Gamma and Gumbel type-II prior. From this study it is clear that for the estimation of δ and α Gumbel prior has smaller posterior risks when compared to the Jeffrey's and gamma priors. Also, based on posterior risks we observed that the LINEX loss function with $m = 0.6$ performs better than the other loss functions. So it is a better choice when considering loss functions. Hence, the combination of Gumbel type-II prior and LINEX loss function with $m = 0.6$ is the suggested combination for the parameter estimation in Weibull distribution.

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