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NUMERICAL SOLUTION OF STOCHASTIC ABEL'S INTEGRAL EQUATIONS USING BOUBAKER WAVELETS STOCHASTIC OPERATIONAL MATRIX OF INTEGRATION

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ABSTRACT. This article provides an effective technique to solve stochastic Abel's integral equations using Boubaker wavelets. These equations can be reduced to a system of algebraic equations with unknown Boubaker coefficients, by using Boubaker wavelets, and these equations can be solved numerically by using well-known numerical methods. Convergence and error analysis of the proposed method is studied. Moreover, the results obtained by the method proposed are compared to the exact solution and the Bernoulli polynomials solution with the number of numerical examples to show that the method described is precise and accurate.

1. Introduction

Various types of integral equations like Fredholm integral equations [3, 5, 21], Volterra integral equations [6,15,17], Volterra-Fredholm integral equations [18,19, 22], integro-differential equations [1, 4, 31], and many other type of integral equations are of great importance in mathematical physics. It has become evident in recent years that certain types of stochatic integral equations, such as stochatic Volterra integral equations [8–11, 14, 16, 26], multidimensional stochastic integral equations [13, 29, 30], stochastic Volterra-Fredholm integral equations [12, 23], stochastic integro-differential equations [2], and many other kind of stochastic integral equations can model various problems more efficiently than by deterministic integral equations. In many cases obtaining the solution of integral and stochastic integral equations is quite difficult and time consuming, therefore, some highly accurate numerical schemes are essential. Therefore several numerical schemes to solve integral and stochastic integral equations were developed in the last few decades [1–6, 8–19, 21–23, 26, 29–31].

The most important type of integral equations, the singular integral equations, arise in many aspects of science and engineering. Since it is hard to find the analytic solution of these kind of equations, the numerical approximation to these

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equations become necessary. Various methods are found in literature to solve these kind of integral equations [7, 20, 25]. Though we find various methods to solve deterministic singular integral equations in literature, even today, very few articles are found related to singular stochastic integral equations. Recently, Nasrin Samadyar and Farshid Mirzaee employed Orthonormal Bernoulli polynomials collocation approach for solving stochastic Itô-Volterra integral equations of Abel's type [27]. Also, Boubaker wavelets operational matrix of integration was introduced by Sarhan et al. [28] in 2020.

Encouraged by most of these works, in this article we have made an attempt for solving stochastic Abel's integral equations using stochastic operational matrix of integration of Boubaker wavelets (SOMIBW).

In this article, we consider the following stochastic Itô-Volterra integral equations of Abel's type:

$$y(x) = f(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}}dt + \int_0^x k(x,t)y(t)dW(t), \quad x \in [0,1), \tag{1.1}$$

where $f(x)$ and $k(x,t)$ are the stochastic processes defined on the probability space $(\Omega,\ F,\ P)$, $W(x)$ denotes the Brownian motion process defined on the probability space $(\Omega,\ F,\ P)$, and $y(x)$ is the unknown to be determined.

The article is organized as follows. In section 2, the definition of Boubaker wavelets and its function approximation is given. In section 3, a new stochastic operational matrix of integration of Boubaker wavelets is derived. A new Boubaker wavelets stochastic operational matrix method for solving stochastic Abel's integral equation is given in section 4. Convergence and error analysis of the proposed method is discussed in section 5. In section 6, some computational experiments are carried out to show the efficiency and reliability of the proposed method. Finally conclusion is drawn in section 7.

2. Boubaker wavelets and Function approximation

2.1. Boubaker wavelets. Boubaker wavelets [28] are defined as follows:

$$\psi_{n,m}(x) = \begin{cases} \sqrt{2m+1}\frac{(2m)!}{(m!)^2}2^{\frac{k+1}{2}}B_m(2^{k+1}x-2n+1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where, k is any positive integer, $n = 1,2,...,2^{k-1}$ is an argument and $m = 0,1,...,M-1$ is the order of Boubaker functions [28]:

$$\begin{aligned} B_0(x) &= 1, \\ B_1(x) &= \frac{1}{2}(2x-1), \\ B_2(x) &= \frac{1}{6}(6x^2-6x+1), \\ B_3(x) &= \frac{1}{20}(20x^3-30x^2+12x-1), \end{aligned}$$

and so on. For instance, for $k = 1$ and $M = 4$, we get the Boubaker wavelet bases as follows:

$$\left.\begin{aligned}
\psi_{1,0}(x) &= 2,\\
\psi_{1,1}(x) &= 2\sqrt{3}(8x-3),\\
\psi_{1,2}(x) &= 2\sqrt{5}(96x^2-72x+13),\\
\psi_{1,3}(x) &= 2\sqrt{7}(1280x^3-1440x^2+528x-63).
\end{aligned}\right\} 0 \le x < 1.$$

2.2. Function approximation. Suppose $f(x) \in L^2[0,1)$ is expanded in terms of the Boubaker wavelets as:

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x). \tag{2.2}$$

Truncating the above infinite series, we get

$$f(x) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) = C^T\psi(x) = f_{\hat{m}}(x), \tag{2.3}$$

where, C and $\psi(x)$ are $\hat{m} \times 1$ ($\hat{m} = 2^{k-1}M$) matrices given by

$$C = \left[c_{1,0}, c_{1,1}, ..., c_{1,M-1}, c_{2,0}, c_{2,1}, ..., c_{2,M-1}, c_{2^{k-1},0}, c_{2^{k-1},1}, ..., c_{2^{k-1},M-1}\right], \tag{2.4}$$

and

$$\begin{aligned}\psi(x) = [&\psi_{1,0}(x), \psi_{1,1}(x), ..., \psi_{1,M-1}(x), \psi_{2,0}(x), \psi_{2,1}(x), ..., \psi_{2,M-1}(x),\\
&\psi_{2^{k-1},0}(x), \psi_{2^{k-1},1}(x), ..., \psi_{2^{k-1},M-1}(x)].\end{aligned} \tag{2.5}$$

3. Stochastic operational matrix of integration of Boubaker wavelets

The SOMIBW P_S is a $\hat{m} \times \hat{m}$ matrix defined as:

$$\int_0^x \psi(t)dW(t) = P_S\psi(x). \tag{3.1}$$

In particular, for $k = 1$ and $M = 4$, the matrix P_S is derived as follows:

$$\begin{aligned}\int_0^x \psi_{1,0}(t)dW(t) &= 2W(x), \;\; 0 \le x < 1\\
&\simeq 2W\left(\frac{1}{2}\right)\psi_{1,0}(x),\end{aligned} \tag{3.2}$$

$$\begin{aligned}\int_0^x \psi_{1,1}(t)dW(t) &= 2\sqrt{3}\left((8x-3)\,W(x) - \int_0^x W(x)dx\right), \;\; 0 \le x < 1\\
&\simeq -\left(\sqrt{3}\int_0^x W(t)dt\right)\psi_{1,0}(x) + W\left(\frac{1}{2}\right)\psi_{1,1}(x),\end{aligned} \tag{3.3}$$

$$\begin{aligned}\int_0^x \psi_{1,2}(t)dW(t) &= 2\sqrt{5}\left(\left(96x^2-72x+13\right)W(x) - \int_0^x (2t+1)W(t)dt\right), 0 \le x < 1\\
&\simeq -\left(\sqrt{5}\int_0^x (2t+1)W(t)dt\right)\psi_{1,0}(x) + W\left(\frac{1}{2}\right)\psi_{1,2}(x),\end{aligned} \tag{3.4}$$

$$\int_0^x \psi_{1,3}(t)dW(t) = 2\sqrt{7}\Big(\big(1280x^3 - 1440x^2 + 528x - 63\big) - \int_0^x \big(3840t^2 - 2880t + 528\big)W(t)dt\Big), \ \ 0 \le x < 1$$
$$\simeq -\left(\sqrt{7}\int_0^x (3840t^2 - 2880t + 528)W(t)dt\right)\psi_{1,0}(x) + W\left(\frac{1}{2}\right)\psi_{1,2}(x). \tag{3.5}$$

Using equations (3.2) to (3.5), we get

$$\int_0^x \psi(t)dW(t) = \begin{bmatrix} \int_0^x \psi_{1,0}(t)dW(t) \\ \int_0^x \psi_{1,1}(t)dW(t) \\ \int_0^x \psi_{1,2}(t)dW(t) \\ \int_0^x \psi_{1,3}(t)dW(t) \end{bmatrix}.$$

Therefore,

$$\int_0^x \psi(t)dW(t) = \underbrace{\begin{bmatrix} 2W\left(\frac{1}{2}\right) & 0 & 0 & 0 \\ -\left(\sqrt{3}\int_0^x W(t)dt\right) & W\left(\frac{1}{2}\right) & 0 & 0 \\ -\left(\sqrt{5}\int_0^x (2t+1)W(t)dt\right) & 0 & W\left(\frac{1}{2}\right) & 0 \\ -\left(\sqrt{7}\int_0^x (3840t^2 - 2880t + 528)W(t)dt\right) & 0 & 0 & W\left(\frac{1}{2}\right) \end{bmatrix}}_{P_S} \psi(x).$$

The SOMIBW are derived here for $k = 1$ and $M = 4$ i.e. for $\hat{m} = 4$ and the same can be extended for different values of k and M i.e. for different values of $\hat{m}$.

Remark 3.1. If F is a $\hat{m}$ vector, then

$$\psi(x)\psi^T(x)F = \tilde{F}\psi(x), \tag{3.6}$$

where, $\psi(x)$ is the Boubaker wavelets coefficient matrix for the collocation point $x_j = \frac{j-0.5}{\hat{m}}$ and $\tilde{F}$ is a $\hat{m} \times \hat{m}$ matrix given by

$$\tilde{F} = \psi(x)\bar{F}\psi^{-1}(x), \tag{3.7}$$

where $\bar{F} = \mathrm{diag}(\psi^{-1}(x)F)$. Also, for a $\hat{m} \times \hat{m}$ matrix C,

$$\psi^T(x)C\psi(x) = \hat{C}^T\psi(x), \tag{3.8}$$

where $\hat{C}^T = X\psi^{-1}(x)$, in which $X = \mathrm{diag}(\psi^T(x)C\psi(x))$.

4. Boubaker wavelets stochastic operational matrix method for the numerical solution of stochastic Abel's integral equations

Let us consider equation (1.1). Approximating $f(x)$, $k(x,t)$, and $y(x)$ with respect to Boubaker wavelets as follows:

$$y(x) \simeq C^T\psi(x), \tag{4.1}$$

where C is given in equation (2.4) and is the unknown to be determined.

$$f(x) \simeq F^T\psi(x), \tag{4.2}$$
$$k(x,t) \simeq \psi^T(x)K\psi(t) = \psi^T(t)K^T\psi(x), \tag{4.3}$$

Substituting equations (4.1), (4.2), and (4.3) in (1.1), we get

$$C^T\psi(x) = F^T\psi(x) + \int_0^x \frac{C^T\psi(t)}{\sqrt{x-t}}dt + \psi^T(x)K^T\left(\int_0^x \psi(t)\psi^T(t)CdW(t)\right). \quad (4.4)$$

From section 2 we see that the bases of Boubaker wavelets are polynomials, and hence we can calculate $\int_0^x \frac{t^n}{\sqrt{x-t}}dt$. In [32] it is given as:

$$\int_0^x \frac{t^n}{\sqrt{x-t}}dt = \frac{\sqrt{\pi}x^{\left(\frac{1}{2}+n\right)}\Gamma(n+1)}{\Gamma\left(n+\frac{3}{2}\right)}. \quad (4.5)$$

And therefore

$$\int_0^x \frac{t^n}{\sqrt{x-t}}dt = P\psi(x), \quad (4.6)$$

where P is $2^{k-1}M \times 2^{k-1}M$ matrix. Using Remark 3.1 and equations (4.4), (4.5), and (4.6) we have

$$C^T\psi(x) = F^T\psi(x) + C^TP\psi(x) + \psi^T(x)K^T\left(\int_0^x \tilde{C}\psi(t)dW(t)\right). \quad (4.7)$$

where $\tilde{C}$ is a $\hat{m}\times\hat{m}$ matrix. Using the SOMIBW, equation (4.7) reduces to:

$$C^T\psi(x) = F^T\psi(x) + C^TP\psi(x) + \psi^T(x)K^T\tilde{C}P_s\psi(x). \quad (4.8)$$

Let $X = K^T\tilde{C}P_S$. Again using Remark 3.1, equation (4.8) reduces to:

$$C^T\psi(x) = F^T\psi(x) + C^TP\psi(x) + \hat{X}^T\psi(x). \quad (4.9)$$

Therefore, equation (4.9) reduces to:

$$C^T - C^TP - \hat{X}^T = F^T. \quad (4.10)$$

Equation (4.10) is linear system of equation in terms of vectors of C. Solving this linear system of equations, we get the unknown vector C. Substituting the obtained vector C in equation (4.1), we get the required solution of equation (1.1).

5. Convergence and Error analysis

Theorem 5.1. *[24] If $y : [a,b] \in \mathbb{R}$ is a continuous function and f is an integrable function such that its sign does not change on the interval $[a,b]$, then there exists a constant $\xi \in (a,b)$ such that:*

$$\int_a^b y(x)f(x)dx = y(\xi)\int_a^b f(x)dx, \quad (5.1)$$

Theorem 5.2. *Let $y(x)$ be the exact solution and $y^*(x)$ be the approximate solution of equation (1.1) obtained by Boubaker wavelets. Also, let us assume that the following conditions hold:*

(1) $\| k(x,t) \|_\infty \leq \kappa$,
(2) $1 - \kappa \| W(x) \|_\infty > 0$.

Then the upper bound is obtained as follows:

$$\| y(x) - y^*(x) \| \leq \frac{2 \mid (y(\xi) - y^*(\xi)) \mid}{(1 - \kappa \| W(x) \|_\infty)}, \quad (5.2)$$

where $0 < \xi < x$.

Proof. Let us consider equation (1.1). If $y^*(x)$ is the approximate solution of equation (1.1) obtained by Boubaker wavelets, then

$$y^*(x) = f(x) + \int_0^x \frac{y^*(t)}{\sqrt{x-t}}dt + \int_0^x k(x,t)y^*(t)dW(t).$$

Therefore,

$$y(x) - y^*(x) = \int_0^x \frac{y(t) - y^*(t)}{\sqrt{x-t}}dt + \int_0^x k(x,t)\left(y(t) - y^*(t)\right)dW(t).$$

Hence,

$$\| y(x) - y^*(x) \|_\infty \leq \| \int_0^x \frac{y(t) - y^*(t)}{\sqrt{x-t}}dt \|_\infty + \| \int_0^x k(x,t)\left(y(t) - y^*(t)\right)dW(t) \|_\infty . \quad (5.3)$$

Using Theorem 5.1, there exist a constant $0 < \xi < x$ such that:

$$\int_0^x \frac{y(t) - y^*(t)}{\sqrt{x-t}}dt = \left(y(\xi) - y^*(\xi)\right)\int_0^x \frac{1}{\sqrt{x-t}}dt = \frac{y(\xi) - y^*(\xi)}{\frac{1}{2}}\sqrt{x}.$$

And therefore,

$$\| \int_0^x \frac{y(t) - y^*(t)}{\sqrt{x-t}}dt \|_\infty = 2 \| y(\xi) - y^*(\xi) \| . \quad (5.4)$$

From equations (5.3), (5.4), and assumption 1, we get

$$\| y(x) - y^*(x) \| \leq 2 \| | y(\xi) - y^*(\xi) | + \kappa \| W(x) \|_\infty \| y(x) - y^*(x) \| . \quad (5.5)$$

From equation (5.5) and assumption 2, we conclude that

$$\| y(x) - y^*(x) \| \leq \frac{2 \mid \left(y(\xi) - y^*(\xi)\right) \mid}{\left(1 - \kappa \| W(x) \|_\infty\right)}.$$

□

6. Computational Experiments

Test problem 6.1. Let us consider the following singular linear stochastic Itô-Volterra integral equation:

$$y(x) = \frac{1}{18} - \int_0^x \frac{y(t)}{\sqrt{x-t}}dt - \int_0^x \sin(t)y(t)dW(t), \quad x \in [0,1). \quad (6.1)$$

Exact solution of equation (6.1) is found to be

$$y(x) = \frac{1}{18}\exp\left(-2\sqrt{x} - \frac{1}{4}x + \frac{1}{8}\sin(2x) - \int_0^x \sin(t)dW(t)\right).$$

Table 1 and table 2 shows the exact solution, approximate solution, absolute errors (AE), and comparison of test problem 6.1 with Bernoulli polynomials solution (BPS) for $\hat{m} = 4$ and $\hat{m} = 8$ respectively. Table 3 shows the comparison of maximum absolute errors of test problem 6.1 with Bernoulli polynomials for $\hat{m} = 4$ and $\hat{m} = 8$. Figure 1 shows the graphs of exact and approximate solution of test problem 6.1 for $\hat{m} = 4$ and $\hat{m} = 8$. Figure 2 shows the graph of comparison of absolute errors of test problem 6.1 with that Bernoulli polynomials for $\hat{m} = 4$ and $\hat{m} = 8$.

TABLE 1. comparison of exact solution, approximate solution, and absolute errors (AE) of test problem 6.1 with Bernoulli polynomials solution (BPS) for $\hat{m} = 4$.

x	**Exact**	**Approximate**	**BPS** [27]	**AE**	**AE of BPS** [27]
0	0.0556	0.0496	0.0453	6.00e-03	1.02e-02
0.1	0.0290	0.0267	0.0379	2.30e-03	8.90e-03
0.2	0.0223	0.0219	0.0323	4.00e-04	1.00e-02
0.3	0.0175	0.0245	0.0281	7.00e-03	1.06e-02
0.4	0.0151	0.0154	0.0249	3.00e-04	9.80e-03
0.5	0.0140	0.0216	0.0223	7.60e-03	8.30e-03
0.6	0.0102	0.0177	0.0200	7.50e-03	9.80e-03
0.7	0.0145	0.0118	0.0180	2.70e-03	3.50e-03
0.8	0.0129	0.0154	0.0163	2.50e-03	3.30e-03
0.9	0.0082	0.0050	0.0148	3.20e-03	6.60e-03

TABLE 2. Comparison of exact solution, approximate solution, and absolute errors (AE) of test problem 6.1 with Bernoulli polynomials solution (BPS) for $\hat{m} = 8$.

x	**Exact**	**Approximate**	**BPS** [27]	**AE**	**AE of BPS** [27]
0	0.0556	0.0523	0.0492	3.30e-03	6.30e-03
0.1	0.0301	0.0313	0.0385	1.20e-03	8.50e-03
0.2	0.0225	0.0291	0.0322	6.60e-03	9.70e-03
0.3	0.0187	0.0245	0.0277	5.80e-03	9.00e-03
0.4	0.0134	0.0154	0.0242	2.00e-03	1.08e-02
0.5	0.0111	0.0132	0.0213	2.10e-03	1.03e-02
0.6	0.0134	0.0148	0.0190	1.40e-03	5.60e-03
0.7	0.0107	0.0130	0.0170	2.30e-03	6.30e-03
0.8	0.0074	0.0054	0.0154	2.00e-03	8.00e-03
0.9	0.0059	0.0060	0.0140	1.00e-04	8.10e-03

TABLE 3. Comparison of maximum absolute errors (E_{max}) of 6.1 with Bernoulli polynomials for $\hat{m} = 4$ and $\hat{m} = 8$.

Methods	Maximum absolute error (E_{max})
Boubaker wavelets method	
$\hat{m} = 4$	7.60e-03
$\hat{m} = 8$	6.60e-03
Bernoulli polynomials method [27]	
$\hat{m} = 4$	1.06e-02
$\hat{m} = 8$	1.08e-02

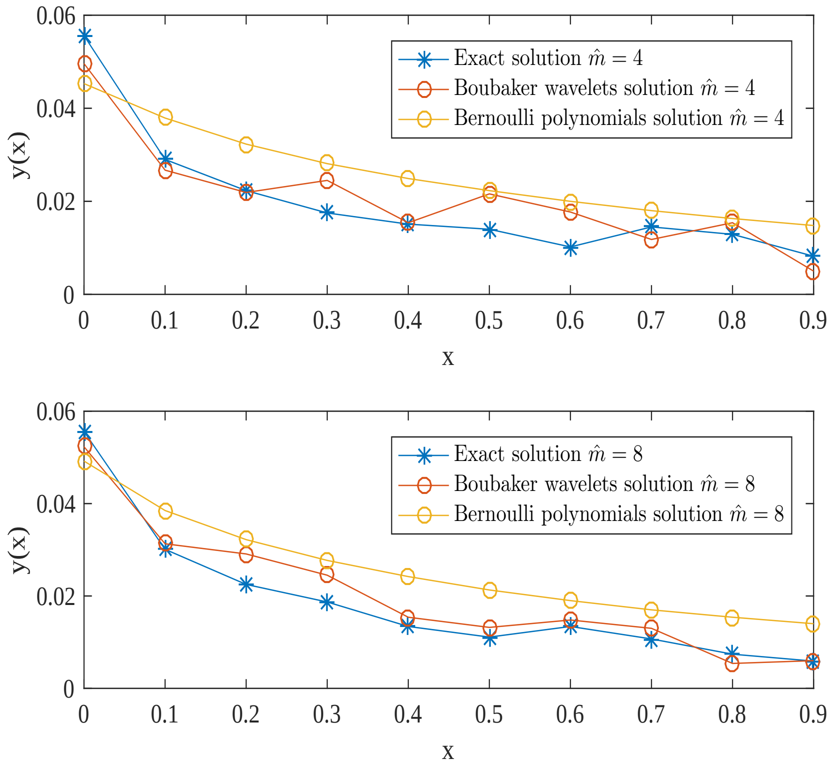

FIGURE 1. Graphs of exact and approximate solution of test problem 6.1 for $\hat{m} = 4$ and $\hat{m} = 8$.

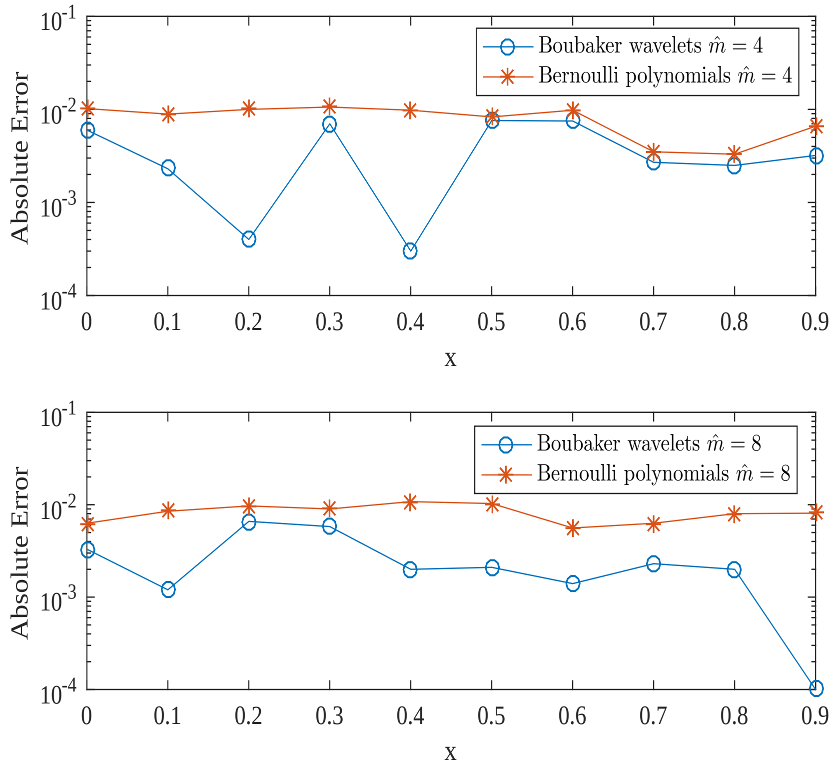

FIGURE 2. Graph of comparison of absolute errors of test problem 6.1 with that Bernoulli polynomials for $\hat{m}=4$ and $\hat{m}=8$.

Test problem 6.2. Let us consider the following singular linear stochastic Itô-Volterra integral equation:

$$y(x)=\frac{1}{36}-\int_0^x \frac{y(t)}{\sqrt{x-t}}dt-\int_0^x \exp(x)\sin(t)y(t)dW(t),\quad x\in[0,1). \qquad (6.2)$$

Exact solution of equation (6.2) is found to be

$$y(x)=\frac{1}{36}\exp\left(-2\sqrt{x}-\frac{1}{4}x\exp(2x)+\frac{1}{8}\exp(2x)\sin(2x)-\int_0^x \exp(x)\sin(t)dW(t)\right).$$

Table 4 and table 5 shows the exact solution, approximate solution, absolute errors (AE), and comparison of test problem 6.2 with Bernoulli polynomials solution (BPS) for $\hat{m}=4$ and $\hat{m}=8$ respectively. Table 6 shows the comparison of maximum absolute errors test problem of 6.2 with Bernoulli polynomials for $\hat{m}=4$ and $\hat{m}=8$. Figure 3 shows the graphs of exact and approximate solution of test problem 6.2 for $\hat{m}=4$ and $\hat{m}=8$. Figure 4 shows the graph of comparison of absolute errors of test problem 6.2 with that Bernoulli polynomials for $\hat{m}=4$ and $\hat{m}=8$.

TABLE 4. comparison of exact solution, approximate solution, and absolute errors (AE) of test problem 6.2 with Bernoulli polynomials solution (BPS) for $\hat{m} = 4$.

x	**Exact**	**Approximate**	**BPS** [27]	**AE**	**AE of BPS** [27]
0	0.0278	0.0267	0.0227	1.10e-03	5.00e-03
0.1	0.0144	0.0140	0.0190	4.00e-04	4.60e-03
0.2	0.0111	0.0119	0.0162	8.00e-04	5.10e-03
0.3	0.0086	0.0092	0.0140	6.00e-04	5.50e-03
0.4	0.0073	0.0066	0.0124	7.00e-04	5.10e-03
0.5	0.0070	0.0064	0.0110	6.00e-04	4.00e-03
0.6	0.0043	0.0017	0.0097	2.60e-03	5.40e-03
0.7	0.0091	0.0088	0.0086	3.00e-04	5.00e-04
0.8	0.0079	0.0078	0.0076	1.00e-04	3.00e-04
0.9	0.0028	0.0016	0.0066	1.20e-03	3.80e-03

TABLE 5. Comparison of exact solution, approximate solution, and absolute errors (AE) of test problem 6.2 with Bernoulli polynomials solution (BPS) for $\hat{m} = 8$.

x	**Exact**	**Approximate**	**BPS** [27]	**AE**	**AE of BPS** [27]
0	0.0278	0.0264	0.0254	1.40e-03	2.30e-03
0.1	0.0146	0.0156	0.0200	1.00e-03	5.40e-03
0.2	0.0115	0.0125	0.0171	1.00e-03	5.60e-03
0.3	0.0097	0.0098	0.0152	1.00e-04	5.40e-03
0.4	0.0090	0.0086	0.0137	4.00e-04	4.70e-03
0.5	0.0077	0.0066	0.0126	1.10e-03	4.90e-03
0.6	0.0075	0.0074	0.0118	1.00e-04	4.30e-03
0.7	0.0106	0.0102	0.0112	4.00e-04	6.00e-04
0.8	0.0099	0.0093	0.0109	6.00e-04	1.00e-03
0.9	0.0083	0.0061	0.0108	2.20e-03	2.60e-03

TABLE 6. Comparison of maximum absolute errors (E_{max}) of 6.2 with Bernoulli polynomials for $\hat{m} = 4$ and $\hat{m} = 8$.

Methods	Maximum absolute error (E_{max})
Boubaker wavelets method	
$\hat{m} = 4$	2.60e-03
$\hat{m} = 8$	2.20e-03
Bernoulli polynomials method [27]	
$\hat{m} = 4$	5.50e-03
$\hat{m} = 8$	5.60e-03

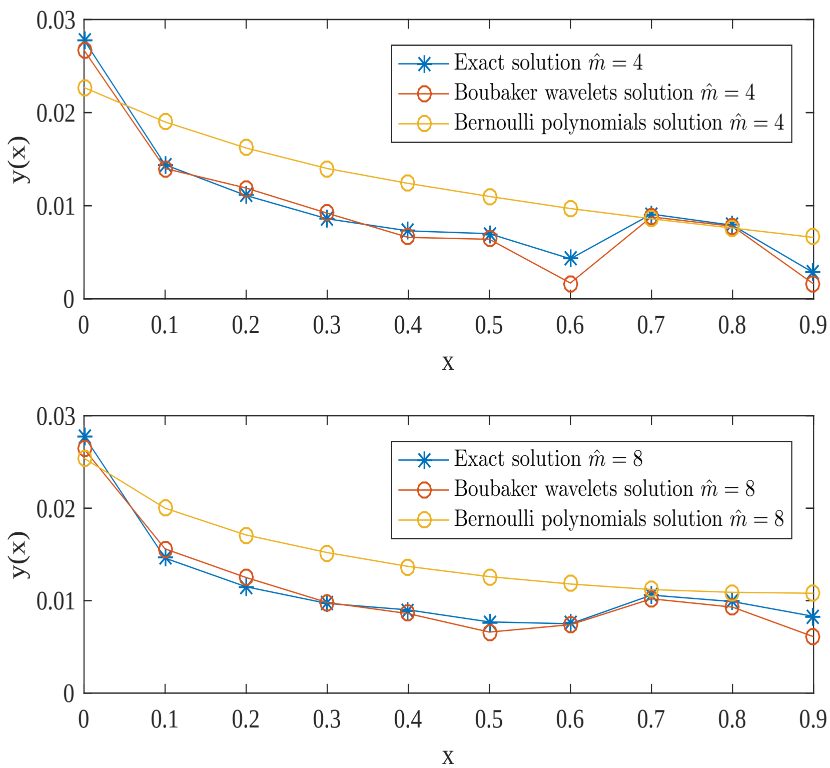

FIGURE 3. Graphs of exact and approximate solution of test problem 6.2 for $\hat{m} = 4$ and $\hat{m} = 8$.

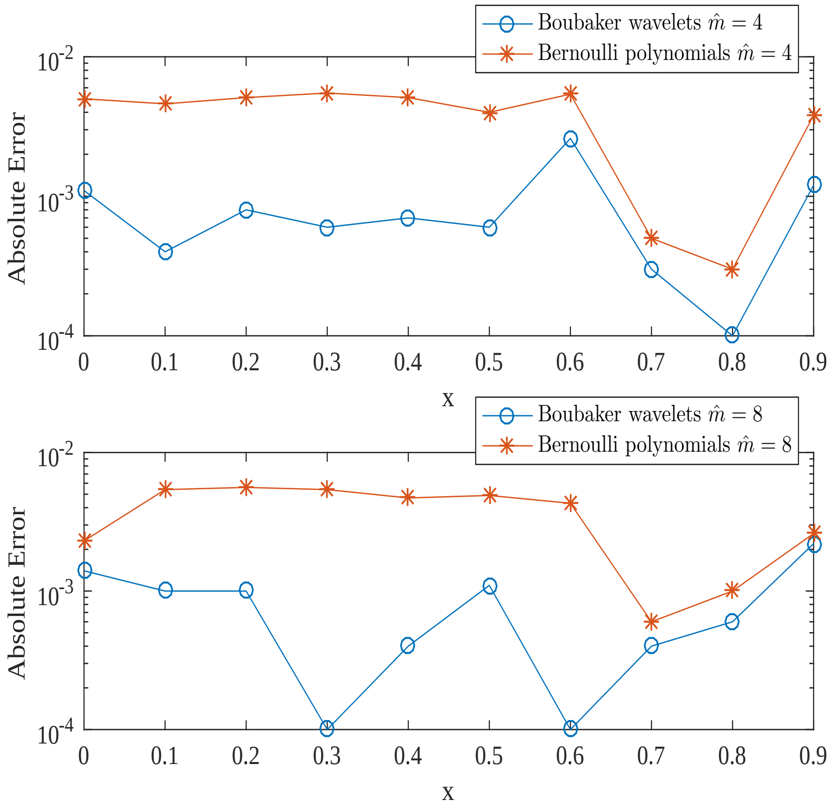

FIGURE 4. Graph of comparison of absolute errors of test problem 6.2 with that Bernoulli polynomials for $\hat{m} = 4$ and $\hat{m} = 8$.

7. Conclusion

In this article, we have provided an effective technique for solving stochastic Abel's integral equations using Boubaker wavelets. These equations are reduced to a system of algebraic equations with unknown Boubaker coefficients, by using Boubaker wavelets, and these equations are solved numerically. Convergence and error analysis of the proposed method is presented. Moreover, the results obtained by the method proposed are compared with the exact solution and the Bernoulli polynomials solution with the number of numerical examples which show that the method presented is precise and accurate.

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