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APPLICATION OF THE DIRECT COMPUTATION METHOD FOR SOLVING A GENERAL FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this study, the general linear and nonlinear Fredholm integro-differential equations are handled using the Direct Computation approach. The proposed method's exact solution is established using a convergence analysis, and the methodology is provided strictly. A formula for computing the solution is derived after establishing existence and uniqueness conditions. The application of the strategy to a few instances using tables and graphs created using MATLAB (2018) Version 9.4 will demonstrate the effectiveness of the method.

1. Introduction

Ivar Fredholm first presented the Fredholm Integro-differential (FID) equation in the early 1900s. Numerous scientific applications call for the use of the Fredholm integro-differential (FID) equations. The ability to derive these equations from boundary value issues was also demonstrated [1, 2, 5, 8, 10]. The Fredholm integro-differential (FID) equations [6, 9, 22-25, 29, 30] are notoriously challenging to solve analytically when the boundary conditions are known. Therefore, various numerical and approximation techniques must be used to solve these difficulties. When studying a population growth model, Fredholm looked at the genetic factors. A specific topic on Fredholm integro-differential (FID) equations emerged from the research work. [16, 26] contains information on the existence and uniqueness of such problems' solutions.

We look at the class of $n^{th}-order$ variable coefficient Fredholm integro-differential (FID) equation

$$\sum_{m=0}^{n} f_m(y)\Xi^m(y) = g(y) + \lambda \int_c^d K(y,t)\Xi(t)dt, \tag{1.1}$$

In this research, we examine the nonlinear Fredholm integro-differential (FID) equations provided by

$$\sum_{m=0}^{n} f_m(y)\Xi^m(y) = g(y) + \int_c^d K(y,t)G(\Xi(t))dt, \qquad y,t \in [c,d], \tag{1.2}$$

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With the initial conditions

$$\Xi^{(s)}(c) = d_s, \qquad s = 0, 1, 2, \ldots, (n-1). \tag{1.3}$$

Where $\Xi^m(y)$ are the m^{th} derivative of the unknown function $\Xi(y), f_m(y)$ is a function of $y, f_n(y) \neq 0$, the initial conditions $\Xi(0), \Xi^{'}(0), \ldots, \Xi^{m-1}(0)$ and the boundary conditions $\Xi(0) = \alpha_1, \Xi^{'}(0) = \alpha_2, \ldots, \Xi(c) = \alpha_{m-1}, \Xi^{'}(d) = \alpha_m$ where $\alpha_i, i = 1, 2, 3, \ldots, m$ are any finite real constants are prescribed. The kernel $K(y,t) \in C([c,d] \times [c,d])$, the function $g(y) \in C[c,d]$ are given real-valued functions, c, d, λ and $G(\Xi(t))$ is nonlinear function of $\Xi(y)$ [15].

The Adomian Decomposition Method (ADM) [4, 7, 12-14, 17, 20, 29, 30], Variational Iteration Method (VIM) [2, 4, 11, 21, 27, 29], He's Homotopy Perturbation Method [5], Homotopy Analysis Method (HAM) [1, 18], Homotopy Perturbation Method [3], the Haar Wavelet method [19], the Laplace Adomian Decomposition Method [28], Modified Adomian Decomposition Method [29, 30] and others have recently been used to solve Integro-Differential Equations.

The primary goal of the current work is to analyse the behaviour of linear or nonlinear Fredholm integro-differential (FID) equation solutions. This behaviour can be explicitly characterised using the Direct Computational Method (DCM), a semi-analytical approximate method. Additionally, we established the Fredholm integro-differential (FID) equations existence and uniqueness results.

2. Derivation of the Direct Computation Method

In order to solve the system of general linear and non-linear Fredholm's integro-differential (FID) equations, this study extends the analysis of the Direct Computational Method (DCM). Many Fredholm integro-differential (FID) equations [29, 30] are typically handled using the classic Direct Computation Method (DCM) [23]. A Fredholm Integro-differential (FID) equation is transformed into an ordinary differential equation (ODE) via the DCM. The acquired ODE's solution is then converted into an algebraic system of equations. By computing the algebraic system of equations' answers and substituting them into the ODE's solution.

Assume the kernel in equation (1.2) has the following shape:

$$K(y,t) = \sum_{k=1}^{l} f_k(y) g_k(t). \tag{2.1}$$

The following Ordinary Differential Equation (ODE) is produced by substituting (2.1) into the system of Fredholm Integro-Differential (FID) equation (1.2)

$$\sum_{m=0}^{n} f_m(y)\Xi^{(m)}(y) = g(y) + \int_c^d \sum_{k=1}^{l} f_k(y) g_k(t) G(\Xi(t)) dt, \qquad y \in [c,d], \tag{2.2}$$

The definite integral at the RHS of (2.2) makes it evident that the integrand depends only on the variable t. Because β is a constant, the definite integral in the RHS (2.2) is identical to the number β. Therefore, (2.2) becomes

$$\Xi^{'}(y) = g(y) + \beta_1 f_1(y) + \beta_2 f_2(y) + \cdots + \beta_l f_l(y)$$

Where

$$\beta_m = \int_c^d g_m(t)\Xi_m^{'}(t)dt, \qquad 1 \leq m \leq l. \tag{2.3}$$

So, (2.2) becomes

$$\sum_{m=0}^{n} f_m(y)\Xi^{(m)}(y) = g(y) + \sum_{k=1}^{l} f_k(y)\beta_m dt \tag{2.4}$$

It is possible to find the constant β_m by simplifying both sides of equation (2.2), integrating $m - times$ from 0 to y, coupling with the initial conditions stated in equation (1.3), and then inserting the resulting equations for $\Xi_m(y)$ into equation (2.3). The solution $\Xi(y)$ of the system of Fredholm integro-differential (FID) equations (1.2) is achieved by plugging the determined numerical value of the constant into the determined equation for $\Xi(y)$.

3. Existence and Uniqueness

Using the initial conditions (1.3) and the uniqueness and existence results of (1.2), we will demonstrate it in this section [16, 26].

We can express (1.2) as follows:

$$\Xi(y) = L^{-1}\left[\frac{g(y)}{f_n(y)}\right] + \sum_{s=0}^{n-1}\frac{(y-c)^s}{s!}d_s + \lambda_1 L^{-1}\left[\int_c^d \frac{1}{f_n(y)}K(y,t)G(\Xi_l(t))dt\right] - L^{-1}\left[\sum_{m=0}^{n-1}\frac{f_m(y)}{f_n(y)}\Xi^{(m)}(y)\right].$$

We are able to write

$$L^{-1}\left[\int_c^d \frac{1}{f_n(y)}K(y,t)G(\Xi_l(t))dt\right] = \int_c^d \frac{(y-t)^n}{n!f_n(y)}K(y,t)G(\Xi_l(t))dt$$

$$\sum_{m=0}^{n-1} L^{-1}\left[\frac{f_m(y)}{f_n(y)}\right]\Xi^{(m)}(y) = \sum_{m=0}^{n-1}\int_c^d \frac{(y-t)^{n-1}f_m(t)}{(n-1)!f_n(t)}\Xi^{(m)}(t)dt.$$

We set

$$\Pi(y) = L^{-1}\left[\frac{g(y)}{f_n(y)}\right] + \sum_{s=0}^{n-1}\frac{(y-c)^s}{s!}d_s.$$

We introduce the following hypothesis before getting started and proving the main results:

(A1) $\exists$ two constants ϕ and $\Phi_m > 0, m = 0, 1, 2, \ldots, n$ such that, for any $\Xi_1, \Xi_2 \in C(M, \mathbb{R})$

$$|G(\Xi_1) - G(\Xi_2)| \leq \phi|\Xi_1 - \Xi_2|$$

And

$$|D^m(\Xi_1) - D^m(\Xi_2)| \leq \Phi_m|\Xi_1 - \Xi_2|.$$

We assume that the nonlinear terms $G(\Xi(y))$ and $D^m(\Xi) = \left(\frac{d^m}{dy^m}\right)\Xi(y) = \sum_{i=0}^{\infty}\Phi_{i_m}$, where D^m is a derivative operator, are Lipschitz continuous for $m = 0, 1, 2, \ldots, n$.

(A2) We consider it to be $\forall\ c \leq t \leq y \leq d$, and $m = 0, 1, 2, \ldots, n$:

$$\left|\frac{\lambda(y-t)^n K(y,t)}{n! f_n(y)}\right| \leq \tau_1, \qquad \left|\frac{\lambda(y-t)^n K(y,t)}{n!}\right| \leq \tau_2,$$

And

$$\left|\frac{(y-t)^{n-1} f_m(t)}{(n-1)! f_n(t)}\right| \leq \tau_3, \qquad \left|\frac{(y-t)^{n-1} f_m(t)}{(n-1)!}\right| \leq \tau_4.$$

(A3) $\exists$ three functions τ_3^*, τ_4^*, and $\Phi^* \in C(D, \mathbb{R}^+)$, the collection of all continuous positive functions $D = \{(y,t) \in \mathbb{R} \times \mathbb{R} : 0 \leq t \leq y \leq 1,\}$ such that:

$$\tau_3^* = max|\tau_3|, \qquad \tau_4^* = max|\tau_4|, \qquad and \qquad \Phi^* = max|\Phi_m|.$$

(A4) $\Pi(y)$ is bounded function $\forall\ y$ in $M = [c, d]$.

Theorem 3.1 *Assume that (A1) - (A4) hold. If* $0 < \Pi = (\phi\tau_1 + n\phi^*\tau_3^*)(d-c) < 1$, *then* $\exists$ *a unique solution* $\Xi(y) \in C(M)$ *to initial value problem (IVP) (1.2) -(3).*

Proof. Let Ξ_1 and Ξ_2 be two different solutions of IVP (1.2)-(1.3), then

$$\begin{aligned}
|\Xi_1 - \Xi_2| &= \left|\int_c^d \frac{\lambda(y-t)^n K(y,t)}{f_n(y) n!}\right|[G(\Xi_1) - G(\Xi_2)]dt \\
&\quad - \sum_{m=0}^{n-1}\int_c^d \frac{(y-t)^{n-1} f_m(t)}{f_n(t)(n-1)!}[D^m(\Xi_1) - D^m(\Xi_2)]dt \\
&\leq \int_c^d \left|\frac{\lambda(y-t)^n K(y,t)}{f_n(y) n!}\right||G(\Xi_1) - G(\Xi_2)|dt \\
&\quad - \sum_{m=0}^{n-1}\int_c^d \left|\frac{(y-t)^{n-1} f_m(t)}{f_n(t)(n-1)!}\right||D^m(\Xi_1) - D^m(\Xi_2)|dt \\
&\leq (\phi\tau_1 + n\phi^*\tau_3^*)(d-c)|\Xi_1 - \Xi_2|
\end{aligned}$$

We get $(1-\Pi)|\Xi_1 - \Xi_2| \leq 0$. Since $0 < \Pi < 1$, so $|\Xi_1 - \Xi_2| = 0$.

Therefore, $\Xi_1 = \Xi_2$ and the proof is completed.

4. Analysis

Demonstrate how the existence theorem can be applied to several examples of FID equations as well as the numerical examples that are solved by DCM in this article. Additionally, some numerical solutions that were obtained converged approximately to the exact solution. To compare the Direct Computational Method (DCM) answer with the exact solution, we provide the absolute which is defined by

$$Error(l) = |\Xi(y) - \Xi_l(y)|.$$

To make the notation simpler in (2.1), we substitute β for β_1 in the situation where $l = 1$.

Example 4.1: Consider the FID equation:

$$\Xi^{''}(y) = -siny + y - \int_0^{\pi/2} yt\Xi(t)dt, \quad \Xi(0) = 0, \quad \Xi^{'}(0) = 1.$$

Uniqueness:

$$|f(y,t,\Xi_1(t)) - f(y,t,\Xi_2(t))| = |y\Xi_1(t) - y\Xi_2(t)| = |y||\Xi_1 - \Xi_2|,$$

Where $K_1(y,t) = y$, since

$$\sup_{t\in[0,\pi/2]} \int_0^{\pi/2} K_1(y,t)dy = \sup_{t\in[0,\pi/2]} \int_0^{\pi/2} ydy \leq 1.$$

Then, the existence theorem allows us to conclude that the problem has a unique solution.

When using DCM, we can pinpoint the exact solution.
Where

$$\beta = \int_0^{\pi/2} t\Xi(t)dt.$$

To find the value of β, $\quad \beta = 1$.
We thus arrive to answer $\Xi(y) = sin(y)$, which is exact.

Example 4.2: Consider the FID equation:

$$\Xi^{'}(y) = -1 + \frac{1}{e} - cosh(y) + ysinh(1) + \int_0^1 (y-t)\Xi^{'}(t)dt, \quad \Xi(0) = 1.$$

Uniqueness:

$$|f(y,t,\Xi_1(t)) - f(y,t,\Xi_2(t))| = |(y-t)\Xi_1(t) - (y-t)\Xi_2(t)| = |y-t||\Xi_1 - \Xi_2|,$$

Where $K_1(y,t) = (y-t)$, since

$$\sup_{t\in[0,1]} \int_0^1 K_1(y,t)dy = \sup_{t\in[0,1]} \int_0^1 (y-t)dy \leq 1.$$

Then, the existence theorem allows us to conclude that the problem has a unique solution.

When using DCM, we can pinpoint the exact solution.
Where

$$\beta_1 = \int_0^1 \Xi^{'}(t)dt, \quad \beta_2 = \int_0^1 t\Xi^{'}(t)dt.$$

To find the value of β_1 and β_2 as

$$\beta_1 = \frac{1}{e} + \frac{1}{2}(-2 + \beta_1 - 2\beta_2 - sinh(1)),$$

$$\beta_2 = \frac{8 + e^2 + 2e\beta_1 - 3e(3 + \beta)}{6e}.$$

Solving these above equations, we obtain

$$\beta_1 = -\frac{-2 + 2e^2 + 9esinh(1)}{13e},$$

$$\beta_2 = -\frac{-12 + 13e - e^2 + 2esinh(1)}{13e}.$$

We thus arrive to answer $\Xi(y) = 1 - sinh(y)$, which is exact.

TABLE 1. The approximate and exact solution of example 4.2

y	**Exact Solution,** $n = 10$	**Approximate Sol.** $n = 10$	$\lvert\Xi(y) - \Xi_m(y)\rvert$
0.2	0.7986640	0.7986450	$1.9129E - 5$
0.4	0.5892480	0.5892040	$4.3296E - 5$
0.6	0.3633460	0.3632880	$5.8509E - 5$
0.8	0.1118940	0.1118440	$4.9678E - 5$

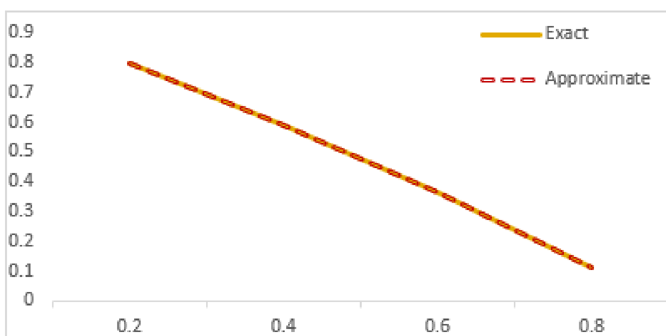

FIGURE 1. Comparison between the exact and approximate solution for example 4.2.

Example 4.3: Consider the FID equation:

$$\Xi^{''}(y) = \frac{9}{4} - \frac{1}{3}y + \int_0^1 (y - t)\Xi(t)dt, \quad \Xi(0) = \Xi^{'}(0) = 0.$$

Uniqueness:

$$|f(y, t, \Xi_1(t)) - f(y, t, \Xi_2(t))| = |(y - t)\Xi_1(t) - (y - t)\Xi_2(t)| = |(y - t)||\Xi_1 - \Xi_2|,$$

Where $K_1(y,t) = (y-t)$, since

$$\sup_{t\in[0,1]} \int_0^1 K_1(y,t)dy = \sup_{t\in[0,1]} \int_0^1 (y-t)dy \leq 1.$$

Then, the existence theorem allows us to conclude that the problem has a unique solution.

When using DCM, we can pinpoint the exact solution.
Where

$$\beta_1 = \int_0^1 \Xi(t)dt, \qquad \beta_2 = \int_0^1 t\Xi(t)dt.$$

To find the value of β_1 and β_2 as

$$\beta_1 = \frac{1}{4}, \qquad \beta_2 = \frac{1}{3}.$$

We thus arrive to answer $\Xi(y) = y^2$, which is exact.

Example 4.4: Consider the FID equation:

$$\Xi^{'}(y) = -\frac{33}{50} siny + \int_{-1/2}^{1} siny\Xi^{'}(t)dt, \quad \Xi(-0.5) = cos(-0.5).$$

Uniqueness:

$$|f(y,t,\Xi_1(t)) - f(y,t,\Xi_2(t))| = |siny\Xi_1(t) - siny\Xi_2(t)| = |siny||\Xi_1 - \Xi_2|,$$

Where $K_1(y,t) = siny$, since

$$\sup_{t\in[-1/2,1]} \int_{-1/2}^1 K_1(y,t)dy = \sup_{t\in[-1/2,1]} \int_{-1/2}^1 siny dy \leq 1.$$

Then, the existence theorem allows us to conclude that the problem has a unique solution.

When using DCM, we can pinpoint the exact solution.
Where

$$\beta = \int_{-1/2}^1 \Xi^{'}(t)dt.$$

To find the value of β as

$$\beta = -0.33728$$

We thus arrive to answer $\Xi(y) = cos(y)$, which is exact.

TABLE 2. The approximate and exact solution of example 4.4

y	Exact Solution, $n = 10$	Approximate Sol. $n = 10$	$\lvert \Xi(y) - \Xi_m(y) \rvert$
-0.35	0.9393730	0.9393730	$2.22045E - 16$
-0.20	0.9800670	0.9800670	$1.11022E - 16$
-0.05	0.9987500	0.9987500	0.000000000
0.10	0.9950040	0.9950040	0.000000000
0.25	0.9689120	0.9689120	$1.11022E - 16$
0.40	0.9210661	0.9210661	$2.22045E - 16$
0.55	0.8525250	0.8525250	$2.22045E - 16$
0.70	0.7648420	0.7648420	$1.11022E - 16$
0.85	0.6599840	0.6599840	0.000000000

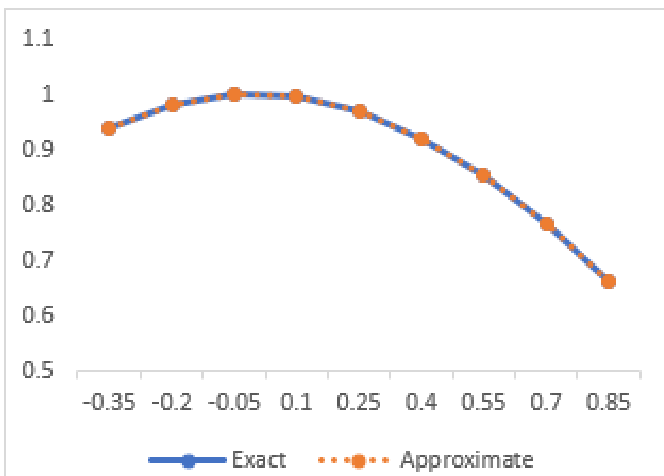

FIGURE 2. Comparison between the exact and approximate solution for example 4.4.

5. Conclusion

In this research, the system of general Fredholm Integro-differential (FID) equations was solved using the Direct Computation Method (DCM). It is crucial to note that additional techniques should be used for systems with different or separate kernels. While any form of kernel can be utilised with the DCM to solve the system of general Fredholm Integro-differential (FID) equations. Additionally, we looked at whether the proposed FID equations had a unique solution and calculated the estimated inaccuracy of the suggested system. The outcomes demonstrate that DCM is very effective, practical, and adaptable to address an array of issues. Tables 1 and 2 demonstrate the consistency between the exact answer and the numerical result produced using DCM.

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