Journal of Mathematical Control Science and Applications
Vol. 7 No. 1 (January-June 2021)

Received: 01st November 2020   Revised: 15th December 2020   Accepted: 10th January 2021

A NOTE ON TYPE 2 POLYEXPONENTIAL EULER POLYNOMIALS AND NUMBERS

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Abstract. In this paper, we construct the type 2 polyexponential Euler polynomials and numbers, are called the type 2 polyexponential Euler polynomials and numbers by using polyexponential functions and derive several properties on the type 2 polyexponential Euler polynomials and numbers. Then, we introduce type 2 unipoly-Euler polynomials by using polyexponential functions and investigate some properties of them. Furthermore, we derive some new explicit expressions and identities of type 2 unipoly-Euler polynomials and related to special numbers and polynomials.

1. Introduction

Special polynomials have their origin in the solution of the differential equations (or partial differential equations) under some conditions. Special polynomials can be defined in a various ways such as by generating functions, by recurrence relations, by p-adic integrals in the sense of the fermionic and bosonic, by degenerate versions, etc.

The aim of this paper is to study the type 2 Euler polynomials and numbers by using polyexponential functions, namely type 2 polyexponential Euler polynomials and numbers, in the spirit of [1]. They were recently introduced by Kim-Kim [13]. We derive their explicit expressions and some identities involving them. Further, we introduce the type 2 unipoly-Euler polynomials and numbers. Again, we deduce their explicit expressions and some identities related to them.

As is well known, the type 2 Bernoulli polynomials $B_n(x), (n \geq 0)$ and the type 2 Euler polynomials $E_n(x), (n \geq 0)$ are, respectively, defined by

$$e^{xt}\frac{t}{2}\csc h\frac{t}{2} = \frac{t}{e^{\frac{1}{2}}(t) - e^{-\frac{1}{2}}(t)}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1.1}$$

and

$$e^{xt}\sec h\frac{t}{2} = \frac{2}{e^{\frac{1}{2}}(t) + e^{-\frac{1}{2}}(t)}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \text{(see [3, 14])}. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 33C45, 11B73, 11B83, 05A19.

Key words and phrases. polyexponential functions, type 2 Euler polynomials, unipoly function.

For $x = 0$, $B_n(0) := B_n$ (or $E_n(0) := E_n$) are called the type 2 Bernoulli (or type 2 Euler) numbers.

For $k \in \mathbb{Z}$, the polylogaritm function is defined by

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, (|\, x \,| < 1), (\text{see [5]}). \tag{1.3}$$

Note that

$$\mathrm{Li}_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x). \tag{1.4}$$

For $k \in \mathbb{Z}$, Kim-Kim considered the polyexponential function, as an inverse to the polylogarithm function to be

$$\mathrm{Ei}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!n^k}, (\text{see [13]}). \tag{1.5}$$

It is notice that

$$e(x, 1/k) = \frac{1}{x}\mathrm{Ei}_k, \ \mathrm{Ei}_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1. \tag{1.6}$$

In [1], Dolgy-Jang introduced the poly-Genocchi polynomials arising from poly-exponential function as

$$\frac{2\mathrm{Ei}_k\left(\log(1+t)\right)}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x) \frac{t^n}{n!}, (k \in \mathbb{Z}). \tag{1.7}$$

In the case when $x = 0$, $G_n^{(k)}(0) = G_n^{(k)}$ are called the poly-Genocchi numbers. Note that $G_n^{(1)}(x) = G_n(x)$ are called the ordinary Genocchi polynomials.

Yoshinori [2] introduced the poly-Euler polynomials and numbers are defined by

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{t(e^t+1)} = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!}. \tag{1.8}$$

In the case when $x = 0$, $E_n^{(k)}(0) = E_n^{(k)}$ are called the poly-Euler numbers. In particular, for $k = 1$, $E_n^{(1)}(x) = E_n(x)$ are called the ordinary Euler numbers.

The Daehee polynomials are defined by

$$\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, (\text{see [12, 17, 19]}). \tag{1.9}$$

When $x = 0$, $D_n(0) = D_n$ are called the Daehee numbers.

For $n \geq 0$, the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \text{(see [4-8])}, \tag{1.10}$$

where $(x)_0 = 1$, and $(x)_n = x(x-1)\cdots(x-n+1), (n \geq 1)$. From (1.10), it is easily to see that

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \ (k \geq 0), \text{(see [9-12])}. \tag{1.11}$$

In the inverse expression to (1.10), the Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l. \tag{1.12}$$

From (1.12), we see that

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=l}^{\infty} S_2(n,l)\frac{t^n}{n!}, \text{(see [1-19])}. \tag{1.13}$$

2. Type 2 polyexponential-Euler polynomials and numbers

In this section, we introduce the type 2 poly-Euler polynomials and numbers employing the polyexponential functions and represent the usual type 2 Euler numbers (more precisely, the value of type 2 Euler polynomials at 1) when $k = 1$. At the same time, we give explicit expressions and identities involving those polynomials.

In view of (1.7) and using the polyexponential functions, we define the type 2 polyexponential Euler polynomials by

$$\frac{2\mathrm{Ei}_k\left(\log(1+t)\right)}{t\left(e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)\right)}e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!}, (k \in \mathbb{Z}). \tag{2.1}$$

In the case when $x = 0$, $E_n^{(k)}(0) := E_n^{(k)}$ are called the type 2 polyexponential Euler numbers.

For $k = 1$, by using (1.2) and (2.1), we see that

$$\frac{2\mathrm{Ei}_1\left(\log(1+t)\right)}{t\left(e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)\right)}e^{xt} = \frac{2}{e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{2.2}$$

where $E_n(x)$ are called the type 2 Euler polynomials.

By (2.1) and (2.2), we get

$$E_n^{(1)}(x) = E_n(x), (n \geq 0).$$

Theorem 2.1. *For $k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$E_n^{(k)} = \sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} E_{n-l}.$$

Proof. From (2.1), we note that

$$\frac{2\mathrm{Ei}_k\left(\log(1+t)\right)}{t\left(e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)\right)} = \frac{2}{e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m+1}^{\infty} S_1(l, m+1) \frac{t^l}{l!}$$

$$= \frac{2}{e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)} \sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{l=m}^{\infty} \frac{S_1(l+1, m+1)}{l+1} \frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} \frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{1}{(m+1)^{k-1}} \frac{S_1(l+1, m+1)}{l+1} E_{n-l} \right) \frac{t^n}{n!}. \tag{2.3}$$

Therefore, by (2.1) and (2.3), we complete the proof. □

Corollary 2.1. For $n \geq 0$, we have

$$E_n = \sum_{l=0}^{n} \binom{n}{l} \sum_{m=0}^{l} \frac{S_1(l+1, m+1)}{l+1} E_{n-l}.$$

Theorem 2.2. *For $k \in \mathbb{Z}$ and $n \geq 0$, we have*

$$E_n^{(k)}(x) = \sum_{m=0}^{n} \binom{n}{m} E_{n-m}^{(k)} x^m = \sum_{m=0}^{n} \binom{n}{m} E_m^{(k)} x^{n-m}.$$

Proof. From (2.1), we note that

$$\sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!} = \frac{2\mathrm{Ei}_k\left(\log(1+t)\right)}{t\left(e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)\right)} e^{xt}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} E_{n-m}^{(k)} x^m \right) \frac{t^n}{n!}$$

$$\text{or} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} E_m^{(k)} x^{n-m} \right) \frac{t^n}{n!}. \tag{2.4}$$

Therefore, by (2.1) and (2.4), we obtain the result. □

Theorem 2.3. *Let $n \geq 0$, and $k \in \mathbb{Z}$, we have*

$$\frac{d}{dx} E_n^{(k)}(x) = n E_{n-m}^{(k)}(x).$$

Proof. By using Theorem 2.2, we observe that

$$\frac{d}{dx}E_n^{(k)}(x) = \frac{d}{dx}\sum_{m=0}^{n}\binom{n}{m}E_{n-m}^{(k)}x^m$$

$$= \sum_{m=1}^{n}\binom{n}{m}E_{n-m}^{(k)}x^{m-1} = \sum_{m=0}^{n-1}\binom{n}{m+1}E_{n-m}^{(k)}(m+1)x^m$$

$$= n\sum_{m=0}^{n-1}\frac{(n-1)!}{m!(n-m-1)}E_{n-m}^{(k)}x^m = nE_{n-m}^{(k)}(x). \qquad (2.5))$$

Therefore, by comparing on both sides of t^n in (2.5), we get the result. □

For the next theorem, we need the following well known identity, (see [1, 17])

$$\left(\frac{t}{\log(1+t)}\right)^r (1+t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n-r+1)}(x)\frac{t^n}{n!},$$

where $B_n^{(n)}$ is the Bernoulli numbers of order n at $x = 0$.

Theorem 2.4. *For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have*

$$E_n^{(k)} = \sum_{m=0}^{n}\binom{n}{m}\sum_{m_1+\cdots+m_{k-1}=m}\binom{m}{m_1,\cdots,m_{k-1}}E_{n-m}$$

$$\times\frac{B_{m_1}^{(m_1)}(0)}{m_1+1}\frac{B_{m_2}^{(m_2)}(0)}{m_1+m_2+1}\cdots\frac{B_{m_{k-1}}^{(m_{k-1})}(0)}{m_1+\cdots+m_{k-1}+1}.$$

Proof. First, we note that

$$\frac{d}{dx}\mathrm{Ei}_k\left(\log(1+x)\right) = \frac{d}{dx}\sum_{n=1}^{\infty}\frac{(\log(1+x))^n}{n^k(n-1)!}$$

$$= \frac{1}{(1+x)\log(1+x)}\sum_{n=1}^{\infty}\frac{(\log(1+x))^n}{n^{k-1}(n-1)!} = \frac{1}{(1+x)\log(1+x)}\mathrm{Ei}_{k-1}(\log(1+x)). \qquad (2.6)$$

Thus, by (2.6), for $k \geq 2$, we get

$$\mathrm{Ei}_k(\log(1+x)) = \int_0^x \frac{1}{(1+t)\log(1+t)}\mathrm{Ei}_{k-1}(\log(1+t))dt$$

$$= \int_0^x \underbrace{\frac{1}{(1+t)\log(1+t)}\int_0^t\cdots\frac{1}{(1+t)\log(1+t)}\int_0^t\frac{1}{(1+t)\log(1+t)}}_{(k-2)-\text{times}}dt\cdots dt$$

$$\times\mathrm{Ei}_1(\log(1+t))dt\cdots dt$$

$$= \int_0^x \underbrace{\frac{1}{(1+t)\log(1+t)}\int_0^t\cdots\frac{1}{(1+t)\log(1+t)}\int_0^t\frac{1}{(1+t)\log(1+t)}}_{(k-2)-\text{times}}dt\cdots dt. \qquad (2.7)$$

From (2.1) and (2.7), we get

$$\sum_{n=0}^{\infty} E_n^{(k)} \frac{x^n}{n!} = \frac{2\mathrm{Ei}_k\left(\log(1+x)\right)}{x\left(e^{\frac{1}{2}}(x)+e^{-\frac{1}{2}}(x)\right)} = \frac{2}{x\left(e^{\frac{1}{2}}(x)+e^{-\frac{1}{2}}(x)\right)}$$

$$\times \int_0^x \underbrace{\frac{1}{(1+t)\log(1+t)} \int_0^t \frac{1}{(1+t)\log(1+t)} \cdots \int_0^t}_{(k-2)-\text{times}} \frac{t}{(1+t)\log(1+t)} dt \cdots dt. \tag{2.8}$$

$$= \frac{2x}{x\left(e^{\frac{1}{2}}(x)+e^{-\frac{1}{2}}(x)\right)} \sum_{m=0}^{\infty} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1,\cdots,m_{k-1}}$$

$$\times \frac{B_{m_1}^{(m_1)}(0)}{m_1+1} \frac{B_{m_2}^{(m_2)}(0)}{m_1+m_2+1} \cdots \frac{B_{m_{k-1}}^{(m_{k-1})}(0)}{m_1+\cdots+m_{k-1}+1} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \sum_{m_1+\cdots+m_{k-1}=m} \binom{m}{m_1,\cdots,m_{k-1}} E_{n-m}$$

$$\times \frac{B_{m_1}^{(m_1)}(0)}{m_1+1} \frac{B_{m_2}^{(m_2)}(0)}{m_1+m_2+1} \cdots \frac{B_{m_{k-1}}^{(m_{k-1})}(0)}{m_1+\cdots+m_{k-1}+1} \frac{x^n}{n!}. \tag{2.9}$$

Therefore, by comparing the coefficients of t^n in (2.9), we arrive the desired result. □

Corollary 2.2. For $n \geq 0$, we have

$$E_n^{(2)} = \sum_{m=0}^{n} \binom{n}{m} \frac{B_{m+1}^{(m)}(0)}{m+1} E_{n-m}.$$

Theorem 2.5. *For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, we have*

$$n \sum_{m=1}^{n} \binom{n-1}{m} \frac{1}{2^{m-1}} \left[E_{n-1-m}^{(k)} + (-1)^m E_{n-1-m}^{(k)}\right] = \sum_{m=1}^{n} \frac{1}{m^{k-1}} S_1(n,m).$$

Proof. From (2.1), we note that

$$2\mathrm{Ei}_k\left(\log(1+t)\right) = t(e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)) \sum_{n=0}^{\infty} E_n^{(k)} \frac{t^n}{n!}$$

$$= t\left(\sum_{m=0}^{\infty} \frac{t^m}{2^m m!} + \sum_{m=0}^{\infty} \frac{(-1)^m t^m}{2^m m!}\right) \sum_{n=0}^{\infty} E_n^{(k)} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \frac{1}{2^m} \left[E_{n-m}^{(k)} + (-1)^m E_{n-m}^{(k)}\right]\right) \frac{t^{n+1}}{n!}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} \binom{n-1}{m} \frac{1}{2^m} \left[E_{n-1-m}^{(k)} + (-1)^m E_{n-1-m}^{(k)}\right]\right) \frac{t^n}{(n-1)!}. \tag{2.10}$$

On the other hand,

$$2\mathrm{Ei}_k\left(\log(1+t)\right) = 2\sum_{m=1}^{\infty}\frac{(\log(1+t))^m}{(m-1)!m^k}$$

$$= 2\sum_{m=1}^{\infty}\frac{(\log(1+t))^m}{(m-1)!m^k}\frac{m!}{m!} = \sum_{m=1}^{\infty}\frac{1}{m^{k-1}}\frac{(\log(1+t))^m}{m!}$$

$$= 2\sum_{m=}^{\infty}\frac{1}{m^{k-1}}\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!} = 2\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{1}{m^{k-1}}S_1(n,m)\right)\frac{t^n}{n!}. \qquad (2.11)$$

Therefore, by (2.10) and (2.11), we obtain the result. □

By (2.1), we have

$$\frac{2\mathrm{Ei}_k\left(\log(1+t)\right)}{t\left(e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)\right)} = \sum_{n=0}^{\infty}E_n^{(k)}\frac{t^n}{n!}, (k\in\mathbb{Z}) \qquad (2.12).$$

Theorem 2.6. *For $n\in\mathbb{N}$, we have*

$$\sum_{m=0}^{n}E_m^{(k)}S_2(n,m) = \sum_{j=0}^{n}\sum_{q_2=0}^{j}\sum_{q_1=0}^{q_2}\binom{j}{q_2}\binom{n}{j}E_{q_1}S_2(q_2,q_1)B_{j-q_2}\frac{1}{(n-j+1)^k}.$$

Proof. Replacing t by e^t-1 in (2.12), we get

$$\sum_{m=0}^{\infty}E_m^{(k)}\frac{(e^t-1)^m}{m!} = \frac{2\mathrm{Ei}_k(t)}{(e^t-1)\left(e^{\frac{1}{2}}(e^t-1)+e^{-\frac{1}{2}}(e^t-1)\right)}$$

$$\frac{2\mathrm{Ei}_k(t)}{e^{\frac{1}{2}}(e^t-1)+e^{-\frac{1}{2}}(e^t-1)} = \frac{2}{e^{\frac{1}{2}}(e^t-1)+e^{-\frac{1}{2}}(e^t-1)}\frac{t}{e^t-1}\frac{\mathrm{Ei}_k(t)}{t}$$

$$= \sum_{q_1=0}^{\infty}E_{q_1}\frac{1}{q_1!}(e^t-1)^{q_1}\sum_{j=0}^{\infty}B_j\frac{t^j}{j!}\sum_{n=0}^{\infty}\frac{t^n}{n!(n+1)^k}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{q_2=0}^{j}\sum_{q_1=0}^{q_2}\binom{j}{q_2}\binom{n}{j}E_{q_1}S_2(q_2,q_1)B_{j-q_2}\frac{1}{(n-j+1)^k}\right)\frac{t^n}{n!}. \qquad (2.13)$$

On the other hand,

$$\sum_{m=0}^{\infty}E_m^{(k)}\frac{(e^t-1)^m}{m!} = \sum_{m=0}^{\infty}E_m^{(k)}\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}E_m^{(k)}S_2(n,m)\right)\frac{t^n}{n!}. \qquad (2.14)$$

By comparing the coefficients of t^n in (2.13) and (2.14), we obtain the result. □

3. Type 2 unipoly-Euler polynomials and numbers

In this section, we introduce type 2 unipoly-Euler polynomials attached to p and derive several properties and explicit expressions of these polynomials.

Let p be any arithmetic function which is a real or complex valued function defined on the set of positive integers $\mathbb{N}$. Then Kim-Kim [13] defined the unipoly function attached to p by

$$u_k(x|p) = \sum_{n=1}^{\infty} \frac{p(n)}{n^k} x^n, (k \in \mathbb{Z}). \tag{3.1}$$

It is well known that

$$u_k(x|1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = \mathrm{Li}_k(x), (\text{see [5]}). \tag{3.2}$$

In view of (3.1), we define the type 2 unipoly-Euler polynomials attached to p by

$$\frac{2u_k\left(\log(1+t)|p\right)}{t\left(e^{\frac{1}{2}}(t) + e^{-\frac{1}{2}}(t)\right)} e^{xt} = \sum_{n=0}^{\infty} E_{n,p}^{(k)}(x) \frac{t^n}{n!}. \tag{3.3}$$

In the case when $x = 0$, $E_{n,p}^{(k)}(0) := E_{n,p}^{(k)}$ are called the type 2 unipoly-Euler numbers attached to p.

If we take $p(n) = \frac{1}{\Gamma(n)}$. Then, we have

$$\sum_{n=0}^{\infty} E_{n,\frac{1}{\Gamma}}^{(k)} \frac{t^n}{n!} = \frac{2}{t\left(e^{\frac{1}{2}}(t) + e^{-\frac{1}{2}}(t)\right)} u_k\left(\log(1+t)|\frac{1}{\Gamma}\right)$$

$$= \frac{2}{t\left(e^{\frac{1}{2}}(t) + e^{-\frac{1}{2}}(t)\right)} \sum_{m=1}^{\infty} \frac{(\log(1+t))^m}{m^k (m-1)!}$$

$$= \frac{2}{t\left(e^{\frac{1}{2}}(t) + e^{-\frac{1}{2}}(t)\right)} u_k\left(\log(1+t)\right) = \sum_{n=0}^{\infty} E_n^{(k)} \frac{t^n}{n!}. \tag{3.4}$$

Thus, by (3.4), we obtain

$$E_{n,\frac{1}{\Gamma}}^{(k)} = E_n^{(k)}, (n \geq 0). \tag{3.5}$$

Theorem 3.1. *For $n \geq 0$ and $k \in \mathbb{Z}$, we have*

$$E_{n,p}^{(k)} = \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \frac{p(m+1)(m+1)!}{(m+1)^k} \frac{S_1(l+1, m+1)}{l+1} E_{n-l}.$$

In particular,

$$E^{(k)}_{n,\frac{1}{\Gamma}} = \sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{E_{n-l}}{(m+1)^{k-1}}\frac{S_1(l+1,m+1)}{l+1}.$$

Proof. From (3.5), we have

$$\sum_{n=0}^{\infty}E^{(k)}_{n,\lambda}\frac{t^n}{n!} = \frac{2}{t\left(e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)\right)}\sum_{m=1}^{\infty}\frac{p(m)(\log(1+t))^m}{m^k}$$

$$= \frac{2}{t\left(e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)\right)}\sum_{m=0}^{\infty}\frac{p(m+1)(\log(1+t))^{m+1}}{(m+1)^k}$$

$$= \frac{2}{t\left(e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)\right)}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+1)^k}\sum_{l=m+1}^{\infty}S_1(l,m+1)\frac{t^l}{l!}$$

$$= \frac{2}{e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)}\sum_{m=0}^{\infty}\frac{p(m+1)(m+1)!}{(m+1)^k}\sum_{l=m}^{\infty}\frac{S_1(l+1,m+1)}{l+1}\frac{t^l}{l!}$$

$$= \left(\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=0}^{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}\right)\frac{t^l}{l!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{l}\binom{n}{l}\frac{p(m+1)(m+1)!}{(m+1)^k}\frac{S_1(l+1,m+1)}{l+1}E_{n-l}\right)\frac{t^n}{n!}.$$

On comparing the coefficients of t, we get the result. □

Theorem 3.2. *For $n \geq 0$ and $k \in \mathbb{Z}$, we have*

$$E^{(k)}_{n,p}(x) = \sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}(x)_m S_2(m,l)E^{(k)}_{n-l,p}.$$

Proof. Using equations (3.3) and (1.13), we obtain

$$\sum_{n=0}^{\infty}E^{(k)}_{n,p}(x)\frac{t^n}{n!} = \frac{2}{t\left(e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)\right)}u_k(\log(1+t)|p)e^{xt}$$

$$= \frac{2}{t\left(e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)\right)}u_k(\log(1+t)|p)\left(e^t-1+1\right)^x$$

$$= \frac{2u_k\left(\log(1+t)|p\right)}{t\left(e^{\frac{1}{2}}(t)+e^{-\frac{1}{2}}(t)\right)}\left(\sum_{m=0}^{\infty}\binom{x}{m}(e^t-1)^m\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\sum_{m=0}^{l}(x)_m S_2(m,l)E^{(k)}_{n-l,p}\right)\frac{t^n}{n!}. \tag{3.6}$$

Thus, we complete the proof of the theorem. □

Theorem 3.3. *For $n \geq 0$ and $k \in \mathbb{Z}$, we have*

$$E_{n,p}^{(k)} = \sum_{l=0}^{n} \sum_{m=0}^{l} \sum_{j=0}^{n-l} \binom{n-l}{j} \binom{n}{l} D_{n-j-l} E_j \frac{p(m+1)m!}{(m+1)^k} S_1(l,m).$$

Proof. Using equations (1.2), (1.9) and (3.5), we see that

$$\sum_{n=0}^{\infty} E_{n,\lambda}^{(k)} \frac{t^n}{n!} = \frac{2}{t\left(e^{\frac{1}{2}}(t) + e^{-\frac{1}{2}}(t)\right)} \sum_{m=1}^{\infty} \frac{p(m)(\log(1+t))^m}{m^k}$$

$$= \frac{2}{t\left(e^{\frac{1}{2}}(t) + e^{-\frac{1}{2}}(t)\right)} \sum_{m=0}^{\infty} \frac{p(m+1)(\log(1+t))^{m+1}}{(m+1)^k}$$

$$= \frac{2\log(1+t)}{t\left(e^{\frac{1}{2}}(t) + e^{-\frac{1}{2}}(t)\right)} \sum_{m=0}^{\infty} \frac{p(m+1)m!}{(m+1)^k} \sum_{l=m}^{\infty} S_1(l,m) \frac{t^l}{l!}$$

$$= \frac{2\log(1+t)}{t\left(e^{\frac{1}{2}}(t) + e^{-\frac{1}{2}}(t)\right)} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{p(m+1)m!}{(m+1)^k} S_1(l,m) \frac{t^l}{l!}$$

$$= \frac{\log(1+t)}{t} \frac{2}{e^{\frac{1}{2}}(t) + e^{-\frac{1}{2}}(t)} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{p(m+1)m!}{(m+1)^k} S_1(l,m) \frac{t^l}{l!}$$

$$= \left(\sum_{n=0}^{\infty} D_n \frac{t^n}{n!}\right) \left(\sum_{j=0}^{\infty} E_j \frac{t^j}{j!}\right) \left(\sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{p(m+1)m!}{(m+1)^k} S_1(l,m) \frac{t^l}{l!}\right).$$

On comparing the coefficients of t, we obtain the result. □

4. Conclusion

In this article, we introduced type 2 polyexponential Euler polynomials by arising from polyexponential functions and derived several properties of these polynomials. Specially, we obtained various expressions of type 2 polyexponential Euler polynomials in terms of Euler, Bernoulli polynomials and Stirling numbers of the first kind and second kind in Theorem 2.1 to 2.6. Finally, we considered type 2 unipoly-Euler polynomials attached to p by using polyexponential functions and investigated some identities of for those polynomials. Besides, we obtained some identities relating between type 2 Euler numbers, Daehee numbers and Stirling numbers of the first and second kind in Theorem 3.1 to 3.3.

Author Contributions: All authors contributed equally to the manuscript and typed, read, and approved final manuscript.

Funding: No.

Acknowledgements: No.

Conflict of Interest: The authors declare no conflict of interest.

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