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BOUNDS OF NON-SYMMETRIC FUZZY DIVERGENCE MEASURE VIA FUZZY KULLBACK- LEIBLER DIVERGENCE MEASURE

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ABSTRACT. Information divergence measure and their bounds plays important role in the literature of information theory.in this research paper we will present a new non-symmetric fuzzy divergence measure and studied of upper and lower bounds of the non-symmetric fuzzy divergence measure in terms of Kullback-Leibler divergence measure .obtained a numerical bounds of fuzzy new divergence measure.

KEYWORDS: Fuzzy Csiszar Divergence Measure ,Kullback-Leibler Divergence Measure, fuzzy Information Inequalities.

1. Introduction

Fuzzy set theory is the tool apply on the various processing that have multiple possibility of truth value. Fuzzy set is an application which can solve problems by the all possible information and gives the best possible decision. Fuzzy set is a extension form of the simple logic which have truth value of one or zero.
Fuzzy set have many application in the field of scientific research and also have fields in theoretical and practical from the engineering field to art and humanities ,computer science to health science etc.

Definition 1.1 In 1961 csiszar introduced a generalized measure is known as csiszar divergence measure. Let $\{P = p_1, p_2, \dots\dots\dots.. p_n\}, P_i \geq 0$ and $\sum_{i=1}^{n} P_i = 1$ is the set of discrete probability distribution.

$$C_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right)$$

Here the function f defined as $f : R^+ \to R^+$ and the function f is convex function. The function normalized also $f(1) = 0$

Definition 1.2 In 1951kullback-Leibler introduced a divergence measure is given as
If consider a function $f(e) = -\log(e)$ then

$$C_f(P,Q) = K(Q,P) = \sum_{i=1}^{n} q_i \log \frac{q_i}{q_i}$$

and also consider a function $f(e) = e\log(e)$ then kullback-Leibler divergence measure

$$C_f(P,Q) = K(P,Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}$$

Definition 1.3 (Fuzzy set) fuzzy set theory was invented by LotfiZaden in 1965 in the article of "Information and Control". Let Y be a universe of discourse and y be the arbitrary elements of Y. the fuzzy set C defined on Y is the collection of order pairs.

$$C = \left\{\left\langle Y, \delta_C(y) / y \in Y \right\rangle\right\}$$

Here $\delta_C(y_i): Y \to [0,1]$ known as membership function of C.

Let us consider the two fuzzy sets C and D in $Y = \{y_1, y_2 ....y_n\}$ and $\delta_C(y_i), \delta_D(y_i), i = 1,2,3.......n$ are the membership values respectively.

- **Fuzzy Csiszar divergence measure**

$$C_f(C,D) = \sum_{i=1}^{n}\left[\delta_D(y_i) f\left(\frac{\delta_C(y_i)}{\delta_D(y_i)}\right) + (1-\delta_D(y_i)) f\left(\frac{1-\delta_C(y_i)}{1-\delta_D(y_i)}\right)\right]$$

- **Fuzzy Kullback-Leibler divergence measure**

$$K(C,D) = \sum_{i=1}^{n}\left[\delta_C(y_i)\log\frac{\delta_C(y_i)}{\delta_D(y_i)} + (1-\delta_C(y_i))\log\frac{1-\delta_C(y_i)}{1-\delta_D(y_i)}\right]$$

And

$$K(D,C) = \sum_{i=1}^{n}\left[\delta_D(y_i)\log\frac{\delta_D(y_i)}{\delta_C(y_i)} + (1-\delta_D(y_i))\log\frac{1-\delta_D(y_i)}{1-\delta_C(y_i)}\right]$$

2. Fuzzy Information Inequalities of Fuzzy Csiszar f-divergence measure

In this section we will obtain the new fuzzy inequalities related with the fuzzy csiszar divergence measure.

Theorem 2.1 let us consider a function $f:(0,\infty) \to R$ is normalized i.e. $f(1)=0$ and satisfies the following assumptions.

i. Function f is twice differentiable on the interval (h,H) where $0 < h \le 1 \le H < \infty$

ii. There exist a real number j,J s. t. $j < J$

$j \le e^{2-a} f''(e) \le J, \forall e \in (h,H), a \in R$ (Set of real number) (2.1)

If $C,D \in Y_i$ are two fuzzy sets then satisfying the following inequality

$$0 < h \le \frac{\delta_C(y_i)}{\delta_D(y_i)} \le H < \infty \forall i \in N \text{ (Set of real number)} \quad (2.2)$$

Then the inequality we have

$jK_a(C,D) \le I_f(C,D) \le JK_a(C,D)$ (2.3)

Now we take $a = 0$ and $a = 1$ then the inequality (2.1) proves

Theorem 2.2let us consider a function $f:(0,\infty) \to R$ is normalized i.e. $f(1)=0$ and satisfies the following assumptions.

i. Function f is twice differentiable on the interval (h,H) where $0 < h \le 1 \le H < \infty$

ii. There exist a real number j,J s. t. $j < J$

$j \le e^2 f''(e) \le J \quad \forall e \in (h,H)$ (2.4)

$j \le ef''(e) \le J \quad e \in (h,H)$ (2.5)

If $\delta_C(y_i)$ and $\delta_D(y_i) \in Y_i$ are two fuzzy sets. Satisfying the following assumptions

$$0 < h \le \frac{\delta_C(y_i)}{\delta_D(y_i)} \le H < \infty \forall i \in N \text{ (set of real number)}$$

Now we takes the values of a, $a = 0$ and $a = 1$ then

$jK(D,C) \le I_f(C,D) \le JK(D,C)$ (2.6)

$jK(C,D) \le I_f(C,D) \le JK(C,D)$ (2.7)

3. Non-symmetric fuzzy-divergence measure

In this section we will introduce a new fuzzy information divergence measure which is the category of csiszar f-divergence measure.

Let us consider a function $f:(0,\infty) \to R$

$$f(e) = \frac{(e^2-1)^2}{e} \quad (3.1)$$

$$f'(e) = \frac{3e^4 - 2e^2 - 1}{e^2}$$

$$f''(e) = \frac{6e^4 + 2}{e^3} > 0 \qquad \forall e > 0 \quad (3.2)$$

The function $f(e)$ is convex from the equation (3.2) and $f(1) = 0$ i.e. normalized. the fuzzy f-divergence measure corresponding to the function (3.1).

$$I_f(C,D) = \sum_{i=1}^{n} \left[\frac{\left(\delta_C^2(y_i) + \delta_D^2(y_i)\right)^2}{\delta_D^2(y_i)\delta_C(y_i)} + \frac{(\delta_C^2(y_i) - \delta_D^2(y_i) + 2\left(\delta_D(y_i) - \delta_C(y_i)\right)^2}{\left(1-\delta_D(y_i)\right)^2\left(1-\delta_C(y_i)\right)} \right]$$

$$= 4\sum_{i=1}^{n} \left[\left(1 \Big/ \frac{2\delta_C(y_i)\delta_D(y_i)}{\delta_C(y_i)+\delta_D(y_i)} \right) \left(\frac{\delta_C(y_i)+\delta_D(y_i)}{2} \right) \left(\frac{\delta_C(y_i)-\delta_D(y_i)}{2} \right) + \left(1 \Big/ \frac{2\left(1-\delta_C(y_i)\right)\left(1-\delta_D(y_i)\right)}{2-\delta_C(y_i)-\delta_D(y_i)} \right) \left(\frac{2-\delta_C(y_i)-\delta_D(y_i)}{2} \right) \left(\frac{\delta_D(y_i)-\delta_C(y_i)}{1-\delta_D(y_i)} \right) \right]$$

$$= N(C,D)$$

Here $N(C,D)$ is the combination of fuzzy Harmonic, fuzzy Arithmetic and Chi-square divergence f-measure.

Now if we will check the convexity of the $N(C,D)$ then we get the result measure $N(C,D)$ also convex and normalize $f(1) = 0$ and have non-negative slope.

4. Bounds of the Fuzzy Kullback-Leibler divergence measure

In this section we established some special cases of the theorem (2.1) by the use of N(C,D).

Result 4.01 Let $C,D \in Y_i$ and $a = 0$ then $\exists h, H$ s. t. $h < H$ and

$$0 < h \le \frac{\delta_C(y_i)}{\delta_D(y_i)} \le H < \infty \forall i \in N \text{ (Set of real number)}$$

i. If $h \in (0, 0.6)$ then

$$4.618K(C,D) \le \max\left\{\frac{6h^2+2}{h}, \frac{6H^2+2}{H}\right\}K(C,D) \quad (4.1)$$

ii. $$\frac{6H^2+2}{H}K(C,D) \le N(C,D) \le \frac{6h^2+2}{h}K(C,D) \quad (4.2)$$

Proof: from the equations (3.2),(3.3) and (4.1) then we gets

$$s(e) = e^2 f''(e) = \frac{6e^4+2}{e}, \quad \forall e > 0$$

Then we have $s'(e) = 18e^2 - \frac{2}{e^2} = 0$

$s'(e) = 0$ then $s_0 = 0.6$

$$s''(e) = 36e + \frac{4}{e^3}$$

And $s''(0.6) = 36(0.6) + \frac{4}{(0.6)^3} = 40.11$ (positive)

This result shows that the function $s(e)$ has minimum real value at $e_0 = 0.6$ and $\min s(e_0) = j$.

now we have two constants

i. If $0 < h \le 0.6$, then

$$j = \inf_{e \in [h,H]} s(e) = s(e_0) = 4.618$$

$$J = \sup_{e \in [h.H]} s(e) = \max\left\{\frac{6h^2+2}{h}, \frac{6H^2+2}{H}\right\} \quad (4.3)$$

ii. If $0.6 < h < \infty$, then

$$j = \inf_{e \in [h,H]} s(e) = \frac{6h^4+2}{h}$$

$$j = \sup_{e \in [h,H]} s(e) = \frac{6H^4+2}{H} \quad (4.4)$$

Result 4.02 Let $C, D \in Y_i$ and $a = 1$ then there exist h, H s.t. $h < H$ and

$$0 < h \le \frac{\delta_C(y_i)}{\delta_D(y_i)} \le H < \infty \forall i \in N \text{ (Set of real number)}$$

i. If $0 < h \le 0.76$, then

$$6.92K(C,D) \le N(C,D) \le \max\left[\frac{6h^4+2}{h^2}, \frac{6H^4+2}{H^2}\right] K(C,D) \quad (4.5)$$

ii. If $0.76 < h < \infty$ then

$$\frac{6e^4+2}{e^2} K(C,D) \le N(C,D) \le \frac{6H^2+2}{H^2} K(C,D) \quad (4.6)$$

Proof: from the equation (3.1) ,(3.2) and (4.2) then we gets

$$s(e) = ef''(e) = \frac{6e^4+2}{e^2}, \quad \forall e > 0$$

Then we have $s'(e) = 12e - \frac{4}{e^3} = 0$

$s'(e) = 0$ gives to $\left(\frac{1}{3}\right)^{1/4} \approx 0.76$

$$s''(e) = 12 + \frac{12}{e^3},$$

And $s''(0.76) =$ (positive)

Which shows that the function $s(e)$ has minimum real values at $e_0 = \left(\frac{1}{3}\right)^{1/4} \approx 0.76$

Here we have two conditions:

i. If $0 < h \le 0.76$ then

$$j = \inf_{e \in [h,H]} s(e) = s(e_0) = 4\sqrt{3} \quad (4.7)$$

$$J = \sup_{e \in [h,H]} = \max\left[\frac{6h^4+2}{h^2}, \frac{6H^4+2}{H^2}\right] \quad (4.8)$$

ii. If $0.76 < h < \infty$ then

$$j = \inf_{e \in [h,H]} s(e) = \frac{6h^4 + 2}{h^2}$$

$$J = \inf_{e \in [h,H]} s(e) = \frac{6H^4 + 2}{H^2} \quad (4.9)$$

Equations (2.6) and (2.7) of theorem (2.2) and using the equation (3.2),(4.7) and (4.9) gives the results (4.5) and (4.6).

5. Special case:

Let $\delta_C(y_i)$ is the fuzzy set and $i = 9$ and $c \in [0, 0.5]$ and also let $\delta_D(y_i)$ another fuzzy set $i = 9$ and $D \in [0.5, 1]$ now discuss the following table.

y_i	1	2	3	4	5	6	7	8	9
$\delta_C(y_i)$	0.004	0.004	0.109	0.219	0.274	0.219	0.109	0.051	0.004
$\delta_D(y_i)$	0.005	0.005	0.104	0.220	0.282	0.220	0.104	0.050	0.005
$\delta_C(y_i)/\delta_D(y_i)$	0.774	0.774	1.0503	0.997	0.968	0.997	1.0503	1.042	0.774
N(C,D)	0.00026	0.00026	0.00018	0.0009	0.00011	0.00018	0.00026	0.00026	0.0008

Table 5.1

Here h = 0.7 and H = 1.05 is the lower and upper bounds, now we will discuss about numerical bounds of fuzzy information divergence measure in terms of Kullback-Leibler divergence measure using the equations (4.1),(4.2),(4.5) and (4.6).

i. If $h \in (1, 0.60)$ then

$$4.618K(C,D) \approx 4.61K(C,D) \le N(C,D) \le \max\left\{\frac{6h^2+2}{h}, \frac{6H^2+2}{H}\right\} K(C,D)$$

$$4.618K(C,D) \approx 4.61K(C,D) \le N(C,D) \le \max\left\{\frac{6(0.77)^2+2}{(0.77)}, \frac{6(1.051)^2+2}{(1.051)}\right\} K(C,D)$$

$$4.618K(C,D) \approx 4.61K(C,D) \le N(C,D) \le \max\{5.336, 8.850\} K(C,D)$$

$$4.618K(C,D) \approx 4.61K(C,D) \le N(C,D) \le 8.850K(C,D)$$

ii. If $h \in (0.6, \infty)$

$$\frac{6H^4+2}{H}K(C,D) \le N(C,D) \le \frac{6h^4+2}{h}K(C,D)$$

$$[8.850]K(C,D) \le N(C,D) \le [5.336]K(C,D)$$

iii. If $0 < h \le 0.76$,then

$$6.92K(C,D) \le N(C,D) \le \max\left[\frac{6h^4+2}{h^2}, \frac{6H^4+2}{H^2}\right]K(C,D)$$

$$6.92K(C,D) \le N(C,D) \le \max\left[\frac{6(0.77)^4+2}{(0.77)^2}, \frac{6(1.05)^4+2}{(1.05)^2}\right]K(C,D)$$

$$6.92K(C,D) \le N(C,D) \le \max[6.930, 8.4290]K(C,D)$$

$$6.92K(C,D) \le N(C,D) \le [8.4290]K(C,D)$$

iv. If $0.76 \le h < \infty$, then

$$\frac{6h^4+2}{h^2}K(C,D) \le N(C,D) \le \frac{6H^4+2}{H^2}K(C,D)$$

$$[6.930]K(C,D) \le N(C,D) \le [8.4290]K(C,D)$$

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