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SOLUTION OF TWO PARAMETERS SINGULAR PERTURBATION PROBLEM USING HIGHER ORDER COMPACT NUMERICAL METHOD

K. PHANEENDRA AND G. MAHESH

Abstract. This paper envisages, a higher order compact numerical method for the solution of two parameters singularly perturbed boundary value problem. The second order singular perturbation problem is written as a system of first order ordinary differential equations. Seventh order compact difference scheme forms a tool to solve this system. Several numerical examples are solved and maximum errors are tabulated with comparison using other methods in the literature to establish the proposed numerical scheme. Effect of the small parameters on the behaviour of the solution is shown graphically.

1. Introduction

Two parameters singularly perturbed boundary value problems are familiar in science and engineering. Typically these problems arise in the fields transport phenomena in chemistry, biology, chemical reactor theory and lubrication theory (Bigge and Bohl, 1985; Bohl, 1981; Chen and O'Malley, 1974). The character of the two-parameter problem was asymptotically examined by O'Malley, 1967; O'Malley, 1991, where the ratio of two parameters have significant role in solution.

It is well known that as perturbation parameter goes to zero, the analytical solution of singularly perturbed boundary value problem approaches a discontinuous limit and boundary or interior layers appears. The Equations of this type exhibit solutions with layers; that is, the domain of the differential equation contains thin regions where the solution derivatives are enormously large. The numerical treatment of singularly perturbed differential equations gives major computational difficulties due to a very small value of perturbation parameter and presence of boundary and/or interior layers. A wide verity of papers and books have been devised, describing various methods for singularly perturbed two-point boundary value problems, among these, we mention Bender and Orszag, 1978; Doolan et al. 1980; Miller et al. 1996.

Kadalbajoo and Jha (2013) proposed a second order defect correction method for the numerical solution of two parameter singular perturbation problem on Bakhvalov-Shishkin mesh. Linß and Roos (2004) studied a model linear convection diffusion reaction problem where both the diffusion term and the convection

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term are multiplied by small parameters. Using a barrier function technique, authors derived sharp bounds for the derivatives of the solution. These bounds are applied in the analysis of a simple upwind difference scheme on Shishkin meshes. Kadalbajoo and Yadaw (2008) presented a uniform convergence of second order B-spline collocation method on piecewise uniform Shishkin mesh for solving a class of two-parameter singularly perturbed boundary value problems.

Torsten Linss (2010) presented streamline-diffusion FEM to solve a singularly perturbed boundary value problem of reaction-convection-diffusion type with two small parameters that give rise to two boundary layers. A robust a posteriori error estimate in the maximum norm is derived. Gracia et al. (2006) constructed a second order monotone numerical method for a singularly perturbed ordinary differential equation with two small parameters affecting the convection and diffusion terms. In this method, the monotone operator is combined with a piecewise uniform Shishkin mesh. An asymptotic error bound in the maximum norm is established theoretically whose error constants are shown to be independent of both singular perturbation parameters.

Zahra and Ashraf (2013) proposed a uniformly convergent numerical method to solve singularly perturbed semi linear boundary value problem with two parameters using exponential spline on Shishkin mesh. Kumar et al. (2013) derived an approximate method to solve two parameters singularly perturbed boundary value problems having boundary layers at both end points. In this method authors divides the region into inner and outer regions. In the outer region the solution is approximated by zeroth order asymptotic expansion while in the inner region B-spline collocation method is used to find the solution.

In this paper, we proposed a higher order compact numerical method method for the solution of two parameters singularly perturbed two-point boundary value problems on a uniform mesh. In section 2, the description of the problem is given. Numerical method for the solution problem is discussed in section 3. Implementation of the method on numerical examples with results are shown in section 4. Finally, conclusions were given in last section.

2. Description of the method

To describe this method, we consider singularly perturbed two point boundary value problems of the form

$$\varepsilon y''(x) + \mu a(x)y'(x) - b(x)y(x) = f(x), \quad 0 \le x \le 1 \tag{2.1}$$

with boundary conditions

$$y(0) = \alpha \tag{2.2}$$

$$y(1) = \beta \tag{2.3}$$

where $0 < \varepsilon \ll 1$ and $0 < \mu \ll 1$ are two small parameters. The functions $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth with $a(x) \ge \tilde{a} > 0$, $b(x) \ge \tilde{b} > 0$, $\frac{b(x)}{a(x)} \ge \tilde{c} > 0$.

Solution of Eq. (2.1) can be described by the roots of the characteristic equation

$$\varepsilon\lambda(x)^2 + \mu a(x)\lambda(x) - b(x) = 0$$

This equation produces two continuous functions

$$\lambda_1(x) = -\frac{\mu a(x)}{2\varepsilon} - \sqrt{\left(\frac{\mu a(x)}{2\varepsilon}\right)^2 + \frac{b(x)}{\varepsilon}} \tag{2.4}$$

$$\lambda_2(x) = -\frac{\mu a(x)}{2\varepsilon} + \sqrt{\left(\frac{\mu a(x)}{2\varepsilon}\right)^2 + \frac{b(x)}{\varepsilon}} \tag{2.5}$$

Put $\theta_1 := \max_{x\in[0,1]} \lambda_1 < -\frac{\mu}{\varepsilon} \le 0$ and $\theta_2 := \min_{x\in[0,1]} \lambda_2(x)$ The decay of the solution in boundary layer region is defined by θ_1 and θ_2.

For $\frac{\varepsilon}{\mu^2} \le 1,\ |\theta_1| = O\left(\frac{\mu}{\varepsilon}\right)$ and $|\theta_2| = O\left(\frac{1}{\mu}\right)$,

$$\frac{\mu^2}{\varepsilon} \le 1,\ |\theta_1| = O\left(\frac{1}{\sqrt{\varepsilon}}\right) \text{ and } |\theta_2| = O\left(\frac{1}{\sqrt{\varepsilon}}\right).$$

At $x = 0$ the layer is governed by the term $e^{-\theta_1 x}$ and at $x = 1$ the layer is governed by $e^{-\theta_2(1-x)}$.

From Gracia et al. (2006) and O'Riordan et al. (2003), we have

$$\theta_1 = \begin{cases} \frac{\sqrt{\gamma\tilde{a}}}{\sqrt{\varepsilon}},\ \frac{\mu^2}{\varepsilon} \le \frac{\gamma}{\tilde{a}} \\ \frac{\tilde{a}\mu}{\varepsilon},\ \frac{\mu^2}{\varepsilon} \ge \frac{\gamma}{\tilde{a}} \end{cases}, \quad \theta_2 = \begin{cases} \frac{\sqrt{\gamma\tilde{a}}}{2\sqrt{\varepsilon}},\ \frac{\mu^2}{\varepsilon} \le \frac{\gamma}{\tilde{a}} \\ \frac{\gamma}{2\mu},\ \frac{\mu^2}{\varepsilon} \ge \frac{\gamma}{\tilde{a}} \end{cases}$$

where, $\tilde{a} = \min_{x\in[0,1]} a(x)$ and $\gamma = \min_{x\in[0,1]} \frac{b(x)}{a(x)}$.

3. Numerical Scheme

We solve Eq. (2.1) subject to the boundary conditions EqS. (2.2), (2.3) by using seventh order compact numerical method (Dianyun Peng [5]) described as follows:

The first order linear system corresponding to the above BVP is

$$Y' = A(x)Y + R(x), x\epsilon[a,b] \tag{3.1}$$

with the boundary conditions$B_1Y(0) + B_2Y(1) = D$, where A, B_1 and B_2 are 2×2 matrices. Y, R, D are two dimensional vectors.

Here $A = \begin{pmatrix} 0 & 1 \\ \frac{-b}{\varepsilon} & \frac{-\mu a}{\varepsilon} \end{pmatrix}$, $Y = \begin{pmatrix} y \\ z \end{pmatrix}$, $R = \begin{pmatrix} 0 \\ \frac{f(x)}{\varepsilon} \end{pmatrix}$, $D = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$,

$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ with $y' = z$.

Decompose the interval [0, 1] into N equal parts with constant mesh length H. Let $0 = x_0, x_1, ..., x_N = 1$ be the mesh points. Then we have $x_i = iH$, for i= 0, 1,. . . , N. Again we divide each subinterval $[x_i,\ x_{i+1}]$into six equal smaller sub intervals with constant mesh length h. Let $t_1, t_2, ..., t_7$are the grids in the subinterval $[x_i,\ x_{i+1}]$and corresponding values of the variables and its derivatives are $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7$ and $Y'_1, Y'_2, Y'_3, Y'_4, Y'_5, Y'_6, Y'_7$.

By considering Taylor's expansions of $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7$ at the fractional grid t_4 , we have,

$$\frac{h^{n+1}}{(n+1)!} Y_4^{(n+1)} = \sum_{j=1}^{7} a_j^n Y_j + a_8^n Y'_4 + O(h^8 Y_4^{(6)}), n = 1, 2, 3, 4, 5, 6 \tag{3.2}$$

where $h = \frac{x_{i+1}-x_i}{6}$ and the coefficients a_j^n are given by:

$$\begin{aligned}
&a_3^1 = a_5^1 = \frac{3}{4}, a_4^1 = \frac{-49}{36}, a_8^1 = 0, a_1^1 = a_7^1 = \frac{1}{180}, a_2^1 = a_6^1 = \frac{-3}{40} \\
&a_1^2 = -a_7^2 = -\frac{1}{540}, a_2^2 = -a_6^2 = \frac{3}{80}, a_3^2 = -a_5^2 = \frac{-3}{4}, a_4^2 = 0 \\
&a_8^2 = \frac{-49}{36}, a_1^3 = a_7^3 = \frac{-1}{144}, a_2^3 = a_6^3 = \frac{1}{12}, a_3^3 = a_5^3 = \frac{-13}{48}, a_4^3 = \frac{7}{18} \\
&a_8^1 = 0, a_1^4 = a_7^4 = \frac{-1}{432}, a_2^4 = -a_6^4 = \frac{-1}{24}, a_3^4 = -a_5^4 = \frac{13}{48}, \\
&a_4^4 = 0, a_8^4 = \frac{7}{18}, a_1^5 = a_7^5 = \frac{1}{720}, a_2^5 = a_6^5 = \frac{-1}{120}, a_3^5 = a_5^5 = \frac{1}{48}, \\
&a_4^5 = \frac{-1}{36}, a_8^5 = 0, a_1^6 = -a_7^6 = \frac{1}{2160}, a_2^6 = -a_6^6 = \frac{1}{240}, a_3^6 = -a_5^6 = \frac{-1}{48}, \\
&a_4^6 = 0, a_8^6 = \frac{-1}{36}
\end{aligned} \tag{3.3}$$

By taking the Taylor's series expansions of $Y_1^{'}, Y_2^{'}, Y_3^{'}, Y_4^{'}, Y_5^{'}, Y_6^{'}, Y_7^{'}$ at the grid point t_4 and substituting Eq. (3.2), we get

$$Y_k^{'} = \frac{1}{h}\sum_{j=1}^{7} b_j^k Y_j + b_8^k Y_4^{'} + O(h^7 Y_4^{(6)}) \text{ for } k = 1,2,3,5,6,7 \tag{3.4}$$

$$\begin{aligned}
b_j^1 &= -6a_j^1 + 27a_j^2 - 108a_j^3 + 405a_j^4 - 1458a_j^5 + 5103a_j^6 + Sgn(j-8) \\
b_j^2 &= -4a_j^1 + 12a_j^2 - 32a_j^3 + 80a_j^4 - 192a_j^5 + 448a_j^6 + Sgn(j-8) \\
b_j^3 &= -2a_j^1 + 3a_j^2 - 4a_j^3 + 5a_j^4 - 6a_j^5 + 7a_j^6 + Sgn(j-8) \\
b_j^5 &= 2a_j^1 + 3a_j^2 + 4a_j^3 + 5a_j^4 + 6a_j^5 + 7a_j^6 + Sgn(j-8) \\
b_j^6 &= 4a_j^1 + 12a_j^2 + 32a_j^3 + 80a_j^4 + 192a_j^5 + 448a_j^6 + Sgn(j-8) \\
b_j^7 &= 6a_j^1 + 27a_j^2 + 108a_j^3 + 405a_j^4 + 1458a_j^5 + 5103a_j^6 + Sgn(j-8) \\
Sgn(x) &= \left\{ \begin{array}{l} 1, x \geq 0 \\ 0, x < 0 \end{array} \right\}.
\end{aligned}$$

The variable Y and its derivative Y'at grids $t_1, t_2, ..., t_7$ subject to equations

$$Y_j^{'} = A_j Y_j + R_j, j = 1,2,3,4,5,6,7 \tag{3.5}$$

where A_j and R_j are values of A and R at grids t_j. Using Eq. (3.5) in Eq. (3.4), we get six linear algebraic equations with respect to seven unknown variables $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7$. By eliminating Y_2, Y_3, Y_4, Y_5, Y_6 from the above equations a relation between Y_1 and Y_7 can be obtained as follows:

$$\frac{1}{h}S_i Y_i + \frac{1}{h}T_i Y_{i+1} = F_i \text{ for } i = 1,2, ...N-1 \tag{3.6}$$

where S_i and T_i are 2x2 matrices and F_i is a two dimensional vector. By assuming

$$
\begin{aligned}
c_1 &= b_5^7 b_3^1 - b_5^1 b_3^7,\\
c_2 &= (b_5^7 b_3^1 - b_5^1 b_3^7)/c_1,\\
c_3 &= (b_2^7 b_5^1 - b_2^1 b_5^7)/c_1,\\
c_4 &= (b_6^1 b_3^7 - b_6^7 b_3^1)/c_1,\\
c_5 &= (b_2^1 b_3^7 - b_2^7 b_3^1)/c_1,\\
W_1 &= ((b_7^7 b_5^1 - b_7^1 b_5^7)I - hb_5^1 A_1)/c_1,\\
W_2 &= ((b_1^7 b_5^1 - b_1^1 b_5^7)I + hb_5^7 A_1)/c_1,\\
W_2 &= ((b_3^5 b_4^1 - b_3^1 b_4^5)I + h(b_6^5 b_4^1 - b_6^1 b_4^5)A_3)/c_1,\\
W_3 &= ((b_4^7 b_5^1 - b_4^1 b_5^7)I + h(b_8^7 b_5^1 - b_8^1 b_5^7)A_4)/c_1,\\
G_1 &= (b_5^7 R_1 - b_5^1 R_7 + (b_8^7 b_5^1 - b_8^1 b_5^7)R_4)/c_1\\
W_4 &= ((b_7^1 b_3^7 - b_7^7 b_3^1)I + hb_3^1 A_7)/c_1,\\
W_5 &= ((b_1^1 b_3^7 - b_1^7 b_3^1)I - hb_3^7 A_1)/c_1,\\
W_6 &= ((b_4^1 b_3^7 - b_4^7 b_3^1)I + h(b_8^1 b_3^7 - b_8^7 b_3^1)A_4)/c_1,\\
G_2 &= (b_3^1 R_7 - b_3^7 R_1 + (b_3^7 b_8^1 - b_3^1 b_8^7)R_4)/c_1,\\
c_6 &= b_2^6 + b_3^6 c_3 + b_5^6 c_5,\\
W_7 &= b_7^6 I + b_3^6 W_1 + b_5^6 W_4,\\
W_8 &= b_1^6 I + b_3^6 W_2 + b_5^6 W_5,\\
W_9 &= b_4^6 I + b_3^6 W_3 + b_5^6 W_6 + hb_8^6 A_4,\\
W_{10} &= (b_3^6 c_2 + b_5^6 c_4 + b_6^6)I - hA_6,\\
G_3 &= R_6 - b_8^6 R_4 - b_3^6 G_1 - b_5^6 G_2,\\
c_7 &= b_6^2 + b_3^2 c_2 + b_5^6 c_4,\\
W_{11} &= b_3^2 W_1 + b_5^2 W_4 + b_7^2 I,\\
W_{12} &= b_1^2 I + b_3^2 W_2 + b_5^2 W_5,\\
W_{13} &= b_4^2 I + b_3^2 W_3 + b_5^2 W_6 + hb_8^2 A_4,\\
W_{14} &= (b_2^2 + b_3^2 c_3 + b_5^2 c_5)I - hA_2,\\
G_4 &= R_6 - b_8^2 R_4 - b_3^2 + b_7^5 I - hW_4 A_5,\\
W_{16} &= b_3^5 W_2 + b_5^5 W_5 + b_5^1 I - hW_5 A_5,\\
W_{17} &= b_3^5 W_3 + b_5^5 W_6 + b_4^5 I + h(b_8^5 A_4 - W_6 A_5),\\
W_{18} &= b_3^5 c_2 + b_5^5 c_4 + b_6^5 I - hc_4 A_5,\\
W_{19} &= b_3^5 c_3 + b_5^5 c_5 + b_2^5 I - hc_5 A_5,\\
W_{20} &= b_3^3 W_1 + b_5^3 W_4 + b_7^3 I - hW_1 A_3,\\
G_5 &= R_5 - b_8^5 R_4 - b_3^5 G_1 - b_5^5 G_2 + hA_5 G_2,\\
W_{21} &= b_3^3 W_2 + b_5^3 W_5 + b_7^3 I - hW_2 A_3,\\
W_{22} &= b_3^3 W_3 + b_5^3 W_6 + b_4^3 I + h(b_8^3 A_4 - W_3 A_3),\\
W_{23} &= (b_3^3 c_2 + b_5^3 c_4 + b_6^3)I - hc_2 A_3,\\
W_{24} &= (b_3^3 c_3 + b_5^3 c_5 + b_2^3)I - hc_3 A_3,
\end{aligned}
$$

$$
\begin{aligned}
G_6 &= R_3 - b_8^3 R_4 - (b_3^3 - hA_3)G_1 - b_5^3 G_2 \\
W_{28} &= W_{10}W_{14} - c_6c_7I, \\
W_{25} &= W_{28}^{-1}(c_6W_{11} - W_7W_{14}), \\
W_{26} &= W_{28}^{-1}(c_6W_{12} - W_8W_{14}), \\
W_{27} &= W_{28}^{-1}(c_6W_{13} - W_9W_{14}), \\
G_7 &= W_{28}^{-1}(c_6G_4 - G_3W_{14}), \\
W_{29} &= -(W_{10}W_{25} + W_7)/c_6, \\
W_{30} &= -(W_{10}W_{26} + W_8)/c_6, \\
W_{31} &= -(W_{10}W_{27} + W_9)/c_6, \\
G_8 &= -(G_3 + W_{10}G_7)/c_6, \\
W_{32} &= W_{19}W_{29} + W_{18}W_{25} + W_{15}, \\
W_{33} &= W_{19}W_{30} + W_{18}W_{26} + W_{16}, \\
W_{34} &= W_{19}W_{31} + W_{18}W_{27} + W_{17}, \\
G_9 &= G_5 + W_{19}G_8 + W_{18}G_7, \\
W_{35} &= W_{24}W_{29} + W_{23}W_{25} + W_{20}, \\
W_{36} &= W_{24}W_{30} + W_{23}W_{26} + W_{21}, \\
W_{37} &= W_{24}W_{31} + W_{23}W_{27} + W_{22}, \\
G_{10} &= G_6 + W_{24}G_8 + W_{23}G_7,
\end{aligned}
$$

We get

$$S_i = W_{36} - W_{37}W_{33}W_{34}^{-1}, T_i = W_{35} - W_{37}W_{32}W_{34}^{-1}, F_i = G_{10} - W_{37}G_9W_{34}^{-1}$$

The formula Eq. (3.6) is the seventh order compact difference scheme of equation Eq. (3.1) in the ith subinterval. The structure of the seventh order scheme is profiled here as the following coefficient matrix of Eq. (3.6):

$$
\begin{bmatrix}
S_1 & T_1 \ldots\ldots\ldots\ldots\ldots\ldots & & F_1 \\
& S_2 & T_2 \ldots\ldots\ldots\ldots\ldots & F_2 \\
& & S_3 \;\; T_3 \ldots\ldots\ldots & F_3 \\
\\
\ldots\ldots\ldots\ldots\ldots S_{N-1} & & T_{N-1} & F_{N-1}
\end{bmatrix}
$$

Solving the system of Eq. (3.6) together with the boundary conditions $y(0) = \alpha$ and $y(1) = \beta$, we get the solution.

4. Numerical examples

To demonstrate the proposed method computationally, we consider three singularly perturbed two parameters boundary value problems. These problems have been chosen because they have been discussed in the literature and because exact solutions are available for comparison. The maximum absolute errors in the solution are estimated using $E_{N,\varepsilon,\mu} = \max\limits_{0\le i\le N} |y(x_i) - y_i|$ where $y(x_i)$ is exact solution and y_i is computed solution.

Example 1. Consider the two parameters boundary value problem

$$\varepsilon y'' + \mu y' - y = -x,\ 0 < x < 1$$

with $y(0) = 1$, $y(1) = 0$.

The exact solution is

$$y(x) = \frac{(1+\mu) + (1-\mu)\, e^{m_2}}{e^{m_2} - e^{m_1}} e^{m_1 x} + \frac{(1+\mu) + (1-\mu)\, e^{m_1}}{e^{m_1} - e^{m_2}} e^{m_2 x} + x + \mu$$

where $m_1 = \frac{-\mu - \sqrt{\mu^2+4\varepsilon}}{2\varepsilon}$, $m_2 = \frac{-\mu + \sqrt{\mu^2+4\varepsilon}}{2\varepsilon}$

The maximum absolute errors are presented in Tables 1, 2, 3 and 4 for different values of ε, μ and h. The boundary layer behaviour in the solution is shown in figure 1.

Example 2. Consider the two parameters singularly perturbed boundary value problem

$$-\varepsilon y'' + \mu y' + y = \cos \pi x,\quad 0 < x < 1 \text{ with } y(0) = 1, y(1) = 0.$$

The exact solution of this problem is

$$y(x) = a\cos\pi x + b\sin\pi x + Ae^{\lambda_1 x} + Be^{-\lambda_2 (1-x)}$$

where

$$a = \frac{\varepsilon\pi^2 + 1}{\mu^2\pi^2 + (\varepsilon\pi^2+1)^2},\ b = \frac{\mu\pi}{\mu^2\pi^2 + (\varepsilon\pi^2+1)^2},$$

$$A = -a\,\frac{1+e^{-\lambda_2}}{1-e^{\lambda_1-\lambda_2}},\ B = a\frac{1+e^{\lambda_1}}{1-e^{\lambda_1-\lambda_2}},\quad \lambda_{1,2} = \frac{\mu \mp \sqrt{\mu^2+4\varepsilon}}{2\varepsilon}$$

The maximum absolute errors are presented in Table 5 for different values of ε, μ and h. The boundary layer behaviour in the solution is shown in figure 2.

Example 3. Consider the singularly perturbed boundary value problem

$$-\varepsilon y'' - \mu y' + y = \exp(1-x)\quad 0 < x < 1 \text{ with} y(0) = 0, y(1) = 0.$$

The exact solution of this problem is

$$y(x) = \frac{\exp(m_2+1) - 1}{D}\exp(m_1 x) + \frac{1 - \exp(m_1+1)}{D}\exp(m_2 x) - \frac{\exp(1-x)}{\varepsilon(m_1+1)(m_2+1)}$$

where

$$D = \varepsilon(\exp(m_2) - \exp(m_1))(m_1+1)(m_2+1),$$

$$m_1 = \frac{-\mu - \sqrt{\mu^2+4\varepsilon}}{2\varepsilon}\quad m_2 = \frac{-\mu + \sqrt{\mu^2+4\varepsilon}}{2\varepsilon}$$

The maximum absolute errors are presented in Tables 6 and 7 for different values of ε, μ and h. The boundary layer behaviour in the solution is shown in figure 3.

Table 1. Maximum absolute errors in the solution of Example 1 for μ=2^{-16}

$N\downarrow \rightarrow\epsilon$	2^{-32}	2^{-36}	2^{-40}	2^{-44}	2^{-48}	2^{-52}
Present method						
2^6	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)
2^7	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)
2^8	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)
2^9	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)
2^{10}	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)	1.525(−5)
Results in [11]						
2^6	3.184(−3)	1.863(−2)	2.278(−2)	2.309(−2)	2.311(−2)	2.311(−2)
2^7	8.790(−4)	5.965(−3)	7.544(−3)	7.665(−3)	7.673(−3)	7.673(−3)
2^8	2.298(−4)	1.720(−3)	2.245(−3)	2.286(−3)	2.289(−3)	2.289(−3)
2^9	5.874(−5)	4.629(−4)	6.132(−4)	6.253(−4)	6.261(−4)	6.262(−4)
2^{10}	1.485(−5)	1.202(−4)	1.606(−4)	1.639(−4)	1.641(−4)	1.641(−4)

Table 2. Maximum absolute errors in the solution of Example 1 for $\varepsilon = 2^{-16}$

$N\downarrow \rightarrow\mu$	2^{-32}	2^{-36}	2^{-40}	2^{-44}	2^{-48}	2^{-52}
Present method						
2^6	4.6577(−5)	4.6577(−5)	4.6577(−5)	4.6577(−5)	4.6577(−5)	4.6577(−5)
2^7	4.1903(−7)	4.1903(−7)	4.1903(−7)	4.1903(−7)	4.1903(−7)	4.1903(−7)
2^8	1.9601(−9)	1.9601(−9)	1.9601(−9)	1.9601(−9)	1.9601(−9)	1.9601(−9)
2^9	7.415(−12)	7.415(−12)	7.415(−12)	7.415(−12)	7.415(−12)	7.415(−12)
2^{10}	3.053(−14)	3.053(−14)	3.053(−14)	3.053(−14)	3.053(−14)	3.053(−14)
Results in [11]						
2^6	2.256(−3)	2.256(−3)	2.256(−3)	2.256(−3)	2.256(−3)	2.256(−3)
2^7	5.811(−4)	5.811(−4)	5.811(−4)	5.811(−4)	5.811(−4)	5.811(−4)
2^8	1.470(−4)	1.470(−4)	1.470(−4)	1.470(−4)	1.470(−4)	1.470(−4)
2^9	3.698(−5)	3.698(−5)	3.698(−5)	3.698(−5)	3.698(−5)	3.698(−5)
2^{10}	9.267(−6)	9.267(−6)	9.267(−6)	9.267(−6)	9.267(−6)	9.267(−6)

5. Conclusion

We have presented a higher order compact finite difference method for the solution of two parameters singularly perturbed two-point boundary value problem. The second order singular perturbation problem is transformed as a system of first order ordinary differential equations. Seventh order compact difference scheme is used to solve this system. The maximum absolute errors in comparison to the standard examples chosen from the literature are tabulated to show the efficiency of the method. Effect of the small parameters on the solution behaviour is shown graphically. We noticed that, that as ε decreases for fixed μ the width of the boundary layer decreases and becomes more rigid at the end points $x = 0$ and $x = 1$. Also, we observed that, the numerical solution meets the exact solution very well.

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Table 3. Maximum absolute errors in the solution of Example 1 for $\varepsilon = 10^{-3}$

$\mu \downarrow N \rightarrow$	2^6	2^7	2^8	2^9	2^{10}
Present method					
10^{-2}	2.334(−11)	8.9817(−14)	1.1990(−14)	4.6629(−15)	8.3267(−15)
10^{-3}	7.6452(−12)	3.1530(−14)	7.1054(−15)	3.7748(−15)	4.6629(−15)
10^{-4}	6.8266(−12)	2.7756(−14)	1.3822(−14)	4.4409(−15)	6.2172(−15)
Results in [13]					
10^{-2}	3.6590(−3)	1.1005(−3)	2.7573(−4)	6.8812(−5)	1.7196(−5)
10^{-3}	3.0262(−3)	7.4023(−4)	1.8406(−4)	4.5953(−5)	1.1484(−5)
10^{-4}	2.9008(−3)	7.0989(−4)	1.7654(−4)	4.4076(−5)	1.1015(−5)

Table 4. Maximum absolute errors in the solution of Example 1 for $\mu = 10^{-4}$

$\epsilon \downarrow N \rightarrow$	2^6	2^7	2^8	2^9	2^{10}
Present method					
10^{-1}	2.5543(−15)	2.1094(−15)	1.4554(−15)	1.2147(−15)	1.1056(−15)
10^{-2}	2.9976(−15)	2.8866(−15)	1.7748(−15)	1.3299(−15)	1.0944(−15)
10^{-3}	6.8266(−12)	2.7756(−14)	1.3822(−14)	4.4409(−15)	2.4403(−15)
Results in [13]					
10^{-1}	1.5752(−5)	3.9408(−6)	9.8514(−7)	2.4628(−7)	6.1570(−8)
10^{-2}	2.8064(−4)	7.0125(−5)	1.7522(−5)	4.3807(−6)	1.0952(−6)
10^{-3}	2.9008(−3)	7.0989(−4)	1.7654(−4)	4.4076(−5)	1.1015(−5)

Table 5. Maximum absolute errors in the solution of Example 2

$\mu\downarrow$	ε=10^{-4} $N = 128$			
	In [12]	In [21]	In [18]	Our method
10^{-3}	9.4446(−3)	4.7598(−3)	5.1964(−3)	9.6737(−4)
10^{-4}	9.0436(−3)	4.2856(−3)	4.1710(−3)	9.6647(−4)
10^{-5}	9.0036(−3)	4.2295(−3)	4.0754(−3)	9.6637(−4)
10^{-6}	8.9996(−3)	4.2238(−3)	4.0659(−3)	9.6636(−4)
10^{-7}	8.9992(−3)	4.2232(−3)	4.0650(−3)	9.6636(−4)

Table 6. Maximum absolute errors in the solution of Example 3 for $\mu = 2^{-5}$

$N \downarrow \epsilon \rightarrow$	2^{-10}	2^{-11}	2^{-12}	$2^{=13}$	2^{-14}
2^7	3.4983(−12)	2.2666(−10)	2.3311(−8)	2.3929(−6)	1.7464(−4)
2^8	2.3759(−14)	9.2926(−13)	8.6895(−11)	1.2165(−8)	1.6781(−6)
2^9	1.4211(−14)	1.6214(−14)	3.3840(−13)	4.5687(−11)	8.2032(−9)
2^{10}	1.4210(−14)	1.8406(−14)	4.7073(−14)	1.5499(−13)	3.0890(−11)

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TABLE 7. Maximum absolute errors in the solution of Example 3 for $\epsilon = 2^{-16}$

$N \downarrow \mu \rightarrow$	2^{-32}	2^{-36}	2^{-40}	2^{-44}	2^{-48}
2^7	1.1390(−6)	1.1390(−6)	1.1390(−6)	1.1390(−6)	1.1390(−6)
2^8	5.3281(−9)	5.3281(−9)	5.3281(−9)	5.3281(−9)	5.3281(−9)
2^9	2.015(−11)	2.015(−11)	2.015(−11)	2.015(−11)	2.015(−11)
2^{10}	8.415(−14)	8.415(−14)	8.415(−14)	8.415(−14)	8.415(−14)

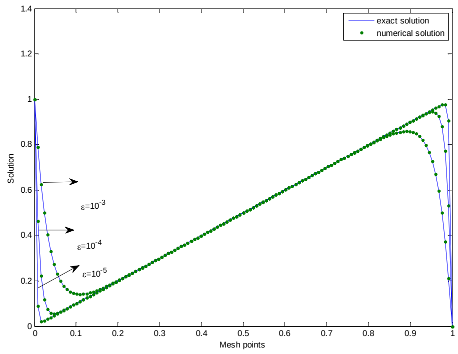

Fig 1. Graphical representation of solution in Example 1 with $\mu = 10^{-4}$

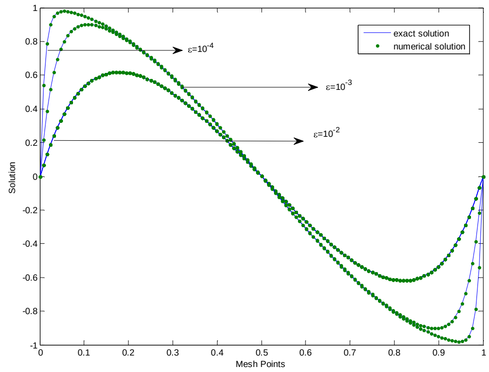

Fig 2. Graphical representation of solution in Example 2 with $\mu = 10^{-4}$

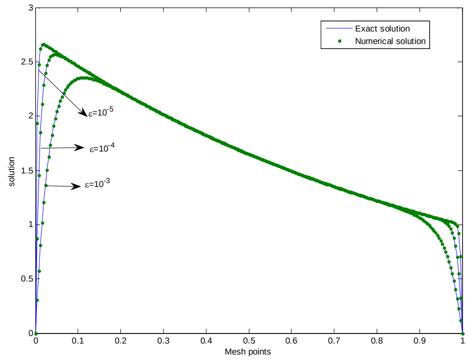

Fig 3. Graphical representation of solution in Example 3 with $\mu = 10^{-4}$

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K. Phaneendra: Dept. of Mathematics, University College of Science, Saifabad, Osmania University, Hyderabad, 500004
E-mail address: kollojuphaneendra@yahoo.co.in

Department of Mathematics, University College of Science, Saifabad, Osmania University, Hyderabad, 500004
E-mail address: gattumahesh790@gmail.com