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A REVIEW ON LEAST-SQUARES METHODS FOR PDE CONSTRAINED OPTIMAL CONTROL PROBLEMS

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Abstract. In this article, a survey of existing least-squares methods for optimal control problems with elliptic partial differential equations (pdes) such as div-curl systems, Stokes and Navier-Stokes equations as constraints has been presented. The goal of this article is to discuss advantages and challenges of least-squares based methods for solving pde constrained optimal control problems.

1. Introduction

Optimal control problems with partial differential equations (pdes) as constraints arise in many scientific applications such as modelling air flow around the body, optimize flight trajectory, maximize fuel efficiency, minimize production cost, optimal shape designing and so on. Here we refer by optimal control problems as optimization problems consisting of state variables, control parameters and pdes as constraints known as state equations. Existing approaches broadly can be divided into two categories as Lagrange multiplier technique and penalization technique. Lagrange multiplier is a widely used technique for solving optimization problems. Hence its use has been extended to pde constrained optimal control problems. In contrast to Lagrange multiplier technique, penalization approach uses a penalty parameter ϵ and constraints are added to the least-squares functional with help of this parameter. A third approach has been introduced in [13] which involves bilevel optimization approach.

Least-squares based methods have been extensively used for solving partial differential equations ([2],[10],[11],[12],[15],[16],[17],[18],[19],[21],[22],[23],[29],[30],[31],[32],[33],[34],[35],[36],[37]). In specific, while dealing with systems of differential equations these methods avoid inf-sup stability conditions making both theory and computation easier compared to Galerkin/ mixed formulation. However this advantage is not straightforward while implementing least-squares principle to pde constrained optimal control problems. Least-squares based methods for solving optimal control problems face many challenges such as indefiniteness and presence of negative norm in the least-squares functional. Here an overview of existing approaches has been presented.

Function spaces and associated norms are introduced in section 2. In section 3, we discuss existing results on div-curl system constrained optimal control problems. In section 4 and 5 we discuss Stokes and Navier-Stokes equations constrained optimal control problems respectively. Available error estimates are discussed in section 6. Concluding remarks have been presented in section 7.

2. Notations and prelimnaries

Here we introduce the functional spaces and associated norms that are used to analyse least-squares formulations for pde constrained optimal control problems. Let

$$H^m(\Omega) = \left\{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \;\text{ for }\; 0 \le \alpha \le m\right\}.$$

Here $D^\alpha u$ refers to distributional partial derivative and α is the multi-index, $|\alpha| = \sum_i \alpha_i$, equipped with norm

$$\|u\|_m = \left\{ \sum_{0\le|\alpha|\le m} \|D^\alpha u\|^2_{L^2(\Omega)} \right\}^{\frac{1}{2}}.$$

We define,

$$L^2_0(\Omega) = \left\{u \in L^2(\Omega) : \int_\Omega u \; d\Omega = 0\right\},$$
$$H^1_0(\Omega) = \left\{u \in H^1(\Omega) : u = 0 \;\text{ on }\; \Gamma\right\},$$
$$H^1_{\boldsymbol{n}}(\Omega) = \left\{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{u}.\boldsymbol{n}|_\Gamma = 0 \;\text{ on }\; \Gamma\right\}.$$

Vectors are denoted by bold letters i.e. $\boldsymbol{u} = (u_1, u_2)$ or (u_1, u_2, u_3) depending on dimension of Ω. Similarly, $\boldsymbol{H}^m(\Omega) = H^m(\Omega) \times H^m(\Omega)$ for $\Omega \subset \mathbb{R}^2$ and $\boldsymbol{H}^m(\Omega) = H^m(\Omega) \times H^m(\Omega) \times H^m(\Omega)$ for $\Omega \subset \mathbb{R}^3$.

Next, we discuss briefly discuss Agmon-Douglas-Nirenberg (ADN) theory, that was introduced in [1] and has been widely used for regularity estimates of general elliptic systems. Since optimality systems arising in case of pde constrained optimal control problems can be fit into ADN system, it is beneficial to have an idea about assumptions and conclusions involved in ADN theory.

2.1. Agmon-Douglas-Nirenberg (ADN) theory. Consider the system

$$\begin{cases} \mathcal{L}U = F \;\text{ in }\; \Omega, \\ \mathcal{R}U = G \;\text{ on }\; \Gamma. \end{cases}$$

Here $\mathcal{L}$ corresponds to the differential operator and $\mathcal{R}$ corresponds to the boundary operator and Γ denotes the boundary of Ω. The estimates are based on two types of conditions named as supplimentary and complementing conditions. Here we are quoting a simpler presentation of ADN theory from [14].

Notations for supplementary condition :

- $\mathcal{L} = \{\mathcal{L}_{ij}\}, i, j = 1, \cdots, N$ be the elliptic differential operator.
- $\mathcal{R} = \{\mathcal{R}_{lj}\}, l = 1, \cdots, m, j = 1, \cdots, N$ be the boundary operator.
- $\{s_i : s_i \le 0\}$ be the indices assigned to equations.
- $\{t_j : t_j \ge 0\}$ be the indices assigned to unknowns.
- $\{r_l : r_l \le 0\}$ be the indices assigned to boundary conditions.
- $\mathcal{L}^p$ be the principal part of $\mathcal{L}$ having terms of $\mathcal{L}_{ij}$ with orders being equal to $s_i + t_j$.
- $\mathcal{R}^p$ be the principal part of $\mathcal{R}$ having terms of $\mathcal{R}_{lj}$ with orders being equal to $r_l + t_j$.

Supplementary conditions on $\mathcal{L}$ are :

- det $\mathcal{L}^p(\boldsymbol{\xi})$ is of even degree in $\boldsymbol{\xi}$.
- For each set of linearly independent real vectors $\boldsymbol{\xi}, \boldsymbol{\xi}^{'}$ the polynomial det $\mathcal{L}^p(\boldsymbol{\xi} + \tau\boldsymbol{\xi}^{'})$ in the complex variable τ has exactly m roots with positive imaginary part.
- Compatibility of a particular set of boundary conditions with given system of differential equations.

Notations for Complimenting condition are :

- Let $\tau_k^+(\boldsymbol{\xi})$ be the m roots of det $\mathcal{L}^p(\boldsymbol{\xi} + \tau\boldsymbol{\xi}^{'})$ with positive imaginary part.
- $M^+(\boldsymbol{\xi}, \tau) = \prod_{k=1}^{m}(\tau - \tau_k^+(\boldsymbol{\xi}))$.
- For any point $P \in \Gamma$, $\boldsymbol{n}$ be the unit outward normal vector at P.
- $\mathcal{L}^{'}$ denotes adjoint of $\mathcal{L}$.
- For $\boldsymbol{\xi} \neq 0$, being tangent to Γ at P , $M^+(\boldsymbol{\xi}, \tau)$ and the elements of the matrix $\sum_{j=1}^{N} \mathcal{R}_{lj}^p(\boldsymbol{\xi} + \tau\boldsymbol{n})\mathcal{L}_{jk}^{'}(\boldsymbol{\xi} + \tau\boldsymbol{n})$ are polynomials in τ.

Complimenting condition on $\mathcal{L}$ and $\mathcal{R}$ is :
The differential and boundary operators $\mathcal{L}$ and $\mathcal{R}$ (respectively) satisfy the complimenting condition if the following condition is satisfied :

$$\sum_{l=1}^{m} c_l \sum_{j=1}^{N} \mathcal{R}_{lj}^p \mathcal{L}_{jk}^{'} \equiv 0(\mathrm{mod} M^+) \ \text{ iff } \ c_l = 0, \forall l.$$

Now we state the regularity result for general elliptic problems based on ADN theory.

Theorem 2.1. $\mathcal{L}U = F$ be uniformly elliptic and satisfies supplementary conditions. The boundary operator satisfies complimenting condition. Let $U \in \prod_{j=1}^{N} H^{q+t_j}(\Omega)$, $F \in \prod_{i=1}^{N} H^{q-s_i}(\Omega), G \in \prod_{l=1}^{m} H^{q-r_l-\frac{1}{2}}(\Gamma)$. Then $\exists\, C > 0$ such that

$$\sum_{j=1}^{N} \|u_j\|_{q+t_j,\Omega} \leq C\bigg(\sum_{i=1}^{N} \|F_i\|_{q-s_i,\Omega} + \sum_{l=1}^{m} \|G_l\|_{q-r_l-\frac{1}{2},\Gamma} + \sum_{j=1}^{N} \|u_j\|_{0,\Omega} \bigg).$$

2.2. Existence, uniqueness and regularity results for first order systems. Here we discuss the solvability and regularity results for general first order elliptic systems based on results from Agmon-Douglas-Nirenberg (ADN) theory [1] as they are crucial in analysing least-sqaures based approach for optimal control problems. Optimality systems can be represented by first order elliptic systems and hence ADN theory based results can be used to obtain regularity estimates.

Consider general first order elliptic system [25]

$$\mathcal{L}\boldsymbol{u} = A\boldsymbol{u}_x + B\boldsymbol{u}_y + C\boldsymbol{u}_z + D\boldsymbol{u} = \boldsymbol{f} \ \text{ in } \ \Omega \tag{2.1}$$

$$\mathcal{R}\boldsymbol{u} = \boldsymbol{g} \ \text{ on } \ \Gamma \tag{2.2}$$

and

$$B_j(\boldsymbol{u}) = a_j, j = 1, 2, ..., N, \tag{2.3}$$

where N is the nullity of the system, on $\Omega \subset \mathbb{R}^3$, a bounded simply connected domain with piecewise continuously differentiable boundary and $A, B, C, D \in \mathbb{R}^{2n\times 2n}$

matrices with C^∞ entries. $\mathcal{R}$ is a given $n\times 2n$ matrix and $B_j:\mathbb{R}^{2n}\to\mathbb{R}$ are linear functionals which span the null space. Differential and boundary conditions satisfy the complementing and supplimentary conditions ([1],[6]). Then following existence, uniqueness and regularity result can be found in [25].

Theorem 2.2. Let $A, B, C, D, \mathcal{R}$ have smooth entries and let $\Omega\in\mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Let $(\mathcal{L},\mathcal{R})$ be regular elliptic and let $B_j(.), j=1,\cdots,N$ be independent bounded linear functionals on $[H^s(\Omega)]^{2n}$. For $s\geq 1$, $\exists$ a constant $C(\Omega,n)$ such that

$$\frac{1}{C}\|\boldsymbol{v}\|_s \leq \|\mathcal{L}\boldsymbol{v}\|_{s-1}+\|R\boldsymbol{v}\|_{s-\frac{1}{2},\partial\Omega}+\sum_{j=1}^{N}|B_j(\boldsymbol{v})|\leq C\|\boldsymbol{v}\|_s, \forall\boldsymbol{v}\in[H^s(\Omega)]^{2n} \quad (2.4)$$

holds. If $\boldsymbol{f}\in[H^{s-1}(\Omega)]^{2n}$ and $\boldsymbol{g}\in[H^{s-\frac{1}{2}}(\partial\Omega)]^n$, for $s\geq 1$, ($\boldsymbol{f},\boldsymbol{g}$ satisfy compatibility conditions) [38]. Then (2.1)-(2.3) has a unique solution $\boldsymbol{u}\in[H^s(\Omega)]^{2n}$ satisfying (2.4).

3. Least-squares formulation for div-curl system

3.1. Lagrange multiplier based least-squares formulation for div-curl system. In [25] Gunzburger and Lee have discussed a Lagrange multiplier based finite element method for three dimensional optimal control problems with first-order elliptic systems (div-curl systems) as constraints.

Consider the quadratic functionals

$$\mathcal{T}_f(\boldsymbol{u},f)=\frac{1}{2}\int_\Omega|\boldsymbol{u}-\boldsymbol{U}|^2d\Omega+\frac{\delta}{2}\int_\Omega|f|^2d\Omega, \quad (3.1)$$

and

$$\mathcal{T}_{\boldsymbol{g}}(\boldsymbol{u},\boldsymbol{g})=\frac{1}{2}\int_\Omega|\boldsymbol{u}-\boldsymbol{U}|^2d\Omega+\frac{\delta}{2}\int_\Omega|\boldsymbol{g}|^2d\Omega, \quad (3.2)$$

subject to

$$\begin{cases} \nabla.\boldsymbol{u}=f \quad \text{in} \quad \Omega, \\ \nabla\times\boldsymbol{u}=\boldsymbol{g} \quad \text{in} \quad \Omega, \end{cases} \quad (3.3)$$

with boundary conditions

$$\boldsymbol{u}.\boldsymbol{n}=0 \ \text{ on } \ \Gamma, \quad (3.4)$$

or

$$\boldsymbol{u}\times\boldsymbol{n}=0 \ \text{ on } \ \Gamma. \quad (3.5)$$

The functions f and $\boldsymbol{g}$ need to satisfy the following compatibility conditions

$$\int_\Omega f\, \mathrm{d}\Omega=0 \text{ and } \nabla.\boldsymbol{g}=0 \ \text{ in } \ \Omega \ \text{ for boundary condition (3.4)}$$

and

$$\int_\Omega f\ \mathrm{d}\Omega=0 \text{ and } \nabla.\boldsymbol{g}=0 \ \text{ in } \ \Omega, \ \ \boldsymbol{g}.\boldsymbol{n}=0 \text{ on } \Gamma \ \text{ for boundary condition (3.5).}$$

Since the div-curl system $\begin{bmatrix} \nabla.\boldsymbol{u} \\ \nabla\times\boldsymbol{u} \end{bmatrix}$ is not elliptic an auxiliary function p is added and the extended elliptic operator is defined as

$$\mathcal{L}\begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \begin{bmatrix} \nabla.\boldsymbol{u} \\ \nabla\times\boldsymbol{u}+\nabla p \end{bmatrix} = \begin{bmatrix} f \\ \boldsymbol{g} \end{bmatrix}.$$

Boundary condition (3.4) is extended as

$$R\begin{bmatrix} \boldsymbol{u} \\ p \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & n_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The optimal control problem for the newly defined div-curl-grad system is defined as :

For $\boldsymbol{U}$ being target function, find the control f and the state variables $\boldsymbol{u}$ and p such that the cost functional (3.1) subject to

$$\begin{cases} \nabla.\boldsymbol{u} = f \ \text{ in } \ \Omega, \\ \nabla\times\boldsymbol{u}+\nabla p = 0 \ \text{ in } \ \Omega, \\ \boldsymbol{u}.\boldsymbol{n} = 0 \ \text{ on } \ \Gamma, \\ p = 0 \ \text{ on } \ \Gamma, \\ \int_\Omega f \ d\Omega = 0. \end{cases}$$

Set of admissibility solutions is defined as

$$\mathcal{U}_{ad} = \left\{(\boldsymbol{u},p,f)\in \boldsymbol{H}^1_n(\Omega)\times H^1_0(\Omega)\times L^2(\Omega) : \mathcal{T}_f(\boldsymbol{u},f)<\infty\right\}.$$

Optimal control problem is formulated as a constrained minimization problem as $\min_{(\boldsymbol{u},p,f)\in\mathcal{U}_{ad}} \mathcal{T}_f(\boldsymbol{u},f)$. Existence and uniqueness of the optimal solution has been established in $\mathcal{U}_{ad}$. Then method of Lagrange multipliers has been used to modify the constrained optimization process to unconstrained one. The optimality system is given by eight first order differential equations in eight unknowns with boundary conditions and framed into ADN theory of ellipticity and hence regularity estimates have been obtained. A least-squares functional based on this first order optimality system has been defined whose minimizer is same as solution of the optimality system. Finite element approximations of solutions have been discussed and error estimates have been derived. Four different optimal control problems have been considered and the compatibility conditions are added as constraints in the system. Optimality systems for these problems have been provided.

3.2. Penalty based least-squares formulation for div-curl system. Gunzburger and Lee [26] have proposed a penalty parameter based method for minimization of (3.1) with first order pdes as constraints as (3.3) and (3.4) as boundary constraints which is defined as follows:

Given a target function $\boldsymbol{U}\in L^2(\Omega)$ and the boundary data $\boldsymbol{g}\in L^2(\Omega)$ such that $\nabla.\boldsymbol{g}=0$, find the control $\boldsymbol{f}\in L^2_0(\Omega)$ and the state variable $\boldsymbol{u}\in \boldsymbol{H}^1_n(\Omega)$ such that the cost functional

$$\mathcal{T}_\epsilon(\boldsymbol{u},f) = \mathcal{T}_f(\boldsymbol{u},f) + \frac{1}{2\epsilon}\left\{\|\nabla.\boldsymbol{u}-f\|^2 + \|\nabla\times\boldsymbol{u}-\boldsymbol{g}\|^2\right\} \tag{3.6}$$

is minimized. Here ϵ is the positive penalty parameter. The admissibility set $\mathcal{U}_{ad}^{\epsilon}$ is defined by

$$\mathcal{U}_{ad}^{\epsilon} = \left\{(\boldsymbol{u}, f) \in \boldsymbol{H}_n^1(\Omega) \times L_0^2(\Omega) : \mathcal{T}_{\epsilon}(\boldsymbol{u}, f) < \infty\right\}.$$

The optimal control problem is converted to an unconstrained minimization problem

$$\min_{(\boldsymbol{u},f)\in\mathcal{U}_{ad}^{\epsilon}} \mathcal{T}_{\epsilon}(\boldsymbol{u}, f).$$

Existence and uniqueness is established in $\mathcal{U}_{ad}^{\epsilon}$. A gradient based method is used to solve the minimization problem.

Lee and Choi ([27]) have proposed a least-squares finite element method for optimal control problems with second order pdes as constraints on two dimensional domains. However the second order elliptic system

$$\begin{cases} -\Delta u + u = f \quad \text{in} \quad \Omega, \\ \nabla u.\boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

has been converted to the first order elliptic system using the transformation $\nabla u = \boldsymbol{p}$ as following

$$\begin{cases} -\nabla.\boldsymbol{p} + u = f \quad \text{in} \quad \Omega, \\ \nabla \times \boldsymbol{p} = 0 \quad \text{in} \quad \Omega, \\ \nabla u - \boldsymbol{p} = 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{p}.\boldsymbol{n} = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Optimal control problem with penalty parameter is defined as follows: Given a target function U, find the control f and the state variables u and $\boldsymbol{p}$ such that the cost functional

$$\mathcal{T}_{\epsilon}(u, \boldsymbol{p}, f) = \mathcal{T}(u, f) + \frac{1}{2\epsilon}\left(\|\nabla.\boldsymbol{p} - u + f\|_0^2 + \|\nabla \times \boldsymbol{p}\|_0^2 + \|\nabla u - \boldsymbol{p}\|_0^2\right)$$

is minimized. Here ϵ is a positive penalty parameter. The admissibility set $\mathcal{U}_{ad}^{\epsilon}$ is given by

$$\mathcal{U}_{ad}^{\epsilon} = \left\{(u, \boldsymbol{p}, f) \in H^1(\Omega) \times \boldsymbol{H}_n^1(\Omega) \times L^2(\Omega) : \mathcal{T}_{\epsilon}(u, \boldsymbol{p}, f) < \infty\right\}$$

and the optimal control problem is formulated as an unconstrained minimization problem

$$\min_{(u,\boldsymbol{p},f)\in\mathcal{U}_{ad}^{\epsilon}} \mathcal{T}_{\epsilon}(u, \boldsymbol{p}, f).$$

Similar gradient method based analysis as in ([26]) has been carried out for optimality and finite element approximation with $O(h)$ accuracy. First order derivatives of the proposed functional are forced to vanish.

4. Least squares methods for optimal control problems with Stokes equations

Bochev and Gunzburger ([7],[8],[9]) have discussed various aspects of implementing least-squares finite element methods for control problems with constraints as Stokes equations. Approximations using both Lagrange multipliers and penalization techniques have been discussed. Lagrange multiplier based method leads to an indefinite matrix and penalization approach leads to an ill conditioned matrix, hence a different approach has been introduced.

Let's consider the cost functionals

$$\mathcal{T}_1(\boldsymbol{u},\boldsymbol{\theta}) = \frac{1}{2}\int_\Omega |\nabla \times \boldsymbol{u}|^2 d\Omega + \frac{\delta}{2}\int_\Omega |\boldsymbol{\theta}|^2 d\Omega \tag{4.1}$$

and

$$\mathcal{T}_2(\boldsymbol{u},\boldsymbol{\theta},\hat{\boldsymbol{u}}) = \frac{1}{2}\int_\Omega |\boldsymbol{u} - \hat{\boldsymbol{u}}|^2 d\Omega + \frac{\delta}{2}\int_\Omega |\boldsymbol{\theta}|^2 d\Omega. \tag{4.2}$$

Here $\delta > 0$ is a given constant, $\hat{\boldsymbol{u}} \in [L^2(\Omega)]^s$ is a given function. These functional with Stokes system as constraints is termed as velocity tracking problem with distributed controls for the Stokes system. For the first cost functional $\mathcal{T}_1$, problem is to find a distributed control function $\boldsymbol{\theta}$ such that vorticity is minimized in L^2 sense. For the cost functional $\mathcal{T}_2$, find the velocity $\boldsymbol{u}, \boldsymbol{\theta}$ such that $\boldsymbol{u}$ matches with $\hat{\boldsymbol{u}}$ as much as possible. The second term is used to limit the size of the control function $\boldsymbol{\theta}$.

In [7], least-squares methods for optimal control problems (cost functional $\mathcal{T}_2$) with Stokes system

$$\begin{cases} -\Delta \boldsymbol{u} + p - \boldsymbol{\theta} = 0 \ \text{ in } \ \Omega, \\ \nabla.\boldsymbol{u} = 0 \ \text{ in } \ \Omega \\ \boldsymbol{u} = 0 \ \text{ on } \ \Gamma, \end{cases} \quad \text{and} \quad \textstyle\int_\Omega p \ d\Omega = 0, \tag{4.3}$$

as constraints have been proposed. General results about constrained optimization problems have been discussed for Lagrange multiplier based solutions and penalty paramter based least-squares solutions. Direct penalization of least-squares functional leads to minimization of

$$\mathcal{T}_\epsilon(\boldsymbol{u},p,\boldsymbol{\theta}) = \frac{1}{2}\int_\Omega |\boldsymbol{u} - \hat{\boldsymbol{u}}|^2 d\Omega + \frac{\delta}{2}|\boldsymbol{\theta}|^2 d\Omega + \frac{1}{2\epsilon}(\|-\Delta \boldsymbol{u} + \nabla p - \boldsymbol{\theta}\|_{-1}^2 + \|\nabla.\boldsymbol{u}\|_0^2). \tag{4.4}$$

However for its finite element approximation, the approximating spaces need to go through inf-sup stability conditions which negate the advantage of choosing least-squares approach over Galerkin approach. Hence another version of least-squares approach is proposed as

$$\min_{\boldsymbol{u},\boldsymbol{\theta} \in [\boldsymbol{H}_0^1(\Omega)] \times [L^2(\Omega)]} \mathcal{T}_2(\boldsymbol{u},\boldsymbol{\theta})$$

$$\text{subject to}$$

$$\min_{\boldsymbol{u},\boldsymbol{\theta} \in [\boldsymbol{H}_0^1(\Omega)] \times [L_0^2(\Omega)]} \frac{1}{2}\big(\|-\Delta \boldsymbol{u} + \nabla p - \boldsymbol{\theta}\|_{-1}^2 + \|\nabla.\boldsymbol{u}\|_0^2\big).$$

In [8] authors have proposed six different ways to implement least-squares methods for solving pde constrained optimal control problems.

- Apply Lagrange multiplier rule to optimization problem $\rightarrow$ Finite element formulation of optimality system using Galerkin approach
- Apply Lagrange multiplier rule to optimization problem $\rightarrow$ Least squares formulation of the optimality system $\rightarrow$ Discretize using finite element approach
- Apply Lagrange multiplier rule $\rightarrow$ Perturb optimality system using penalty parameter $\rightarrow$ Discretize using finite element approach $\rightarrow$ Eliminate discrete Lagrange multiplier
- Penalize cost functional by least-squares functional $\rightarrow$ Optimize $\rightarrow$ Discretize optimality equations using finite element approach
- Constrain the cost functional by least-squares formulation of state equations $\rightarrow$ Apply Lagrange multiplier rule to get the optimality system $\rightarrow$ Discretize using finite element approach
- Constrain cost functional with least-squares formulation of state equations $\rightarrow$ Apply Lagrange multiplier rule $\rightarrow$ Perturb optimality system using a penalty parameter$\rightarrow$ Discretize using finite element method $\rightarrow$ Eliminate discrete Lagrange multiplier
- Constrain the cost functional by least-squares formulation of state equations $\rightarrow$ Penalize of cost functional $\rightarrow$ Optimize $\rightarrow$ Discretize using finite element method

In [9], Bochev and Gunzburger have proposed a least-squares based finite element method for pde governed optimal control problems with quadratic functionals. The major advantage of the proposed method is the uncoupling of discrete optimality equations and their iterative solutions. As an example, least-squares finite element formulation has been discussed for optimal control problems with Stokes equations as constraints. The optimal control problem is stated as follows : Minimize the functionals $\mathcal{T}_1$ or $\mathcal{T}_2$ with

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p + \boldsymbol{\theta} = \boldsymbol{g}, \\ \nabla.\boldsymbol{u} = 0, \\ \boldsymbol{u} = 0, \\ \int_\Omega p d\Omega = 0. \end{cases}$$

For the sake of practiality (to manage the condition number and avoid C^1 continuity requirement) of the method, Stokes system has been transformed to vorticity based first order system (velocity-vorticity-pressure), as follows

$$\begin{cases} \nabla \times \boldsymbol{w} + \nabla p + \boldsymbol{\theta} = \boldsymbol{g}, \\ \nabla.\boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \\ \nabla \times \boldsymbol{u} - \boldsymbol{w} = 0, \\ \boldsymbol{u} = 0 \quad \text{on} \quad \Gamma, \\ \int_\Omega p \; d\Omega = 0. \end{cases}$$

The least-squares functional involves a negative norm, which results calculation of H^{-1} inner product in the corresponding bilinear form. Hence different alternatives circumventing calculation of negative norm has been proposed. Use of L^2 norms disturbs the coercivity and stability of the least-squares functional. Proposed alternatives are :

- Switch to a different first order formulation. (velocity velocity gradient-pressure)
- Replace H^{-1} norm by mesh-dependent (weighted) L^2 functional.
- Replace continuous negative norm by equivalent discrete L^2 norms.

In [20], a vorticity based first order formulation of Stokes equations has been used to study optimal control problems with Stokes equations as constraints. Symmetricity and positive definiteness have been preserved with the resulting linear system. V-cycle multigrid method is used for computation.

Ryu et al. [39] have discussed a least-squares method for optimal control problems with Stokes equations as constraints using Lagrange multiplier technique and multigrid approach with objective functional as 4.2.

5. Least squares methods for optimal control problems with Navier-Stokes equations

Least-squares principles have been applied to optimal flow control problems in [3]. A least-squares finite element method for boundary control of Navier-Stokes equations (steady state) on two dimensional domains with Lipschitz continuous boundary has been developed. Separation problem for driven cavity problems has been investigated as a numerical experiment.

Navier-Stokes equations describing steady, viscous, incompressible flow is given by

$$\begin{cases} -\nu(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) + \boldsymbol{u}.\nabla \boldsymbol{u} + \nabla p = \boldsymbol{f} \ \text{ in } \ \Omega, \\ \nabla.\boldsymbol{u} = 0 \ \text{ in } \ \Omega, \\ \boldsymbol{u} = \boldsymbol{g} \ \text{ on } \ \Gamma, \end{cases}$$

where $\boldsymbol{u}$ is the velocity field, p is pressure, $\boldsymbol{f}$ is given body force, ν is kinematic viscosity, $\boldsymbol{g}$ is the boundary data satisfying $\int_\Gamma \boldsymbol{n}.\boldsymbol{g} \ d\Gamma = 0$. Two model optimal control problems have been considered with driven cavity flow on $[0,1] \times [0,1]$ with zero body force. $\Gamma_L, \Gamma_R, \Gamma_B, \Gamma_T$ are denoted as left, right, bottom, top surfaces of Ω respectively. $\boldsymbol{u}|_{\Gamma_B}, \boldsymbol{u}|_{\Gamma_T}$ are denoted as $\boldsymbol{u}_B, \boldsymbol{u}_T$ respectively. With these notations two model problems are defined as follows :

5.1. Problem 1.

$$\text{Minimize } \ \mathcal{T}_1(\boldsymbol{u}, p, \boldsymbol{g}) = \int_{\Gamma_S} |\boldsymbol{u}_2|^2 d\Gamma$$

with controls of the form $\boldsymbol{g} = \boldsymbol{g}_0 + \boldsymbol{u}_T \boldsymbol{g}_1$

$$\boldsymbol{g}_0 = \begin{cases} (\boldsymbol{u}_B, 0) & \text{on } \ \Gamma_B \\ (0,0) & \text{otherwise}, \end{cases} \quad \boldsymbol{g}_1 = \begin{cases} (1,0) & \text{on } \ \Gamma_T \\ (0,0) & \text{otherwise.} \end{cases}$$

5.2. Problem 2.

$$\text{Minimize } \mathcal{T}_2(\boldsymbol{u}, p, \boldsymbol{g}) = \int_{\Gamma_S} |\nabla \times \boldsymbol{u}|^2 d\Gamma$$

with controls of the form $\boldsymbol{g} = \boldsymbol{g}_0 + l_1\boldsymbol{g}_1 + l_2\boldsymbol{g}_2$ on a subdomain $\Omega_1 \subset \Omega$

$$\boldsymbol{g}_0 = \begin{cases} (1,0) & \text{on } \Gamma_T \\ (0,0) & \text{otherwise,} \end{cases} \quad \boldsymbol{g}_1 = \begin{cases} (0,1) & \text{on } \hat{\Gamma}_B \\ (0,0) & \text{otherwise,} \end{cases} \quad \boldsymbol{g}_2 = \begin{cases} (1,0) & \text{on } \hat{\Gamma}_R \\ (0,0) & \text{otherwise.} \end{cases}$$

Here $\hat{\Gamma}_B, \hat{\Gamma}_R$ are the bottom and right boundary of the sub-domain Ω_1. Here $\Omega_1 = [0.75, 1.00] \times [0, 0.25]$.

Incompressible second order Navier-Stokes equations have been transformed into vorticity based first order system

$$\begin{cases} \nu\nabla \times \omega + \omega \times \boldsymbol{u} + \nabla p = f & \text{in } \Omega, \\ \nabla \times \boldsymbol{u} - \omega = 0 & \text{in } \Omega, \\ \nabla.\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{g} & \text{on } \Gamma, \end{cases}$$

and the weighted least-squares functional is defined using H^{-1} and L^2 norms. Newton's method is used to obtain numerical solutions. Numerical results with specific weights ($\alpha_1 = \alpha_2 = \alpha_3 = 1, \alpha_4 = \frac{1}{h}$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$) have been provided.

Bochev et al. [5] have proposed a robust least-squares method for control problems with Navier-Stokes equations as constraints. Numerical results addressing control of driven cavity flow have been presented. Proposed method uses a weighted least-squares approach, i.e. state equations are constrained with weights, after transforming into vorticity based first order formulations. Weights are determined based on regularity of solutions.

6. Error estimates

Error estimates for optimal control problems with velocity-vorticity-pressure of Stokes equations as constraints have been presented in [13]. Using continuous finite element approximation of degree r for all the variables $O(h^r)$ accuracy has been achieved. For velocity variable H^1 norm has been used and for rest of the variables errors have been calculated using L^2 norm. Error estimates [39] represent $O(h^r)$ accuracy for vorticity variable, $O(h^{r+1})$ accuracy for velocity variable and $O(h^r)$ accuracy for pressure variable, where r is the order of approximation used. Infimums of both L^2 and H^1 have been used for error analysis. Gunzburger and Lee [25] have obtained error estimates of $O(h^r)$ for unknowns in the optimality system, where r is the polynomial order. H^1 norm has been used for error estimates. Bochev [4] has presented error estimates for optimal control problems with Navier-Stokes equations as constraints in velocity- vorticity-pressure form. Presented accuracy is of order $O(h^r)$, velocity is of order $O(h^{r+1})$ and error in pressure is of order $O(h^r)$, where r is the order of polynomial approximation. Both L^2 norm and H^1 norm have been used for error analysis. Fu and Rui [24] have obtained a priori error estimates for optimal control problems with first order elliptic system as constraints. L^2 norm and H^1 norm have been used to obtain error estimates.

7. Conclusion

A survey of existing least-squares based methods for elliptic pde constrained optimal control problems has been given. Least-squares based methods for solving pdes are gaining its popularity since last few decades because of many theoretical and computational advantages. However, use of least-squares method for constrained optimization problems is not trivial as it comes with many challenges, such as loss of positive definiteness, use of negative norms, ill-conditioning issue with penalization approach. While solving Navier-Stokes equations constrained optimal control problems, Reynolds number higher than 612.6 diverges which alerts for development of more robust/efficient least-squares formulation for pde constrained optimal control problems [3]. Hence proper attention should be given while defining the least-squares functional in order to maintain the attractive features of least squares methods. Existing literature shows the development of lower order accurate finite element methods, however higher order methods can be developed for spectral accuracy which is an ongoing research work. After obtaining optimality systems, if can be fit into ADN theory for elliptic systems, regularity estimates can be obtained which will be useful for obtaining corresponding least-squares formulations. Development of an efficient way of computing negative norms might be of great help in analyzing least-squares based solvers for pde constrained optimal control problems.

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