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NON-LOCAL PROBLEM FOR A LOADED PARABOLIC-HYPERBOLIC EQUATION INVOLVING CAPUTO AND ERDELYI-KOBER OPERATORS

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Abstract. This research work devoted to the formulation and the study of non-local boundary value problem with integral gluing condition for the loaded parabolic-hyperbolic equation with Caputo fractional derivatives.

Loaded terms of a considering equation are Erdelyi-Kober integrals, which involves a trace of solution on the line of changing type (i.e. $u(x, 0)$). Formulated problem is investigates, considering to cases related with the coefficients of a non-local condition. Unique solvability of the investigated problem, is proved reducing to the second kind Volterra type integral equations.

1. Introduction

In recent years the theory of fractional differential equation (FDE)s has been significant development, thanks to the monographs of A. A. Kilbas, H. M. Srivastava, J. J. Trujillo [1], K. S. Miller and B. Ross [2], I. Podlubny [3], S. G. Samko, A. A. Kilbas, O. I. Marichev [4] and the references therein. It's related with the applications of FDEs to the various phenomena in physics, like diffusion in a disordered or fractal medium, image analysis, risk management and others. In general, there exists no method that yields an exact solution for these equations. Indeed, we can find numerous applications in viscoelasticity, neurons, electrochemistry, control, porous media, electromagnetism, etc., (for details, see [5], [6], [7], [8], [9]). In research papers [10], [11] the authors considered some classes of initial value problems for fractional differential equations involving Riemann-Liouville and Caputo derivatives of fractional order. Boundary value problem(BVP)s for the mixed type equations involving the Caputo and the Riemann-Liouville fractional differential operators were investigated by many authors (see [12], [13], [14],[15],[16] and references therein).

On the other hand the theory of loaded functional, differential and integral-differential equations has been advanced, also. These equations describe problems in optimal control, regulation of the layer of soil water and ground moisture, underground fluid and gas dynamics, mathematical biology, economics, ecology, and pure mathematics [19], [20],[21],[22]. BVPs for integral-differential and differential

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equations with non-local boundary conditions arise in various fields of mechanics, physics, biology, biotechnology, chemical engineering, medical science, finance and others (see [23], [24],[25]). Local and non-local BVPs with continuously and integral gluing conditions for the loaded parabolic-hyperbolic equations involving Caputo, Riemann-Liouville and other integral-differential operators were investigated in works [17],[18],[26]. We would like to note that, BVPs for the loaded parabolic-hyperbolic equations involving Caputo and Erdelyi-Kober operators was investigated in case, without smaller terms (see [27],[28] and others).

2. Preliminaries

2.1. Integral and differential operators fractional order. Definition 1. Let $f(x)$ be an absolutely continuous function over(a, b). Then the left and right Riemann-Liouville fractional integrals order α $(\alpha \in R^+)$ (respectively) are (see [1].p.69)

$$\left(I_{a+}^{\alpha} f\right) x = \frac{1}{\Gamma(\alpha)} \int\limits_a^x f(t)(x-t)^{\alpha-1} dt, \quad x > a \tag{2.1}$$

$$\left(I_{-b}^{\alpha} f\right) x = \frac{1}{\Gamma(\alpha)} \int\limits_x^b f(t)(t-x)^{\alpha-1} dt, \quad x < b. \tag{2.2}$$

The Riemann-Liouville fractional derivatives $D_{ax}^{\alpha} f$ and $D_{xb}^{\alpha} f$ of order $\alpha(\alpha \in R^+)$ are defined by (see [1].p.26):

$$\left(D_{ax}^{\alpha} f\right) x = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int\limits_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \quad n = [\alpha]+1, \quad x > a; \tag{2.3}$$

$$\left(D_{xb}^{\alpha} f\right) x = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int\limits_x^b \frac{f(t)}{(t-x)^{\alpha-n+1}} dt, \quad n = [\alpha]+1, \quad x < b; \tag{2.4}$$

respectively, where $[\alpha]$ is the integer part of α.

In particular, for $\alpha = N \cup \{0\}$ we have

$$\left(D_{ax}^{0} f\right) x = f(x), \quad \left(D_{xb}^{0} f\right) x = f(x), \quad \left(D_{ax}^{n} f\right) x = f^{(n)}(x);$$
$$\left(D_{xb}^{n} f\right) x = (-1)^n f^{(n)}(x), \quad n \in N.$$

where $f^{(n)}(x)$ is the usual derivative of $f(x)$of order n.

Definition 2. Caputo fractional derivatives ${}_C D_{ax}^{\alpha} f$ and ${}_C D_{xb}^{\alpha} f$of order $\alpha > 0$ $(\alpha \notin N \cup \{0\})$ are defined by (see [1].p.92):

$$\left({}_C D_{ax}^{\alpha} f\right) x = \frac{1}{\Gamma(n-\alpha)} \int\limits_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad n = [\alpha]+1, \quad x > a; \tag{2.5}$$

$$\left({}_C D_{xb}^{\alpha} f\right) x = \frac{(-1)^n}{\Gamma(n-\alpha)} \int\limits_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, \quad n = [\alpha]+1, \quad x < b; \tag{2.6}$$

respectively.

From (2.1)-(2.6), as a conclusion we will have:

$$\left({}_C D_{ax}^{\alpha} f\right) x = \mathrm{si}\,gn^k (x-a) \left(I_{ax}^{\alpha-k} f^{(k)} \right) x, \quad k-1<\alpha \le k, \; k \in N;$$

consequently, while for $\alpha \in N \cup \{0\}$ we have

$$\left({}_C D_{ax}^{0} f\right) x = f(x), \quad \left({}_C D_{xb}^{0} f\right) x = f(x), \quad \left({}_C D_{ax}^{n} f\right) x = f^{(n)}(x);$$

$$\left({}_C D_{xb}^{n} f\right) x = (-1)^n f^{(n)}(x), \quad n \in N.$$

Definition 3. The right- and left-hand sided Erdelyi-Kober fractional integrals of the orders δ and α, respectively, are defined by [29]

$$\left(I_{\beta}^{\gamma,\delta} f \right)(x) = \frac{\beta}{\Gamma(\delta)} x^{-\beta(\gamma+\delta)} \int\limits_0^x \left(x^{\beta} - t^{\beta} \right)^{\delta-1} t^{\beta(\gamma+1)-1} f(t) dt, \quad \delta, \beta > 0, \gamma \in R, \tag{2.7}$$

$$\left(J_{\beta}^{\gamma,\alpha} f \right)(x) = \frac{\beta}{\Gamma(\alpha)} x^{\beta\gamma} \int\limits_x^{\infty} \left(t^{\beta} - x^{\alpha} \right)^{\alpha-1} t^{-\beta(\gamma+\alpha-1)-1} f(t) dt, \quad \alpha, \beta > 0, \gamma \in R, \tag{2.8}$$

where $0 < \beta, \gamma, \delta < 1, \;\; 0 < \gamma + \delta < 1.$

2.2. Bessel functions and their properties. Definition 4. ν^{th} order first kind Bessel function has a form [30]:

$$J_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+\nu}}{\Gamma(n+1)\Gamma(\nu+n+1)}. \tag{2.9}$$

where ν is an arbitrary real number. From (2.9), easy to spot, that

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{(n!)^2}, \;\; J_0(0) = 1$$

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+1}}{(n!)(n+1)!}, \;\; J_1(0) = 0.$$

$$\frac{d}{dx} \left[x^{-\nu} J_{\nu}(x) \right] = -x^{-\nu} J_{\nu+1}(x), \tag{2.10}$$

2.3. Integral operators with the Bessel functions. We enter following designations [31]

$$A_{mx}^{n,\mu} \left[f(x) \right] \equiv f(x) - \int\limits_m^x f(t) \left(\frac{t-m}{x-m} \right)^n \frac{\partial}{\partial t} J_0 \left[\mu \sqrt{(x-m)(x-t)} \right] dt, \tag{2.11}$$

$$B_{mx}^{n,\mu} \left[f(x) \right] \equiv f(x) + \int\limits_m^x f(t) \left(\frac{t-m}{x-m} \right)^{n-1} \frac{\partial}{\partial x} J_0 \left[\mu \sqrt{(t-m)(t-x)} \right] dt, \tag{2.12}$$

$$C_{mx}^{0,\mu} \left[f(x) \right] \equiv sign(x-m) \left\{ \frac{d}{dx} f(x) + \mu \int\limits_m^x \frac{f(t)}{(x-t)} J_1 \left[\mu (x-t) \right] \right\} dt, \tag{2.13}$$

Lemma 2.1. *If $f(x) \in C[a,b]$, then for all $m \in [a,b]$ and $x \in (a,b)$ takes place following equality* [31]*:*

$$A_{mx}^{n,\mu}\left\{B_{mx}^{n,\mu}\left[f(x)\right]\right\} = f(x), \;\; B_{mx}^{n,\mu}\left\{A_{mx}^{n,\mu}\left[f(x)\right]\right\} = f(x).$$

Lemma 2.2. *If $f(x) \in C[a,b]$, then for all $m \in [a,b]$ and $x \in (a,b)$ takes place following equality* [31]*:*

$$A_{mx}^{0,\mu}\left\{\int\limits_m^x B_{mx}^{1,\mu}\left[f(x)\right]\right\} = \int\limits_m^x f(t)J_0[\mu(x-t)]dt.$$

Lemma 2.3. *Let $f(x)$ be a function, which $f(x) \in C[a,b] \cap C^1(a,b)$, $f'(x) \in L_1[a,b]$ and $f(a) = 0$. Then a solution of the integral equation*

$$\int\limits_a^x g(t)J_0[\mu(x-t)]dt = f(x)$$

is exists, unique and represents as [31],

$$g(x) = f'(x) + \mu^2 \int\limits_a^x f(t)\frac{J_1[\mu(x-t)]}{\mu(x-t)}dt.$$

3. Problem formulation

Main goal of this work is an investigation of unequivocal solvability of a non-local problem with an integral gluing condition, for equation

$$0 = \begin{cases} u_{xx} - {}_C D_{ot}^{\alpha}u - \mu_1^2 u(x,t) + p_1(x,t)\left(I_{\beta_1}^{\gamma_1,\delta_1}u\right)(x,0) & t > 0 \\ u_{xx} - u_{tt} - \mu_2^2 u(x,t) + p_2(x,t)\left(I_{\beta_2}^{\gamma_2,\delta_2}u\right)(x-t,0), & t < 0 \end{cases} \tag{3.1}$$

with operators (see (2.5) and (2.7)):

$${}_C D_{oy}^{\alpha}u = \frac{1}{\Gamma(1-\alpha)}\int\limits_0^y \frac{u_t(x,t)}{(y-t)^{\alpha}}dt, \tag{3.2}$$

$$\left(I_{\beta}^{\gamma,\delta}u\right)x = \frac{\beta}{\Gamma(\delta)}x^{-\beta(\gamma+\delta)}\int\limits_0^x \frac{t^{\beta(\gamma+1)-1}}{\left(x^{\beta}-t^{\beta}\right)^{1-\delta}}u(t,0)dt \tag{3.3}$$

Let Ω be a domain, bounded with segments: $A_1A_2 = \{(x,t): \;\; x = l, \; 0 < t < h\}$, $B_1B_2 = \{(x,t): \;\; x = 0, \; 0 < t < h\}$, $B_2A_2 = \{(x,t): \;\; t = h, \; 0 < x < l\}$ at $t > 0$, and characteristics: $A_1C: x - t = l$, $B_1C: x + t = 0$ of (3.1) at $t < 0$, where $A_1(l;0)$, $A_2(l;h)$, $B_1(0;0)$, $B_2(0;h)$, and $C\left(\frac{l}{2};\frac{-l}{2}\right)$.

We enter designations: $\Omega_1 = \Omega \cap (t > 0)$, $\Omega_2 = \Omega \cap (t < 0)$, $I = \{x: \; 0 < x < l\}$.

Problem NL$_{\mu}$. To find a solution $u(x,t)$ of equation (3.1) from the class of functions: $W = \{u(x,t): \; u(x,t) \in C(\bar{\Omega}) \cap C^2(\Omega_2); \; u_{xx}, {}_C D_{ot}^{\alpha}u \in C(\Omega_1);$ $u_x \in C^1(\bar{\Omega}_1 \setminus A_2B_2)\}$ satisfies, boundary:

$$a_1u(l,t) + a_2u_x(l,t) = \varphi_1(t), \;\; b_1u(0,t) + b_2u_x(0,t) = \varphi_2(t), \;\; 0 \le t < h; \tag{3.4}$$

$$c_1A_{0x}^{0,\mu_2}\left[u\left(\frac{x}{2},-\frac{x}{2}\right)\right]+c_2A_{x1}^{0,\mu_2}\left[u\left(\frac{x+l}{2},\frac{x-l}{2}\right)\right]+c_3u(x,0)=\psi(x),\ x\in\overline{I}; \tag{3.5}$$

and integral gluing conditions:

$$\lim_{t\to+0}{}_cD_{0t}^{\alpha}u(x,t)=\lambda_1(x)u_t(x,-0)+\lambda_2(x)u_x(x,-0)+$$

$$+\lambda_3(x)u(x,0)+\lambda_4(x)\int\limits_0^x r(t)u(t,0)dt+\lambda_5(x),\ 0<x<l \tag{3.6}$$

where $a_i,b_i,c_i,c_3=const$, $\psi(x)$, $\varphi_i(t)\,(i=1,2)$ and $\lambda_k(x)\,(k=\overline{1,5})$ are given functions, besides $a_1^2+a_2^2\neq 0$, $b_1^2+b_2^2\neq 0$, $c_1^2+c_2^2\neq 0$ and $\sum\limits_{k=1}^{4}\lambda_k^2(x)\neq 0$.

3.1. Main functional relations. Notice, that a solution of the Cauchy problem for equation (3.1) in Ω_2 with the initial dates $u(x,0)=\tau(x)$ and $u_t(x,-0)=\nu^-(x)$, has a form:

$$u(x,t)=\frac{1}{2}\tau(x+t)+\frac{1}{2}\tau(x-t)-\frac{1}{2}\int\limits_{x+t}^{x-t}\nu^-(z)J_0\left[\mu_2\sqrt{(x-z)^2-t^2}\right]dz-$$

$$-\frac{\mu_2 t}{2}\int\limits_{x+t}^{x-t}\tau(z)\frac{J_1\left[\mu_2\sqrt{(x-z)^2-t^2}\right]}{\sqrt{(x-z)^2-t^2}}dz-$$

$$-\frac{1}{4}\int\limits_{x+t}^{x-t}\left(I_{\beta_2}^{\gamma_2,\delta_2}\tau\right)(\eta)d\eta\int\limits_{x+t}^{\eta}p_2\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)J_0\left[\mu_2\sqrt{(x-\xi)^2-(t-\eta)^2}\right]d\xi \tag{3.7}$$

Using by (3.7),owing to (2.12), easy to spot, that

$$u\left(\frac{x}{2},-\frac{x}{2}\right)=\frac{1}{2}\tau(0)+\frac{1}{2}B_{0x}^{0,\mu_2}\left[\tau(x)\right]-\frac{1}{2}\int\limits_0^x B_{0x}^{1,\mu_2}\left[\nu^-(t)\right]dt-$$

$$-\frac{1}{4}\int\limits_0^x\left(I_{\beta_2}^{\gamma_2,\delta_2}\tau\right)(\eta)\int\limits_0^{\eta}p_2\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)J_0\left[\mu_2\sqrt{(\eta+\xi)(\xi-\eta-x)}\right]d\xi \tag{3.8}$$

$$u\left(\frac{x+l}{2},\frac{x-l}{2}\right)=\frac{1}{2}\tau(l)+\frac{1}{2}B_{lx}^{0,\mu_2}\left[\tau(x)\right]-\frac{1}{2}\int\limits_x^l B_{lt}^{1,\mu_2}\left[\nu^-(t)\right]dt-$$

$$-\frac{1}{4}\int\limits_x^l\left(I_{\beta_2}^{\gamma_2,\delta_2}\tau\right)(\eta)d\eta\int\limits_{\eta}^{l}p_2\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)J_0\left[\mu_2\sqrt{(x-\eta-\xi)(l+\eta-\xi)}\right]d\xi. \tag{3.9}$$

Applying operators A_{0x}^{1,μ_2} and A_{1x}^{1,μ_2} to the equalities (3.8) and (3.9), respectively, based on the Lemma 1. and Lemma 2., we receive

$$A_{0x}^{0,\mu_2}\left[u\left(\frac{x}{2},-\frac{x}{2}\right)\right]=\frac{1}{2}\tau(0)J_0\left[\mu_2 x\right]+\frac{1}{2}\tau(x)-\frac{1}{2}\int\limits_0^x\nu^-(t)J_0\left[\mu_2(x-t)\right]dt-$$

$$-\frac{1}{4}A_{0x}^{1,\mu_2}\left\{\int\limits_0^x\left(I_{\beta_2}^{\gamma_2,\delta_2}\tau\right)(\eta)d\eta\int\limits_0^\eta p_2\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)\times\right.$$

$$\left.\times J_0\left[\mu_2\sqrt{(\eta+\xi)(\xi-\eta-x)}\right]d\xi\right\}. \tag{3.10}$$

$$A_{lx}^{1,\mu_2}\left[u\left(\frac{x+l}{2},\frac{x-l}{2}\right)\right]=\frac{\tau(l)}{2}J_0\left[\mu_2(l-x)\right]+\frac{1}{2}\tau(x)-$$

$$-\frac{1}{2}\int\limits_x^l\nu^-(t)J_0\left[\mu_2(t-x)\right]dt-\frac{1}{4}A_{lx}^{1,\mu_2}\left\{\int\limits_x^l\left(I_{\beta_2}^{\gamma_2,\delta_2}\tau\right)(\eta)d\eta\int\limits_\eta^l p_2\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)\times\right.$$

$$\left.\times J_0\left[\mu_2\sqrt{(x-\eta-\xi)(l+\eta-\xi)}\right]d\xi\right\}. \tag{3.11}$$

Considering (3.10) and (3.11), from the non-local condition (3.5), we have:

$$(c_1+c_2+2c_3)\tau(x)+c_1\tau(0)J_0\left[\mu_2 x\right]+c_2\tau(l)J_0\left[\mu_2(l-x)\right]-$$

$$-\frac{c_1}{4}A_{0x}^{1,\mu_2}\left\{\int\limits_0^x\left(I_{\beta_2}^{\gamma_2,\delta_2}\tau\right)(\eta)d\eta\times\right.$$

$$\left.\times\int\limits_0^\eta p_2\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)J_0\left[\mu_2\sqrt{(\eta+\xi)(\xi-\eta-x)}\right]d\xi\right\}-$$

$$-\frac{c_2}{4}A_{xl}^{1,\mu_2}\left\{\int\limits_x^l\left(I_{\beta_2}^{\gamma_2,\delta_2}\tau\right)(\eta)d\eta\times\right.$$

$$\left.\times\int\limits_\eta^l p_2\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)J_0\left[\mu_2\sqrt{(x-\eta-\xi)(l+\eta-\xi)}\right]d\xi\right\}-$$

$$-c_1\int\limits_0^x\nu^-(t)J_0\left[\mu_2(x-t)\right]dt-c_2\int\limits_x^l\nu^-(t)J_0\left[\mu_2(t-x)\right]dt=2\psi(x), \tag{3.12}$$

To find a main functional relation between $\tau(x)$ and $\nu^-(x)$, we consider following two cases:
i) $c_2=0$, $c_1\neq 0$, $c_1+c_3\neq 0$ and *ii*) $c_1=0$, $c_2\neq 0$, $c_2+c_3\neq 0$.

In cases *i*) and *ii*) from (3.12), we get

$$(c_1+2c_3)\tau(x)+\frac{c_1}{c_1+c_3}\psi(0)J_0\left[\mu_2 x\right]-\frac{c_1}{4}A_{0x}^{1,\mu_2}\left\{\int\limits_0^x\left(I_{\beta_2}^{\gamma_2,\delta_2}\tau\right)(\eta)d\eta\times\right.$$

$$\times \int_0^\eta p_2\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) J_0\left[\mu_2\sqrt{(\eta+\xi)(\xi-\eta-x)}\right] d\xi \Bigg\} -$$

$$-c_1 \int_0^x \nu^-(t) J_0\left[\mu_2(x-t)\right] dt = 2\psi(x), \tag{3.13}$$

$$\tau(0) = \frac{1}{c_1+c_3}\psi(0), \tag{3.14}$$

and

$$(c_2+2c_3)\tau(x) + \frac{c_2}{c_2+c_3}\psi(l) J_0\left[\mu_2(l-x)\right] - \frac{c_2}{4} A_{xl}^{1,\mu_2}\Bigg\{ \int_x^l \left(I_{\beta_2}^{\gamma_2,\delta_2}\tau\right)(\eta) d\eta \times$$

$$\times \int_\eta^l p_2\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) J_0\left[\mu_2\sqrt{(x-\eta-\xi)(l+\eta-\xi)}\right] d\xi \Bigg\} -$$

$$-c_2 \int_x^l \nu^-(t) J_0\left[\mu_2(t-x)\right] dt = 2\psi(x), \tag{3.15}$$

$$\tau(l) = \frac{1}{c_2+c_3}\psi(l), \tag{3.16}$$

respectively.

Based on the Lemma 3, taking (2.13) into account, from (3.13) and (3.15), we find

$$\nu^-(x) = \frac{c_1+2c_3}{c_1}\tau'(x) + \mu_2^2 \frac{c_1+2c_3}{c_1} \int_0^x \tau(t) \frac{J_1[\mu_2(x-t)]}{\mu_2(x-t)} dt +$$

$$+\frac{\psi(0)}{c_1+c_3} C_{0x}^{0,\mu_2}\left\{J_0\left[\mu_2 x\right]\right\} - \frac{1}{4} C_{0x}^{0,\mu_2}\left\{A_{0x}^{1,\mu_2}\left[A_1(x)\right]\right\} - \frac{2}{c_1} C_{0x}^{0,\mu_2}\left[\psi(x)\right], \tag{3.17}$$

$$\nu^-(x) = -\frac{c_2+2c_3}{c_2}\tau'(x) + \mu_2^2 \frac{c_2+2c_3}{c_2} \int_x^l \tau(t) \frac{J_1[\mu_2(t-x)]}{\mu_2(t-x)} dt +$$

$$+\frac{\psi(l)}{c_2+c_3} C_{lx}^{0,\mu_2}\left\{J_0\left[\mu_2(l-x)\right]\right\} - \frac{1}{4} C_{lx}^{0,\mu_2}\left\{A_{xl}^{1,\mu_2}\left[A_2(x)\right]\right\} - \frac{2}{c_2} C_{lx}^{0,\mu_2}\left[\psi(x)\right], \tag{3.18}$$

respectively, where

$$A_1(x) = \int_0^x \left(I_{\beta_2}^{\gamma_2,\delta_2}\tau\right)(\eta) d\eta \int_0^\eta p_2\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) J_0\left[\mu_2\sqrt{(\eta+\xi)(\xi-\eta-x)}\right] d\xi, \tag{3.19}$$

$$A_2(x) = \int_x^l \left(I_{\beta_2}^{\gamma_2,\delta_2}\tau\right)(\eta) d\eta \times$$

$$\times \int_{\eta}^{l} p_2\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) J_0\left[\mu_2\sqrt{(l+\eta-\xi)(x-\xi-\eta)}\right] d\xi. \tag{3.20}$$

Further, from the equation (3.1) at $t \to +0$ considering (3.2), (3.3) and (3.6) we obtain:

$$\tau''(x) - \lambda_1(x)\nu^-(x) - \lambda_2(x)\tau'(x) - \lambda_3(x)\tau(x) - \lambda_4(x)\int_0^x r(t)\tau(t)dt+$$

$$+\mu_1^2\tau(x) + p_1(x,0)\left(I_{\beta_1}^{\gamma_1,\delta_1}\tau\right)(x) = 0, \quad 0 < x < l. \tag{3.21}$$

Now, we will investigate the **Problem NL**$_\mu$ in case $i)$. Substituting (3.17) into equation (3.21), we get

$$\tau''(x) - (c_0\lambda_1(x) + \lambda_2(x))\tau'(x) - (\lambda_3(x) - \mu_1^2)\tau(x) - \lambda_4(x)\int_0^x r(t)\tau(t)dt-$$

$$-\mu_2^2 c_0\lambda_1(x)\int_0^x \tau(t)\frac{J_1[\mu_2(x-t)]}{\mu_2(x-t)}dt+$$

$$\frac{1}{4}\lambda_1(x)C_{0x}^{0,\mu_2}\left\{A_{0x}^{1,\mu_2}\left[A_1(x)\right]\right\} + p_1(x,0)\left(I_{\beta_1}^{\gamma_1,\delta_1}\tau\right)(x) =$$

$$= \frac{\psi(0)}{c_1+c_3}\lambda_1(x)C_{0x}^{0,\mu_2}\left\{J_0\left[\mu_2 x\right]\right\} - \frac{2}{c_1}\lambda_1(x)C_{0x}^{0,\mu_2}\left[\psi(x)\right], \tag{3.22}$$

where $c_0 = \frac{c_1+2c_3}{c_1}$.

Taking the class W and (3.14), (3.16) into account, from the boundary conditions (3.4), we get

$$b_2\tau'(0) = \frac{b_1}{c_1+c_3}\psi(0) - \phi_1(0), \tag{3.23}$$

$$a_2\tau'(l) = \frac{a_1}{c_2+c_3}\psi(l) - \phi_2(l). \tag{3.24}$$

Hence, to find a unknown function $\tau(x)$ we receive initial problem for the integral-differential equation (3.22) with the initial dates (3.14) and (3.23) (or (3.16) and (3.24)).

Investigation of the formulated problem, be reduced to the unequivocal solvability of the second kind Volterra type integral equation with respected to $\tau(x)$, in case $i)$. Second kind Volterra type integral equations, follows from the Eq. (3.22), considering conditions (3.14), (3.23).

Now, we will investigate $C_{0x}^{0,\mu_2}\left\{A_{0x}^{1,\mu_2}\left[A_1(x)\right]\right\}$. Considering (2.13), from (3.19), we have:

$$C_{0x}^{0,\mu_2}\left\{A_{0x}^{1,\mu_2}\left[A_1(x)\right]\right\} = \frac{d}{dx}\left\{A_{0x}^{1,\mu_2}\left[A_1(x)\right]\right\} +$$

$$+\mu_2\frac{d}{dx}\int_0^x \left\{A_{0t}^{1,\mu_2}\left[A_1(t)\right]\right\}\frac{J_1[\mu_2(x-t)]}{x-t}dt =$$

$$= A_1'(x) - \frac{d}{dx}\int\limits_0^x A_1(t)\frac{t}{x}\frac{\partial}{\partial t}J_0\left[\mu_2\sqrt{x(x-t)}\right]dt + \mu_2\int\limits_0^x A_1(t)\frac{J_1[\mu_2(x-t)]}{(x-t)}dt-$$

$$-\mu_2\int\limits_0^x \frac{J_1[\mu_2(x-t)]}{(x-t)}dt\int\limits_0^t A_1(z)\frac{z}{t}\frac{\partial}{\partial z}J_0\left[\mu_2\sqrt{t(t-z)}\right]dz.$$

Further, considering $\lim\limits_{x\to 0}\frac{J_1(x)}{x} = \frac{1}{2}$ and (2.10), we get:

$$C_{0x}^{0,\mu_2}\left\{A_{0x}^{1,\mu_2}\left[A_1(x)\right]\right\} = A_1'(x) - \frac{\mu_2 x}{4}A_1(x) + \mu_2^2\int\limits_0^x A_1(t)K_1(x,t)dt, \qquad (3.25)$$

where

$$K_1(x,t) = \frac{J_1[\mu_2(x-t)]}{\mu_2(x-t)} - \frac{t}{2}\int\limits_t^x \frac{J_1[\mu_2(x-z)]}{(x-z)}\frac{J_1\left[\mu_2\sqrt{z(z-t)}\right]}{\sqrt{z(z-t)}}dz-$$

$$-\frac{t(2x-t)}{4x(x-t)}J_2\left[\mu_2\sqrt{x(x-t)}\right]. \qquad (3.26)$$

Considering definition of the Erdelyi-Kober operator (see (3.3)), from (3.19), we receive

$$A_1(x) = \frac{\beta_2}{\Gamma(\delta_2)}\int\limits_0^x t^{\beta_2(\gamma_2+1)-1}\tau(t)dt\int\limits_t^x \frac{\eta^{-\beta_2(\gamma_2+\delta_2)}}{\left(\eta^{\beta_2}-t^{\beta_2}\right)^{1-\delta_2}}P_2(x,\eta)d\eta,$$

and

$$A_1'(x) = \frac{\beta_2 P_2(x,x)}{\Gamma(\delta_2)}x^{-\beta_2(\gamma_2+\delta_2)}\int\limits_0^x \frac{t^{\beta_2(\gamma_2+1)-1}\tau(t)}{\left(x^{\beta_2}-t_2^{\beta}\right)^{1-\delta_2}}dt+$$

$$+\frac{\beta_2\mu_2}{\Gamma(\delta_2)}\int\limits_0^x t^{\beta_2(\gamma_2+1)-1}\tau(t)dt\int\limits_t^x \frac{\eta^{-\beta_2(\gamma_2+\delta_2)}}{\left(\eta^{\beta_2}-t^{\beta_2}\right)^{1-\delta_2}}\frac{\partial P_2(x,\eta)}{\partial x}d\eta,$$

where $P_2(x,\eta) = \int\limits_0^\eta p_2\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)J_0\left[\mu_2\sqrt{(\xi+\eta)(\xi-\eta-x)}\right]d\xi$. Hence, taking (3.25) and (3.3) into account, equation (3.22), we rewrite as follows:

$$\tau''(x) - (c_0\lambda_1(x)+\lambda_2(x))\tau'(x) - (\lambda_1(x)-\mu_1^2)\tau(x)-$$

$$-\lambda_4(x)\int\limits_0^x r(t)\tau(t)dt - \int\limits_0^x \tau(t)K_2(x,t)dt = f(x), \qquad (3.27)$$

where

$$K_2(x,t) = \lambda_4(x)+\mu_2 c_0\lambda_1(x)\frac{J_1[\mu_2(x-t)]}{x-t}+\frac{\beta_1}{\Gamma(\delta_1)}p_1(x,0)\frac{x^{-\beta_1(\gamma_1+\delta_1)}t^{\beta_1(\gamma_1+1)-1}}{(x^{\beta_1}-t^{\beta_1})^{1-\delta_1}}+$$

$$+\frac{\beta_2}{4\Gamma(\delta_2)}\lambda_1(x)P_2(x,x)\frac{x^{-\beta_2(\gamma_2+\delta_2)}t^{\beta_2(\gamma_2+1)-1}}{(x^{\beta_2}-t^{\beta_2})^{1-\delta_2}}+$$

$$\frac{\beta_2\mu_2}{16\Gamma(\delta_2)}\lambda_1(x)(4-x)t^{\beta_2(\gamma_2+1)-1}\int\limits_t^x \frac{\eta^{-\beta_2(\gamma_2+\delta_2)}}{(\eta^{\beta_2}-t^{\beta_2})^{1-\delta_2}}P_2(x,\eta)d\eta+$$

$$+\frac{\beta_2\mu_2^2}{4\Gamma(\delta_2)}\lambda_1(x)t^{\beta_2(\gamma_2+1)-1}\int\limits_t^x K_1(x,s)ds\int\limits_t^x \frac{\eta^{-\beta_2(\gamma_2+\delta_2)}}{(\eta^{\beta_2}-t^{\beta_2})^{1-\delta_2}}P_2(x,\eta)d\eta, \tag{3.28}$$

$$f(x)=\frac{\psi(0)}{c_1+c_3}\lambda_1(x)C_{0x}^{0,\mu_2}\left\{J_0\left[\mu_2 x\right]\right\}-\frac{2}{c_1}\lambda_1(x)C_{0x}^{0,\mu_2}\left[\psi(x)\right]. \tag{3.29}$$

Let $b_2\neq 0$. We integrate equation (3.27), using by conditions (3.14) and (3.23):

$$\tau(x)-\int\limits_0^x K(x,t)\tau(t)dt=f_1(x) \tag{3.30}$$

$$K(x,t)=(x-t)(c_0\lambda_1'(t)+\lambda_2'(t)-\lambda_3'(t)+\mu_1^2)-c_0\lambda_1(x)-\lambda_2(x)-$$

$$-r(t)\int\limits_t^x \lambda_4(z)(x-z)dz-\int\limits_t^x K_2(z,t)(x-z)dz \tag{3.31}$$

$$f_1(x)=\int\limits_0^x (x-t)f(t)dt+\frac{\psi(0)}{c_1+c_2}\left(1-x(c_0\lambda_1(0)+\lambda_2(0))\right)+$$

$$+\frac{b_1 x}{b_2(c_1+c_3)}\psi(0)-\frac{\phi_1(0)x}{b_2}. \tag{3.32}$$

Theorem 3.1. *Let valid* $0<\alpha,\beta_i,\gamma_i,\delta_i<1,\ 0<\gamma_i+\delta_i<1\,(i=1,2)$, *and the following conditions fulfilled:*

$$p_i(x,t)\in C\left(\bar{\Omega}_i\right)\cap C^1\left(\Omega_i\right),\ \varphi_i(t)\in C\left[0,h\right]\cap C^1(0,h),\ \psi(x)\in C[0,l]\cap C^2\left(0,l\right) \tag{3.33}$$

$$\lambda_k(x)\in C\left[0,l\right]\cap C^1\left(0,l\right),\ r(x)\in C[0,l]\cap C^1(0,l),\ (k=1,2,...,5) \tag{3.34}$$

then the Problem $\boldsymbol{NL}_\mu$ *has unique solution.*

Proof. Considering the class of the given functions (3.33), (3.34) and taking (2.9),

$$|p_i(x,t)|\le p_{i0},\ \ 0\le t\le x\le l;\ \left|\frac{J_1(x)}{x}\right|\le j_{10},\ \ \textit{for all } x\in[0,l] \tag{3.35}$$

into account, from (3.26), we infer

$$|K_1(x,t)|\le j_{10}+\left|\frac{t}{2}\int\limits_t^x \frac{J_1[\mu_2(x-z)]}{(x-z)}\frac{J_1\left[\mu_2\sqrt{z(z-t)}\right]}{\sqrt{z(z-t)}}dz\right|+$$

$$+\left|\frac{t(2x-t)}{4x(x-t)}J_2\left[\mu_2\sqrt{x(x-t)}\right]\right|\le$$

$$\le j_{10}+const\,|t(x-t)|+const\,|t(2x-t)|\le const,\ \ \textit{for all},\ \ 0\le t\le x\le l, \tag{3.36}$$

where $p_{i0},\ \ j_{10}=const>0(i=1,2)$.

Considering (3.35), (3.36) and (3.28) we will estimate $\int\limits_t^x K_2(z,t)(x-z)dz$:

$$\left|\int\limits_t^x K_2(z,t)(x-z)dz\right| \le c_1 + c_2 + \frac{\beta_1 p_{10} t^{\beta_1(\gamma_1+1)-1}}{\Gamma(\delta_1)}\left|\int\limits_t^x \frac{z^{-\beta_1(\gamma_1+\delta_1)}(x-z)}{\left(z^{\beta_1}-t^{\beta_1}\right)^{1-\delta_1}}dz\right| +$$

$$+\frac{\beta_2 p_{20} t^{\beta_1(\gamma_1+1)-1}}{\Gamma(4\delta_2)}\left|\int\limits_t^x \lambda_1(z)\frac{z^{-\beta_2(\gamma_2+\delta_2)}(x-z)}{\left(z^{\beta_2}-t^{\beta_2}\right)^{1-\delta_2}}dz\right| +$$

$$+\frac{\beta_2 \mu_2 p_{20} t^{\beta_2(\gamma_2+1)-1}}{\Gamma(16\delta_2)}\left|\int\limits_t^x \lambda_1(z)(4-z)(x-z)dz\int\limits_t^z \frac{s^{-\beta_2(\gamma_2+\delta_2)}}{\left(s^{\beta_2}-t^{\beta_2}\right)^{1-\delta_2}}ds\right| +$$

$$+\frac{\beta_2 \mu_2^2 p_{20} t^{\beta_2(\gamma_2+1)-1}}{\Gamma(4\delta_2)}\cdot const\left|\int\limits_t^x \lambda_1(z)(x-z)(z-t)dz\int\limits_t^z \frac{s^{-\beta_2(\gamma_2+\delta_2)}}{\left(s^{\beta_2}-t^{\beta_2}\right)^{1-\delta_2}}ds\right|$$

Further, based on the conditions of the theorem, we can conclude, that

$$\left|\int\limits_t^x K_2(z,t)(x-z)dz\right| \le const\cdot t^{-\varepsilon},\ \ 0<\varepsilon<1. \tag{3.37}$$

Similarly, taking (3.33), (3.34) and (3.35) into account, from (3.29), we have

$$|f(x)| \le const,\ for\ all\ x\in[0,l] \tag{3.38}$$

Thus, based on the estimates (3.37), (3.38), owing to (3.31) and (3.32), we infer that the equation (3.30) is an the second kind Volterra type integral equation, involving continuous right-hand side and a kernel with weak feature. Hence, integral equation (3.30) has unique solution in the class of continuous functions, and this solution is presented as:

$$\tau(x) = f_1(x) + \int\limits_0^x R(x,t)f_1(t)dt,$$

besides, $\tau(x)\in C[0,l]\cap C^2(0,l)$.

After determining $\tau(x)$, from (3.17) we can find $\nu^-(x)\in C[0,1]\cap C(0,1)$, and finally, unique solution of the investigated problem in the domain Ω_2, we restore as a solution of the Cauchy problem (see (3.7)). Further, using by a solution of the second boundary problem for equation (3.1) in domain Ω_1, we obtain an existence of solution of the Problem NL_μ. □

Remark 1. *In case ii) The **Problem** NL_μ is equivalent to the unequivocal solvability of the problem* (3.22),(3.16), (3.24)*. Notice, that the problem* (3.22),(3.16), (3.24)*, will be reduced to the second kind Volterra and Fredholm type integral equations with respected to $\tau(x)$, at $\lambda_4(x)\equiv 0$ and $\lambda_4(x)\neq 0$, respectively. Solvability of the Volterra type integral equation is proved in a similar way.*

Remark 2. *If $c_1, c_2 \neq 0$, then the **Problem NL_μ** will be reduced to the second kind Fredholm type integral equation. Under certain conditions to the given functions and to the constant μ can be proved unequivocal solvability of a Fredholm type integral equation.*

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