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BOUNDS FOR ENERGY OF BINARY LABELED GRAPH

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ABSTRACT. Let G be a graph with vertex set $V(G)$ and edge set $X(G)$ and consider the set $A = \{0, 1\}$. A mapping $l : V(G) \to A$ is called binary vertex labeling of G and $l(v)$ is called the label of the vertex v under l. The label energy of G is the sum of the absolute values of the label eigenvalues. In this paper, we establish bounds for label energy, largest label eigenvalue and label spectral radius.

1. Introduction

Let $G(V, X)$ be a connected graph with n vertices and m edges and let $A = A(G)$ be its adjacency matrix. The eigenvalues of the adjacency matrix A are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$ assumed in non increasing order. The energy of graph G was first introduced by Ivan Gutman [6] in 1978 as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. For details on energy of graph refer [1, 2, 3, 5, 7, 8, 9, 11, 10, 12, 13, 14, 15].

P. G. Bhat and S. D'Souza in [4] have introduced label matrix denoted as $A_l(G) = [l_{ij}]$ of order n, whose entries l_{ij} are defined as follows:

$$l_{ij} = \begin{cases} a, & \text{if } v_iv_j \in X \text{ and } l(v_i) = l(v_j) = 0, \\ b, & \text{if } v_iv_j \in X \text{ and } l(v_i) = l(v_j) = 1, \\ c, & \text{if } v_iv_j \in X \text{ and } l(v_i) = 0,\ l(v_j) = 1 \text{ or vice-versa}, \\ 0, & \text{otherwise.} \end{cases}$$

where $a,\ b,\ c$ are distinct nonzero real numbers.
The label eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of G are assumed in non increasing order. The label energy of a graph G is defined as $E_l(G) = \sum_{i=1}^{n} |\lambda_i|$. Since $A_l(G)$ is a real symmetric matrix with zero trace, these eigenvalues of binary labeled graph are real with sum equal to zero. Some well known properties of graph label eigenvalues are

$$\sum_{i=1}^{n} \lambda_i = 0$$

$$\sum_{i=1}^{n} \lambda_i^2 = 2Q \tag{1.1}$$

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where $Q = n_1a^2 + n_2b^2 + n_3c^2$ and n_1, n_2, n_3 denote number of edges of G whose end vertex labels are $(0,0)$, $(1,1)$ and $(0,1)$ respectively.

And

$$\det(A) = \prod_{i=1}^{n} \lambda_i. \tag{1.2}$$

This paper is organized as follows. In Section 2, we present some bounds for spectral radius and label energy. Bounds for largest label eigenvalue are established.

2. Bounds for energy of binary labeled graph

Proposition 2.1. *Let $G(m_1, n)$ and $H(m_2, n)$ be two labeled graphs with n vertices. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda_1^{'} \geq \lambda_2^{'} \geq \cdots \geq \lambda_n^{'}$ are label eigenvalues of G and H respectively, then*

$$\sum_{i=1}^{n} \lambda_i\lambda_i^{'} \leq 2\sqrt{(n_1a^2 + n_2b^2 + n_3c^2)(n_1^{'}a^2 + n_2^{'}b^2 + n_3^{'}c^2)},$$

where $n_1^{'}$, $n_2^{'}$, $n_3^{'}$ denote number of edges of H whose end vertex labels are $(0,0)$, $(1,1)$ and $(0,1)$ respectively. Equality holds if G or H is $\overline{K_n}$.

Proof. By Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Setting $a_i = \lambda_i$ and $b_i = \lambda_i^{'}$ in the above inequality, we get

$$\left(\sum_{i=1}^{n} \lambda_i\lambda_i^{'}\right)^2 \leq \left(\sum_{i=1}^{n} \lambda_i^2\right)\left(\sum_{i=1}^{n} \lambda_i^{'2}\right)$$
$$= 4QQ^{'}, \quad \text{where } Q^{'} = n_1^{'}a^2 + n_2^{'}b^2 + n_3^{'}c^2.$$

Hence,

$$\left(\sum_{i=1}^{n} \lambda_i\lambda_i^{'}\right) \leq 2\sqrt{QQ^{'}}.$$

Therefore,

$$\sum_{i=1}^{n} \lambda_i\lambda_i^{'} \leq 2\sqrt{(n_1a^2 + n_2b^2 + n_3c^2)(n_1^{'}a^2 + n_2^{'}b^2 + n_3^{'}c^2)}.$$

Equality holds, when G or $H \cong \overline{K_n}$, we have m_1 or $m_2 = 0$ thus $E_l(G)$ or $E_l(H) = 0$. □

Theorem 2.2. [4] *Let G be a labeled graph with n vertices, m edges. Then*

$$\sqrt{2(n_1a^2 + n_2b^2 + n_3c^2) + n(n-1)p^{\frac{2}{n}}} \leq E_l(G) \leq \sqrt{2n(n_1a^2 + n_2b^2 + n_3c^2)}.$$

In [4], the upper and lower bounds for $E_l(G)$ are attained. Using Theorem 2.2, we find the following bounds for $E_l(G)$.

Theorem 2.3. *Let G be a connected labeled graph with n vertices and m edges. Then*

$$2\sqrt{n_1a^2 + n_2b^2 + n_3c^2} \leq E_l(G) \leq 2\sqrt{m(n_1a^2 + n_2b^2 + n_3c^2)}$$

with left equality holding if G is K_2, $\overline{K_n}$, S_n, complete bipartite graph and right equality holding if and only if G is $\frac{n}{2}K_2$, $\overline{K_n}$.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the label eigenvalues of G. Since,

$$\sum_{i=1}^{n} \lambda_i = 0$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = 2Q$$

we have

$$\sum_{i<j} \lambda_i \lambda_j = -Q. \tag{2.1}$$

Now consider

$$\begin{aligned}
[E_l(G)]^2 &= \left(\sum_{i=1}^{n} |\lambda_i| \right)^2 \\
&= \sum_{i=1}^{n} |\lambda_i| \ \sum_{j=1}^{n} |\lambda_j| \\
&= \sum_{i=1}^{n} |\lambda_i|^2 \ + \ 2 \sum_{1\leq i<j\leq n} |\lambda_i||\lambda_j| \\
&\geq \sum_{i=1}^{n} |\lambda_i|^2 \ + \ 2 \ |\sum_{i<j} \lambda_i \lambda_j| \\
&\geq 2Q + 2Q \ \text{ using equations (1.1) and (2.1).}
\end{aligned}$$

Hence, $E_l(G) \geq 2\sqrt{Q}$.
From Theorem 2.2, we have $E_l(G) \leq \sqrt{2nQ}$. Since $n \leq 2m$,
we have

$$E_l(G) \leq 2\sqrt{mQ}$$

Thus,

$$2\sqrt{Q} \leq E_l(G) \leq 2\sqrt{mQ}.$$

Therefore,

$$2\sqrt{n_1a^2 + n_2b^2 + n_3c^2} \leq E_l(G) \leq 2\sqrt{m(n_1a^2 + n_2b^2 + n_3c^2)}.$$

Left equality holds, when

(i) $G \cong K_2$, an edge whose end vertex labels are $(0,0)$ or $(0,1)$ or $(1,1)$.
(ii) $G \cong \overline{K_n}$ and $E_l(G) = 0$.
(iii) $G \cong S_n$, either n_1 or $n_2 = 0$.
(iv) $G \cong K_{m,m}$, each edge whose end vertex labels are $(0,0)$, $(0,1)$ or $(1,1)$, $(0,1)$.

Right equality holds, when

(i) $G \cong \frac{n}{2}K_2$, each edge whose end vertex labels are $(0,0)$ or $(0,1)$ or $(1,1)$.
(ii) $G \cong \overline{K_n}$ and $E_l(G) = 0$.

□

Now we give few bounds for label spectral radius and obtain bounds for label energy.

Proposition 2.4. *Let G be a labeled graph (n, m)- graph and $\rho_l(G) = \max_{1 \le i \le n} \{|\lambda_i|\}$ be the label spectral radius of G. Then*

$$\sqrt{\frac{2(n_1a^2 + n_2b^2 + n_3c^2)}{n}} \le \rho_l(G) \le \sqrt{2(n_1a^2 + n_2b^2 + n_3c^2)}$$

with left equality holding if and only if G is $\frac{n}{2}K_2$, $\overline{K_n}$ and right equality holds if G is $\overline{K_n}$.

Proof. Consider

$$\rho_l^2(G) = \max_{1 \le i \le n} \{|\lambda_i|^2\}$$
$$\le \sum_{j=1} n\lambda_j^2 = 2Q.$$
$$\rho_l(G) \le \sqrt{2Q}. \tag{2.2}$$

Next consider

$$n\ \rho_l^2(G) \ge \sum_{i=1}^{n} \lambda_i^2$$
$$\ge 2Q.$$

We have

$$\rho_l(G) \ge \sqrt{\frac{2Q}{n}}. \tag{2.3}$$

Combining expression (2.2) and (2.3)

$$\sqrt{\frac{2Q}{n}} \le \rho_l(G) \le \sqrt{2Q}.$$

Therefore,

$$\sqrt{\frac{2(n_1a^2 + n_2b^2 + n_3c^2)}{n}} \le \rho_l(G) \le \sqrt{2(n_1a^2 + n_2b^2 + n_3c^2)}.$$

Left equality holds, when
(i) $G \cong \frac{n}{2}K_2$, each edge whose end vertex labels are $(0,0)$ or $(0,1)$ or $(1,1)$.
(ii) $G \cong \overline{K_n}$.

Right equality holds, when $G \cong \overline{K_n}$. □

Theorem 2.5. *Let G be a labeled graph and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the label eigenvalues of G. If $n \le 2(n_1a^2 + n_2b^2 + n_3c^2)$ and $\lambda_1 \ge \frac{2(n_1a^2 + n_2b^2 + n_3c^2)}{n}$, then*

$$E_l(G) \le \frac{2(n_1a^2 + n_2b^2 + n_3c^2)}{n} + \sqrt{(n-1)\left[2(n_1a^2 + n_2b^2 + n_3c^2) - \left(\frac{2(n_1a^2 + n_2b^2 + n_3c^2)}{n}\right)^2\right]}.$$

Proof. We have

$$\sum_{i=2}^{n} \lambda_i^2 = 2Q - \lambda_1^2 \tag{2.4}$$

By a special case of Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \le n \sum_{i=1}^{n} |\lambda_i|^2.$$

Thus,

$$\left(\sum_{i=2}^{n} |\lambda_i|\right)^2 \le (n-1) \sum_{i=2}^{n} |\lambda_i|^2$$

and hence,

$$\left(\sum_{i=2}^{n} |\lambda_i|\right) \le \sqrt{(n-1) \sum_{i=2}^{n} |\lambda_i|^2}. \tag{2.5}$$

Employing (2.4) in (2.5), we obtain

$$E_l(G) - \lambda_1 \le \sqrt{(n-1)[2Q - \lambda_1^2]}$$

that is,

$$E_l(G) \le \lambda_1 + \sqrt{(n-1)[2Q - \lambda_1^2]}.$$

Consider, the function

$$F(x) = x + \sqrt{(n-1)[2Q - x^2]}.$$

Then,

$$F'(x) = 1 - \frac{x\sqrt{(n-1)}}{\sqrt{2Q - x^2}}.$$

We observe that, $F(x)$ is decreasing in the interval

$$\left(\sqrt{\frac{2Q}{n}}, \sqrt{2Q}\right).$$

Since, $n \le 2Q$ and $\frac{2Q}{n} \le \lambda_1$, we have

$$\sqrt{\frac{2Q}{n}} < \frac{2Q}{n} \le \lambda_1 \le \sqrt{2Q}.$$

Last inequality follows from Proposition 2.4.
Hence,

$$E_l(G) \le \frac{2Q}{n} + \sqrt{(n-1)\left[2Q - \left(\frac{2Q}{n}\right)^2\right]}.$$

Therefore,

$$E_l(G) \le \frac{2(n_1a^2 + n_2b^2 + n_3c^2)}{n} + \sqrt{(n-1)\left[2(n_1a^2 + n_2b^2 + n_3c^2) - \left(\frac{2(n_1a^2 + n_2b^2 + n_3c^2)}{n}\right)^2\right]}.$$

□

As the proof of the following theorem is similar to that of Theorem 2.5 we omit the proof.

Theorem 2.6. *If* $n \leq 2Q$ *and* $\sqrt{\frac{2(n_1a^2+n_2b^2+n_3c^2)}{n}} \leq \rho_l(G) \leq \frac{2(n_1a^2+n_2b^2+n_3c^2)}{n}$, *then*

$$E_l(G) \geq \frac{2(n_1a^2+n_2b^2+n_3c^2)}{n} + \sqrt{(n-1)\left[2(n_1a^2+n_2b^2+n_3c^2) - \left(\frac{2(n_1a^2+n_2b^2+n_3c^2)}{n}\right)^2\right]}.$$

Now we prove the following theorem which is useful to obtain bounds for the largest label eigenvalue of a graph G.

Theorem 2.7. *Let* G *be a labeled graph with* n *vertices and* m *edges and* H *be a* (n, m_1)*-graph. If* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *are label eigenvalues of* G *and* $\lambda_1^{'} \geq \lambda_2^{'} \geq \cdots \geq \lambda_n^{'}$ *are eigenvalues of* H *then*

$$\sum_{i=1}^{n} \lambda_i \lambda_i^{'} \leq 2\sqrt{(n_1a^2+n_2b^2+n_3c^2)m_1}.$$

Equality holds if G *or* H *is* $\overline{K_n}$.

Proof. By Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} \lambda_i \lambda_i^{'}\right)^2 \leq \left(\sum_{i=1}^{n} \lambda_i^2\right)\left(\sum_{i=1}^{n} {\lambda_i^{'}}^2\right) \tag{2.6}$$

From equation (1.1), we know that $\sum_{i=1}^{n} \lambda_i^2 = 2Q$. It is well-known that $\sum_{i=1}^{n} {\lambda_i^{'}}^2 = 2m_1$. Using these in expression (2.6) we obtain

$$\left(\sum_{i=1}^{n} \lambda_i \lambda_i^{'}\right) \leq 2\sqrt{Qm_1}$$

Therefore,

$$\left(\sum_{i=1}^{n} \lambda_i \lambda_i^{'}\right) \leq 2\sqrt{(n_1a^2+n_2b^2+n_3c^2)m_1}$$

Equality holds when G or $H \cong \overline{K_n}$, we have $m = 0$ or $m_1 = 0$ and $E_l(G) = 0$. ☐

If we know the spectrum of a graph H with n vertices and m_1 edges, then we can find an upper bound for the largest label eigenvalue of the labeled graph G with n vertices.

Using Theorem 2.7 we establish bounds for the largest label eigenvalue.

Proposition 2.8. *If* G *is a labeled* (n, m)*-graph and* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ *are label eigenvalues of* G, *then*

$$\lambda_1 \leq \frac{1}{p-1}\left[\sqrt{2(n_1a^2+n_2b^2+n_3c^2)p(p-1)} + \sum_{i=2}^{p} \lambda_{n-p+i}\right]$$

where p *is any integer,* $1 < p \leq n$.

Proof. Let $H = K_p \cup \overline{K_{n-p}}$. Then the Spectrum of H is $\begin{pmatrix} (p-1) & 0 & -1 \\ 1 & n-p & p-1 \end{pmatrix}$. Then by Theorem 2.7 we have

$$\lambda_1(p-1)+\lambda_2(0)+\lambda_3(0)+\cdots+\lambda_{n-p+1}(0)+\lambda_{n-p+2}(-1)+\cdots+\lambda_n(-1) \leq 2\sqrt{\frac{Qp(p-1)}{2}}.$$

Thus,

$$(p-1)\lambda_1 \leq \sqrt{2Qp(p-1)} + \sum_{i=2}^{p} \lambda_{n-p+i}.$$

Hence,

$$\lambda_1 \leq \frac{1}{p-1}\left[\sqrt{2Qp(p-1)} + \sum_{i=2}^{p} \lambda_{n-p+i}\right].$$

Therefore,

$$\lambda_1 \leq \frac{1}{p-1}\left[\sqrt{2(n_1a^2+n_2b^2+n_3c^2)p(p-1)} + \sum_{i=2}^{p} \lambda_{n-p+i}\right].$$

□

Remark 2.9. If $p = n$ in the above proposition, then

$$\lambda_1 \leq \sqrt{\frac{2(n_1a^2+n_2b^2+n_3c^2)(n-1)}{n}}.$$

Remark 2.10. If $p = 2$ in the above proposition, then

$$\lambda_1 - \lambda_n \leq 2\sqrt{2(n_1a^2+n_2b^2+n_3c^2)}.$$

Proposition 2.11. *If G is a labeled (n,m)-graph and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are label eigenvalues of G, then*

$$\sum_{i=1}^{k} \lambda_i \leq \sqrt{\frac{2(n_1a^2+n_2b^2+n_3c^2)k(p-1)}{p}}$$

where p is any integer $1 \leq p \leq n$ and $k = \frac{n}{p}$.

Proof. Let H be a graph with n vertices and k components, each is a complete graph K_p. Then $n = pk$ and H has $\frac{kp(p-1)}{2}$ edges. Thus spectrum of H is $\begin{pmatrix} (p-1) & -1 \\ k & k(p-1) \end{pmatrix}$. Then by Theorem 2.7 we have

$$(p-1)\lambda_1+(p-1)\lambda_2+\cdots+(p-1)\lambda_k+(-1)\lambda_{k+1}+\cdots+(-1)\lambda_n \leq 2\sqrt{\frac{Qkp(p-1)}{2}}.$$

Thus,

$$p\sum_{i=1}^{k}\lambda_i - \sum_{i=1}^{n}\lambda_i \leq \sqrt{2Qkp(p-1)}$$

and

$$\sum_{i=1}^{k}\lambda_i \leq \sqrt{\frac{2Qk(p-1)}{p}}$$

Therefore,

$$\sum_{i=1}^{k} \lambda_i \leq \sqrt{\frac{2(n_1a^2+n_2b^2+n_3c^2)k(p-1)}{p}}$$

□

Remark 2.12. If $k = 1$ in the above proposition, then

$$\lambda_1 \leq \sqrt{\frac{2(n_1a^2+n_2b^2+n_3c^2)(p-1)}{p}}$$

Proposition 2.13. *If G is a labeled (n,m)-graph and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are label eigenvalues of G, then*

$$\left[\sum_{i=1}^{k} \lambda_i - \sum_{i=1}^{k} \lambda_{n-k+i}\right] \leq 2\sqrt{(n_1a^2+n_2b^2+n_3c^2)k},$$

where $1 \leq k < n$ and $k|n$.

Proof. Let H be a graph with n vertices and k components, each is a complete bipartite graph $K_{p,q}$. Then $n = k(p+q)$ and H has kpq edges.
Thus, the spectrum of H is $\begin{pmatrix} \sqrt{pq} & 0 & -\sqrt{pq} \\ k & k(p+q-2) & k \end{pmatrix}$. Then, by Theorem 2.7 we have

$\sqrt{pq}\lambda_1 + \cdots + \sqrt{pq}\lambda_k + (0)\lambda_{k+1} + \cdots + (0)\lambda_{k+k(p+q-2)} + (-\sqrt{pq})\lambda_{k(p+q-1)+1} + \cdots + (-\sqrt{pq})\lambda_n \leq 2\sqrt{Qkpq}$.

Thus,

$$\left[\sum_{i=1}^{k} \lambda_i - \sum_{i=1}^{k} \lambda_{n-k+i}\right] \leq 2\sqrt{Qkpq},$$

and

$$\left[\sum_{i=1}^{k} \lambda_i - \sum_{i=1}^{k} \lambda_{n-k+i}\right] \leq 2\sqrt{Qk}.$$

Therefore,

$$\left[\sum_{i=1}^{k} \lambda_i - \sum_{i=1}^{k} \lambda_{n-k+i}\right] \leq 2\sqrt{(n_1a^2+n_2b^2+n_3c^2)k}.$$

□

Remark 2.14. If $k = 1$ in the above proposition, then

$$\lambda_1 - \lambda_n \leq 2\sqrt{n_1a^2+n_2b^2+n_3c^2}.$$

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