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EXACT AND APPROXIMATE SOLUTIONS FOR SOME KIND OF PERIODIC SYSTEMS

A. ACOSTA, P. GARCÍA, R. VELASTEGUI, AND A. LÓPEZ

ABSTRACT. In this paper, we consider a system of linear ordinary differential equations with periodic coefficients where the matrix of the system is antisymmetric and depends on some parameters. Our work is mainly devoted to obtain exact periodic solutions and approximated almost periodic solutions. We are able to construct a catalog, that depends on the election of the parameters, showing exact periodic solutions. Changing the parameters our system has almost periodic solutions and we obtain approximations for them using our catalog.

1. Introduction

The problem of determining the characteristic multipliers or exponents of linear periodic systems is an extremely difficult one. As is pointed out in [1], except for scalar second order equations and, more generally , Hamiltonian and canonical systems, very little is known at all. Of course, also there are few cases where it is possible to obtain exact solutions for linear periodic systems and for those it is not needed to determine the characteristic multipliers or exponents.

Even with the difficulties described above, the study of periodic solutions of non-autonomous ordinary differential equations by using different approaches is a topic very interesting, see for instance [2]. Also, many of the problems treated by researchers are related to applications. Atom optics [3], quantum chaos [4], Hamiltonian systems [5, 6] or epidemic models [7] are examples illustrating the above affirmation.

This work is mainly devoted to obtain exact periodic solutions and approximate almost periodic solutions for some kind of periodic systems of the type $\dot{x} = A(t)x$, where the matrix $A(t)$ is antisymmetric and its size is 4×4. As a matter of fact, under suitable conditions of the parameters appearing in $A(t)$, we are able to obtain exact periodic solutions. Our approach begins by considering trigonometric polynomials, with undetermined coefficients, in the form of partial sums of Fourier series. This allows us to study different cases, depending on the number of undetermined coefficients, which lead to obtain periodic solutions. We only consider two cases. Then, for other configurations of the parameters, periodic series solutions are obtained. Two cases, which follow ideas form section 2.5 in [8], are treated. In addition, for some changes on the parameters and using the exact solutions we are able to approximate some almost periodic solutions of the perturbed system.

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Some specific situations related with our work arise from equations of the type $iU_t = H(t)U$, the conventional form of the time dependent Schrödinger equation, which are important in physical problems such as the scattering of an atom by a sequence of standing light waves [3], see expression (16) in this reference, or the study of some time-periodic models of quantum chaos [4]. Also, a real system with the characteristic that we consider may appear in the study of classical Hamiltonian systems, see for instance [5].

The rest of the paper is organized as follows. In section 2, we set precisely the system to be considered and established our problem. In section 3, by considering different cases (four cases) we construct a catalogue of exact periodic solutions. Section 4, provides a theoretical result that allows to estimate the separation between periodic exact solutions and almost periodic solutions. Some numerical simulations are performed. Finally, in section 5 we give some concluding remarks and discuss about further analytical work to do, which is a subject of a future research.

2. Setting of the problem

For $a, b, c \in \mathbf{R}$, $\omega > 0$, we consider the linear system of differential equations

$$\begin{aligned}
\dot{x_1} &= -ax_2 + c\sin(\omega t)x_3 - b\cos(\omega t)x_4 \\
\dot{x_2} &= ax_1 + b\cos(\omega t)x_3 + c\sin(\omega t)x_4 \\
\dot{x_3} &= -c\sin(\omega t)x_1 - b\cos(\omega t)x_2 + ax_4 \\
\dot{x_4} &= b\cos(\omega t)x_1 - c\sin(\omega t)x_2 - ax_3
\end{aligned} \tag{2.1}$$

Notice that $x_1\dot{x_1} + x_2\dot{x_2} + x_3\dot{x_3} + x_4\dot{x_4} = 0$, which implies that solutions of (2.1) have constant norm.

In a matrix form we have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = A(t; a, b, c, \omega) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

where $A(t; a, b, c, \omega)$ is the antisymmetric matrix given by

$$A(t; a, b, c, \omega) = \begin{pmatrix} 0 & -a & c\sin(\omega t) & -b\cos(\omega t) \\ a & 0 & b\cos(\omega t) & c\sin(\omega t) \\ -c\sin(\omega t) & -b\cos(\omega t) & 0 & a \\ b\cos(\omega t) & -c\sin(\omega t) & -a & 0 \end{pmatrix}.$$

Our problem is divided in two parts. The first part is devoted to discuss and obtain expressions in the simplest possible form for exact periodic solutions of system (2.1). Specifically we are looking for solutions having the form

$$x_i = \alpha_{i0} + \sum_{j=1}^{m} (\alpha_{ij}\cos(j\omega t) + \beta_{ij}\sin(j\omega t)),\ i \in \{1, 2, 3, 4\}.$$

We divide the discussion in the following cases

Case	Description
i	$a \neq 0, b \neq 0, c \neq 0$
ii	$a = 0, b \neq 0, c \neq 0$
iii	$a = 0, b \neq 0, c = 0$
iv	$a = 0, b = 0, c \neq 0$.

In the second part we want to apply the knowledge of the exact solutions to obtain numerically approximate almost periodic solutions in other scenarios.

Note that according to [9], for any linear system with $A(t)$ periodic and antisymmetric, all the solutions are quasi-periodic, so the task in this case is to approximate some of those solutions using exact periodic solutions with properly chosen parameters.

3. Exact periodic solutions

3.1. Case i: $a \neq 0, b \neq 0, c \neq 0$. In this case it is established that there are non trivial solutions having the form

$$\begin{aligned} x_1 &= \alpha_{10} + \alpha_{11}\cos(\omega t) + \beta_{11}\sin(\omega t) \\ x_2 &= \alpha_{20} + \alpha_{21}\cos(\omega t) + \beta_{21}\sin(\omega t) \\ x_3 &= \alpha_{30} + \alpha_{31}\cos(\omega t) + \beta_{31}\sin(\omega t) \\ x_4 &= \alpha_{40} + \alpha_{41}\cos(\omega t) + \beta_{41}\sin(\omega t). \end{aligned} \tag{3.1}$$

In fact, after replacing these expressions in (2.1), we are able to obtain an algebraic homogeneous system $\mathcal{A}\mathcal{X} = \mathcal{O}$ of twenty equations and twelve unknowns, where

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 & 2a & 0 & 0 & 0 & 0 & -c & 0 & b & 0 \\ 0 & \omega & 0 & 0 & 0 & -a & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & a & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & -b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & b & 0 \\ 2a & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & c \\ 0 & 0 & a & 0 & \omega & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & a & 0 & 0 & 0 & -\omega & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b & 0 & 0 & 0 & c \\ 0 & 0 & -c & 0 & -b & 0 & 0 & 0 & 0 & 2a & 0 & 0 \\ -c & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & a \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & \omega & 0 & -a & 0 \\ 0 & c & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 & 0 & c & 2a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & 0 & 0 & 0 & -a & 0 & \omega & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & -\omega \\ 0 & 0 & b & 0 & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathcal{X} = \begin{pmatrix} \alpha_{10} \\ \alpha_{11} \\ \beta_{11} \\ \alpha_{20} \\ \alpha_{21} \\ \beta_{21} \\ \alpha_{30} \\ \alpha_{31} \\ \beta_{31} \\ \alpha_{40} \\ \alpha_{41} \\ \beta_{41} \end{pmatrix}$$

and $\mathcal{O}$ is the column zero matrix of size 20×1.

It is obtained that if $b^2 - c^2 \neq 0$, then the system (2.1) has only one solution of the form given in (3.1) and that is the trivial solution. We consider just the case $b = c$ and obtain, without any other restriction on the parameters, an equivalent system which associated matrix is

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{b}{a} \\
0 & 1 & 0 & 0 & 0 & 0 & -\frac{a}{b} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-a}{b} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{b}{a} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{-a}{b} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{a}{b} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (a-\omega)(a^2+b^2+a\omega) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a^2+b^2+a\omega & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (a-\omega)(a^2+b^2+a\omega) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^2+b^2+a\omega & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^2+b^2+a\omega & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^2+b^2+a\omega \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{3.2}$$

From this, it is easy to check that if $a = \omega$ or $a^2 + b^2 + a\omega \neq 0$, the only solution having the form (3.1) is the trivial one. Now, we assume that

$$a \neq \omega \text{ and } a^2 + b^2 + a\omega = 0.$$

Under those conditions, after interchanging some rows, the previous matrix becomes in a matrix which the last twelve rows just contains entries that are zero and the first eight rows are described in the following matrix

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{b}{a} \\
0 & 1 & 0 & 0 & 0 & 0 & -\frac{a}{b} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-a}{b} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{b}{a} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{-a}{b} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{a}{b} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}. \tag{3.3}$$

From this, it is easy to see that the general solution for the homogeneous algebraic system associated to (3.1) could be given as: $\alpha_{10} = -\frac{b}{a}\beta_{41}$, $\alpha_{11} = \frac{a}{b}\alpha_{30}$, $\beta_{11} = \frac{a}{b}\alpha_{40}$, $\alpha_{20} = -\frac{b}{a}\alpha_{41}$, $\alpha_{21} = \frac{a}{b}\alpha_{40}$, $\beta_{21} = -\frac{a}{b}\alpha_{30}$, $\alpha_{31} = \beta_{41}$ and $\beta_{31} = -\alpha_{41}$; with $\alpha_{30}, \alpha_{40}, \alpha_{41}$ and β_{41} taking arbitrary real values (free variables).

Next, we obtain four linearly independent solutions, X_i with $i \in \{1, 2, 3, 4\}$, of the system (2.1), with $b = c$, by choosing the free variables as in the following table

	1	2	3	4
α_{30}	1	0	0	0
α_{40}	0	1	0	0
α_{41}	0	0	1	0
β_{41}	0	0	0	1

Thus,

$$X_1 = \begin{pmatrix} \frac{a}{b}\cos(\omega t) \\ -\frac{a}{b}\sin(\omega t) \\ 1 \\ 0 \end{pmatrix}, \; X_2 = \begin{pmatrix} \frac{a}{b}\sin(\omega t) \\ \frac{a}{b}\cos(\omega t) \\ 0 \\ 1 \end{pmatrix}$$

$$X_3 = \begin{pmatrix} 0 \\ -\frac{b}{a} \\ -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix}, \; X_4 = \begin{pmatrix} -\frac{b}{a} \\ 0 \\ \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}.$$

Now,

$$\tilde{X}(t) := \begin{pmatrix} \frac{a}{b}\cos(\omega t) & \frac{a}{b}\sin(\omega t) & 0 & -\frac{b}{a} \\ -\frac{a}{b}\sin(\omega t) & \frac{a}{b}\cos(\omega t) & -\frac{b}{a} & 0 \\ 1 & 0 & -\sin(\omega t) & \cos(\omega t) \\ 0 & 1 & \cos(\omega t) & \sin(\omega t) \end{pmatrix}$$

is a fundamental matrix solution of (2.1) and $X(t) =: \tilde{X}(t)\tilde{X}^{-1}(0)$ is the principal matrix solution at $t = 0$, its expression is given by

$$X(t) = \frac{1}{a^2+b^2} \begin{pmatrix} a^2\cos(\omega t) + b^2 & a^2\sin(\omega t) & ab(\cos(\omega t) - 1) & ab\sin(\omega t) \\ -a^2\sin(\omega t) & a^2\cos(\omega t) + b^2 & -ab\sin(\omega t) & ab(\cos(\omega t) - 1) \\ ab(1-\cos(\omega t)) & ab\sin(\omega t) & a^2\cos(\omega t) + b^2 & -a^2\sin(\omega t) \\ -ab\sin(\omega t) & ab(1-\cos(\omega t)) & a^2\sin(\omega t) & a^2\cos(\omega t) + b^2 \end{pmatrix}. \tag{3.4}$$

The previous discussion allow us, in this case, to obtain the following theorem

Theorem 3.1. *If $b = c$, $a \neq \omega$ and $a^2 + b^2 + a\omega = 0$, then all the solutions of the system (2.1) are periodic and have the form given in (3.1). Moreover, each solution can be obtained through the matrix $X(t)$ given in (3.4).*

3.2. Case ii: $a = 0, b \neq 0, c \neq 0$**.** In this case it is obtained that there exists only one solution having the form (3.1) and it is the trivial solution. We look for non trivial solutions having the form

$$\begin{aligned} x_1 &= \alpha_{10} + \alpha_{11}\cos(\omega t) + \beta_{11}\sin(\omega t) + \alpha_{12}\cos(2\omega t) + \beta_{12}\sin(2\omega t) \\ x_2 &= \alpha_{20} + \alpha_{21}\cos(\omega t) + \beta_{21}\sin(\omega t) + \alpha_{22}\cos(2\omega t) + \beta_{22}\sin(2\omega t) \\ x_3 &= \alpha_{30} + \alpha_{31}\cos(\omega t) + \beta_{31}\sin(\omega t) + \alpha_{32}\cos(2\omega t) + \beta_{32}\sin(2\omega t) \\ x_4 &= \alpha_{40} + \alpha_{41}\cos(\omega t) + \beta_{41}\sin(\omega t) + \alpha_{42}\cos(2\omega t) + \beta_{42}\sin(2\omega t) \end{aligned} \tag{3.5}$$

After replacing these expressions in (2.1), we are able to obtain an algebraic homogeneous system of twenty eight equations and twenty unknowns.

It is obtained that if $b^2 - c^2 \neq 0$, then the system (2.1) has only one solution of the form given in (3.5) and that is the trivial solution. We consider the case $b = c$ and obtain, without any other restriction on the parameters, a system for which the associated matrix contains just entries zero in the last eight rows and the first twenty rows are described in the following matrix

$$\left(\begin{array}{cccccccccccccccccccc}
2b\ \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^2-2\omega^2 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\ \omega \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^2-2\omega^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^2-2\omega^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^2-2\omega^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b^2-2\omega^2
\end{array}\right). \tag{3.6}$$

From this, it is easy to check that if $b^2 - 2\omega^2 \neq 0$, the only solution having the form (3.5) is the trivial one. If $b^2 - 2\omega^2 = 0$, a discussion given as in the foregoing case allows us to obtain the following fundamental matrix solution

$$\tilde{X}(t) := \begin{pmatrix} -\frac{\omega}{b}\sin(2\omega t) & -\frac{b}{\omega}\sin(\omega t) & \frac{\omega}{b}\cos(2\omega t) & \frac{2\omega}{b}\cos(\omega t) \\ -\frac{\omega}{b}\cos(2\omega t) & \frac{b}{\omega}\cos(\omega t) & -\frac{\omega}{b}\sin(2\omega t) & \frac{2\omega}{b}\sin(\omega t) \\ \sin(\omega t) & \text{-}\sin(2\omega t) & \text{-}\cos(\omega t) & \cos(2\omega t) \\ \cos(\omega t) & \cos(2\omega t) & \sin(\omega t) & \sin(2\omega t) \end{pmatrix}.$$

Here, the principal matrix solution $X(t) := \tilde{X}(t)\tilde{X}^{-1}(0)$ is given by

$$X(t) = \frac{1}{3}\begin{pmatrix} 2c(\omega t) + c(2\omega t) & \text{-}2s(\omega t) + s(2\omega t) & \frac{2\omega}{b}(c(\omega t) - c(2\omega t)) & \text{-}\frac{b}{\omega}s(\omega t) - \frac{2\omega}{b}s(2\omega t) \\ 2s(\omega t) - s(2\omega t) & 2c(\omega t) + c(2\omega t) & \frac{2\omega}{b}(s(\omega t) + s(2\omega t)) & \frac{b}{\omega}c(\omega t) - \frac{b}{w}c(2\omega t) \\ \frac{2b}{\omega}(-c(\omega t) + c(2\omega t)) & \text{-}\frac{2\omega}{b}(s(\omega t) + s(2\omega t)) & 2c(\omega t) + c(2\omega t) & 2s(\omega t) - s(2\omega t) \\ \frac{b}{\omega}(s(\omega t) + s(2\omega t)) & \frac{b}{\omega}(-c(\omega t) + c(2\omega t)) & \text{-}2s(\omega t) + s(2\omega t) & 2c(\omega t) + c(2\omega t) \end{pmatrix}. \tag{3.7}$$

where, $c(u) = \cos(u)$ y $s(u) = \sin(u)$

We summarize the discussion of this case in the following theorem

Theorem 3.2. *If $b = c$ and $b^2 - 2\omega^2 = 0$, then all the solutions of the system (2.1) are periodic and have the form given in (3.5). Moreover, each solution can be obtained through the matrix $X(t)$ given in (3.7).*

3.3. Case iii: $a = 0, b \neq 0, c = 0$**.** For this case we face the system

$$\begin{aligned}\dot{x_1} &= -b\cos(\omega t)x_4\\ \dot{x_2} &= b\cos(\omega t)x_3\\ \dot{x_3} &= -b\cos(\omega t)x_2\\ \dot{x_4} &= b\cos(\omega t)x_1.\end{aligned} \tag{3.8}$$

Two uncoupled systems, which are the same, are identified

$$\begin{aligned}\dot{x_1} &= -b\cos(\omega t)x_4\\ \dot{x_4} &= b\cos(\omega t)x_1\\ \dot{x_3} &= -b\cos(\omega t)x_2\\ \dot{x_2} &= b\cos(\omega t)x_3\end{aligned} \tag{3.9}$$

To obtain four linearly independent solutions of system (3.8), we consider four initial value problems for which the initial conditions, IC_i with $i \in \{1, 2, 3, 4\}$, are given in the following table

	IC_1	IC_2	IC_3	IC_4
$x_1(0)$	1	0	0	0
$x_2(0)$	0	1	0	0
$x_3(0)$	0	0	1	0
$x_4(0)$	0	0	0	1

.

Notice that uniqueness of solutions of the decoupling given in (3.9) imply, for each initial condition, for all $t \in \mathbf{R}$ the following:

$$IC_1\ x_2(t) = x_3(t) = 0,\ IC_2\ x_1(t) = x_4(t) = 0,\ IC_3\ x_1(t) = x_4(t) = 0$$

and $IC_4\ x_2(t) = x_3(t) = 0$.

Now, our problem reduces to solve two initial value problem

$$(IVP)_1\ \begin{array}{l}\dot{y_1} = -b\cos(\omega t)y_2\\ \dot{y_2} = b\cos(\omega t)y_1\\ y_1(0) = 1, y_2(0) = 0\end{array} \qquad (IVP)_2\ \begin{array}{l}\dot{y_1} = -b\cos(\omega t)y_2\\ \dot{y_2} = b\cos(\omega t)y_1\\ y_1(0) = 0, y_2(0) = 1\end{array}$$

For both of them, we look for solutions having the representation

$$\begin{aligned}y_1 = \varphi(t) &= \sum_{k=0}^{\infty} b^k \varphi_k(t)\\ y_2 = \psi(t) &= \sum_{k=0}^{\infty} b^k \psi_k(t)\end{aligned} \tag{3.10}$$

Substituting this into the differential equations corresponding to the previous initial value problems yields

$$\dot{\varphi}(t) = \sum_{k=0}^{\infty} b^k \dot{\varphi}_k(t) = -\sum_{k=0}^{\infty} b^{k+1}\cos(\omega t)\psi_k(t)$$

and

$$\dot{\psi}(t) = \sum_{k=0}^{\infty} b^k \dot{\psi}_k(t) = \sum_{k=0}^{\infty} b^{k+1} \cos(\omega t)\varphi_k(t) \ .$$

Equating equal powers of b yields $\dot{\varphi}_0(t) = 0$, $\dot{\psi}_0(t) = 0$ and for $k = 1, 2, 3, \ldots$

$$\begin{aligned} \dot{\varphi}_k(t) &= -\cos(\omega t)\psi_{k-1}(t) \\ \dot{\psi}_k(t) &= \cos(\omega t)\varphi_{k-1}(t). \end{aligned} \tag{3.11}$$

Now, the $(IVP)_1$ is considered and in order to satisfy the initial conditions, we impose the requirements that $\varphi_0(0) = 1, \varphi_k(0) = 0, k \geq 1$ and $\psi_k(0) = 0, k \geq 0$.

Hence, direct computations from (3.11) produces $\varphi_0(t) = 1$, $\psi_0(t) = 0$, $\varphi_1(t) = 0, \psi_1(t) = \frac{1}{\omega}\sin(\omega t), \varphi_2(t) = \frac{1}{\omega^2}(-\frac{1}{4} + \frac{1}{4}\cos(2\omega t)), \psi_2(t) = 0$, $\varphi_3(t) = 0, \psi_3(t) = \frac{1}{\omega^3}(-\frac{1}{8}\sin(\omega t) + \frac{1}{24}\sin(3\omega t)), \cdots\cdots$, and it is observed that $\varphi_{2k+1}(t) = 0$ and $\psi_{2k}(t) = 0$, $k \geq 0$.

Using mathematical induction it is obtained that

$$|\varphi_{2k}(t)| \leq \frac{2^{2k}}{\omega^{2k}} \text{ and } |\psi_{2k+1}(t)| \leq \frac{2^{2k+1}}{\omega^{2k+1}}, \ k = 0, 1, 2, \ldots. \tag{3.12}$$

Indeed, with $k = 0$, $|\varphi_0(t)| \leq 1$ and $|\psi_1(t)| \leq \frac{2}{\omega}$. Now, we assume that (3.12) holds with $k-1$, $k \geq 1$, and consider φ_{2k} :

$$\begin{aligned} \varphi_{2k}(t) &= \int_0^t -\cos(\omega s_1)\psi_{2k-1}(s_1)ds_1 \\ &= \int_0^t \cos(\omega s_1)\left(\int_0^{s_1} -\cos(\omega s_2)\varphi_{2k-2}(s_2)ds_2\right)ds_1. \end{aligned}$$

Taking the last integral as a value of a double integral and interchanging the order of integration, we get

$$\begin{aligned} \varphi_{2k}(t) &= \int_0^t \left(\int_{s_2}^t -\cos(\omega s_1)\cos(\omega s_2)\varphi_{2k-2}(s_2)ds_1\right)ds_2 \\ &= \frac{1}{\omega}\left(\sin(\omega s_2) - \sin(\omega t)\right)\int_0^t \cos(\omega s_2)\varphi_{2k-2}(s_2)ds_2 \\ &= \frac{1}{\omega}\left(\sin(\omega s_2) - \sin(\omega t)\right)\psi_{2k-1}(t). \end{aligned}$$

Thus,

$$|\varphi_{2k}(t)| \leq \frac{2}{\omega} \cdot \frac{2^{2k-1}}{\omega^{2k-1}} = \frac{2^{2k}}{\omega^{2k}} \ .$$

In a similar way one may obtain the estimate corresponding to ψ_{2k+1}.
The previous discussion allows us to conclude that, under the assumption $|\frac{2b}{\omega}| < 1$, the series given in (3.10) are absolute and uniform convergent. Hence the solution of the $(IVP)_1$ is given by

$$y_1 = \sum_{k=0}^{\infty} b^{2k}\varphi_{2k}(t), y_2 = \sum_{k=0}^{\infty} b^{2k+1}\psi_{2k+1}(t) \ ;$$

where

$\varphi_0(t) = 1, \psi_0(t) = 0, \varphi_{2k}(t) = \int_0^t -\cos(\omega s_1)\psi_{2k-1}(s_1)ds_1$ and $\psi_{2k-1}(t) = \int_0^t \cos(\omega s_1)\varphi_{2k-2}(s_1)ds_1,\ k \geq 1$.

In terms of $\varphi_0(t)$ and $\psi_1(t)$, we get that

$$\begin{aligned} \varphi_{2k}(t) \quad = \quad & (-1)^k \int_0^t \int_0^{s_1} \cdots\cdots \\ & \int_0^{s_{2k-1}} \cos(\omega s_1)\cos(\omega s_2)\cdots\cos(\omega s_{2k})\varphi_0(s_{2k})ds_{2k}\cdots ds_2 ds_1 \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \psi_{2k+1}(t) \quad = \quad & (-1)^k \int_0^t \int_0^{s_1} \cdots\cdots \\ & \int_0^{s_{2k-1}} \cos(\omega s_1)\cos(\omega s_2)\cdots\cos(\omega s_{2k})\psi_1(s_{2k})ds_{2k}\cdots ds_2 ds_1. \end{aligned} \tag{3.14}$$

For the $(IVP)_2$, the solution $y_1 = \tilde{\varphi}(t)$, $y_2 = \tilde{\psi}(t)$ is given in terms of the solution of the $(IVP)_1$. In fact, for the representation

$$\begin{aligned} y_1 \quad &= \quad \tilde{\varphi}(t) = \sum_{k=0}^{\infty} b^k \tilde{\varphi}_k(t) \\ y_2 \quad &= \quad \tilde{\psi}(t) = \sum_{k=0}^{\infty} b^k \tilde{\psi}_k(t), \end{aligned}$$

we have that $\tilde{\varphi}_0(t) = 0,\ \tilde{\psi}_0(t) = 1,\ \tilde{\varphi}_1(t) = -\frac{1}{\omega}\sin(\omega t),\ \tilde{\psi}_1(t) = 0,\ \tilde{\varphi}_2(t) = 0,$

$$\tilde{\psi}_2(t) = \frac{1}{\omega^2}(-\frac{1}{4} + \frac{1}{4}\cos(2\omega t)), \tilde{\varphi}_3(t) = \frac{1}{\omega^3}(\frac{1}{8}\sin(\omega t) - \frac{1}{24}\sin(3\omega t)), \tilde{\psi}_3(t) = 0, \ldots$$

It is observed that

$$\tilde{\varphi}_{2k+1}(t) = -\psi_{2k+1}(t), \tilde{\varphi}_{2k}(t) = 0, \tilde{\psi}_{2k}(t) = \varphi_{2k}(t), \tilde{\psi}_{2k+1}(t) = 0 \text{ with } k \geq 0.$$

Now, we can conclude that the matrix $X(t)$ defined by

$$X(t) := \sum_{k=0}^{\infty} \begin{pmatrix} b^{2k}\varphi_{2k}(t) & 0 & 0 & -b^{2k+1}\psi_{2k+1}(t) \\ 0 & b^{2k}\varphi_{2k}(t) & b^{2k+1}\psi_{2k+1}(t) & 0 \\ 0 & -b^{2k+1}\psi_{2k+1}(t) & b^{2k}\varphi_{2k}(t) & 0 \\ b^{2k+1}\psi_{2k+1}(t) & 0 & 0 & b^{2k}\varphi_{2k}(t) \end{pmatrix} \tag{3.15}$$

is a fundamental matrix solution for the present case, with $X(0) = I$. We summarize the discussion of this case in the following

Theorem 3.3. *If the condition* $|\frac{2b}{\omega}| < 1$ *is satisfied, then all the solutions of the system (2.1) are periodic. Moreover, each solution can be obtained through the matrix* $X(t)$ *given in (3.15), where* $\varphi_0(t) = 1$, $\psi_1(t) = \frac{1}{\omega}\sin(\omega t)$ *and, for* $k \geq 1$, $\varphi_{2k}(t)$ *and* $\psi_{2k+1}(t)$, *are giving by the expressions (3.13) and (3.14), respectively.*

3.4. Case iv: $a = 0, b = 0, c \neq 0$**.** We face the system

$$\begin{aligned} \dot{x_1} &= c\sin(\omega t)x_3 \\ \dot{x_2} &= c\sin(\omega t)x_4 \\ \dot{x_3} &= -c\sin(\omega t)x_1 \\ \dot{x_4} &= -c\sin(\omega t)x_2 \end{aligned} \tag{3.16}$$

and using the strategy of the foregoing case, we are able to obtain the following

Theorem 3.4. *If the condition* $|\frac{2c}{\omega}| < 1$ *is satisfied, then all the solutions of the system (2.1) are periodic. Moreover, each solution can be obtained through the matrix*

$$X(t) := \sum_{k=0}^{\infty} \begin{pmatrix} c^{2k}\varphi_{2k}(t) & 0 & -c^{2k+1}\psi_{2k+1}(t) & 0 \\ 0 & c^{2k}\varphi_{2k}(t) & 0 & -c^{2k+1}\psi_{2k+1}(t) \\ c^{2k+1}\psi_{2k+1}(t) & 0 & c^{2k}\varphi_{2k}(t) & 0 \\ & c^{2k+1}\psi_{2k+1}(t) & 0 & c^{2k}\varphi_{2k}(t) \end{pmatrix},$$

where $\varphi_0(t) = 1$, $\psi_1(t) = \frac{1}{\omega}(-1 + \cos(\omega t))$ *and for* $k \geq 1$

$$\begin{aligned} \varphi_{2k}(t) &= (-1)^k \int_0^t \int_0^{s_1} \cdots\cdots \\ & \int_0^{s_{2k-1}} \sin(\omega s_1)\sin(\omega s_2)\cdots\sin(\omega s_{2k})\varphi_0(s_{2k})ds_{2k}\cdots ds_2 ds_1 \end{aligned}$$

and

$$\begin{aligned} \psi_{2k+1}(t) &= (-1)^k \int_0^t \int_0^{s_1} \cdots\cdots \\ & \int_0^{s_{2k-1}} \sin(\omega s_1)\sin(\omega s_2)\cdots\sin(\omega s_{2k})\psi_1(s_{2k})ds_{2k}\cdots ds_2 ds_1. \end{aligned}$$

4. Approximate solutions

We have obtained, theorems 1 and 2, expressions for exact solutions of the system (2.1) in the scenario of the same amplitude, i.e, $b = c$. Being $b \neq c$, we consider two systems like system (2.1) where the matrices giving in (2.2) are $A(t; a, b, b, \omega)$ and $A(t; a, b, b + \epsilon, \omega)$. Thus, we have

$$\begin{aligned}
\dot{x_1} &= -ax_2 + b\sin(\omega t)x_3 - b\cos(\omega t)x_4 \\
\dot{x_2} &= ax_1 + b\cos(\omega t)x_3 + b\sin(\omega t)x_4 \\
\dot{x_3} &= -b\sin(\omega t)x_1 - b\cos(\omega t)x_2 + ax_4 \\
\dot{x_4} &= b\cos(\omega t)x_1 - b\sin(\omega t)x_2 - ax_3
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
\dot{y_1} &= ay_2 + (b+\epsilon)\sin(\omega t)y_3 - b\cos(\omega t)y_4 \\
\dot{y_2} &= ay_1 + b\cos(\omega t)y_3 + (b+\epsilon)\sin(\omega t)y_4 \\
\dot{y_3} &= -(b+\epsilon)\sin(\omega t)y_1 - b\cos(\omega t)y_2 + ay_4 \\
\dot{y_4} &= b\cos(\omega t)y_1 - (b+\epsilon)\sin(\omega t)y_2 - ay_3.
\end{aligned} \tag{4.2}$$

Theorem 4.1. *Let $x = (x_1, x_2, x_3, x_4)'$ and $y = (y_1, y_2, y_3, y_4)'$, where $'$ denotes transpose, be solutions of the systems (4.1) and (4.2), respectively. If $\gamma(t)$ is the angle between $x(t)$ and $\tilde{y}(t)$, where $\tilde{y} = (-y_3, -y_4, y_1, y_2)'$, then*

$$|y(t) - x(t)|^2 = |y(0) - x(0)|^2 + 2\epsilon|x(0)||y(0)| \int_0^t \sin(\omega s)\cos(\gamma(s))ds. \tag{4.3}$$

Proof. Subtracting (4.1) from (4.2) produces

$$\begin{aligned}
(\dot{y_1} - \dot{x_1}) &= -a(y_2 - x_2) + (b+\epsilon)\sin(\omega t)(y_3 - x_3) - b\cos(\omega t)(y_4 - x_4) + \epsilon\sin(\omega t)y_3 \\
(\dot{y_2} - \dot{x_2}) &= a(y_1 - x_1) + b\cos(\omega t)(y_3 - x_3) + (b+\epsilon)\sin(\omega t)(y_4 - x_4) + \epsilon\sin(\omega t)y_4 \\
(\dot{y_3} - \dot{x_3}) &= -(b+\epsilon)\sin(\omega t)(y_1 - x_1) - b\cos(\omega t)(y_2 - x_2) + a(y_4 - x_4) - \epsilon\sin(\omega t)y_1 \\
(\dot{y_4} - \dot{x_4}) &= b\cos(\omega t)(y_1 - x_1) - (b+\epsilon)\sin(\omega t)(y_2 - x_2) - a(y_3 - x_3) - \epsilon\sin(\omega t)y_2.
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{i=1}^{4}(y_i - x_i)(\dot{y_i} - \dot{x_i}) &= \epsilon\sin(\omega t)\left[-y_3x_1 - y_4x_2 + y_1x_3 + y_2x_4\right] \\
&= \epsilon\sin(\omega t)(x \cdot \tilde{y}) \\
&= \epsilon\sin(\omega t)|x||y|\cos(\gamma(t)).
\end{aligned}$$

Thus, due to the fact that solutions of (2.1) have constant norm, we have

$$\frac{d}{dt}\sum_{i=1}^{4}(y_i - x_i)^2 = 2\epsilon\sin(\omega t)|x(0)||y(0)|\cos(\gamma(t)).$$

Integrating between 0 and t lead us to

$$\sum_{i=1}^{4}(y_i - x_i)^2 - \sum_{i=1}^{4}(y_i(0) - x_i(0))^2 = 2\epsilon|x(0)||y(0)| \int_0^t \sin(\omega s)\cos(\gamma(s))ds, \tag{4.4}$$

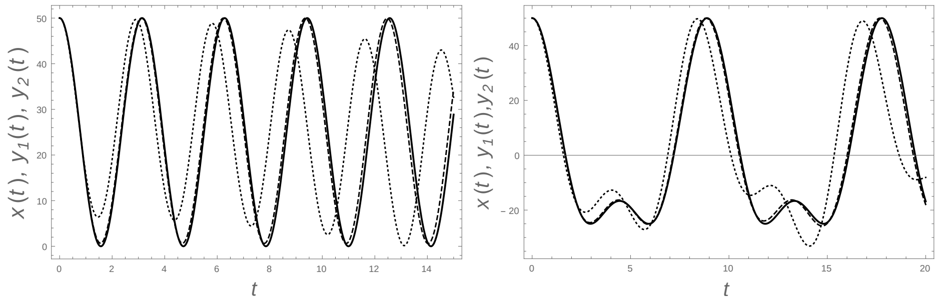

FIGURE 1. In both figures, continuous line represents the exact periodic solution in the cases (i) and (ii). Dashed and dotted line represent the almost periodic solution in the case $b = c + \epsilon$, with $\epsilon = 0.01$ and $\epsilon = 0.1$ respectively.

and the result follows.

$\square$

In particular, when $x(0)$ and $y(0)$ have the same norm we get from (4.3) that

$$|y(t) - x(t)|^2 = 2|x(0)|^2 - 2(x(0) \cdot y(0)) + 2\epsilon|x(0)|^2 \int_0^t \sin(\omega s)\cos(\gamma(s))ds,$$

and the fact $|x(t)| = |y(t)| = |x(0)|$, $t \in \mathbf{R}$, lead us to the estimates

$$0 \leq \epsilon \int_0^t \sin(\omega s)\cos(\gamma(s))ds \leq 2. \tag{4.5}$$

It is deduced, for all t, that either $\int_0^t \sin(\omega s)\cos(\gamma(s))ds \geq 0$ $(\epsilon > 0)$ or $\int_0^t \sin(\omega s)\cos(\gamma(s))ds \leq 0$ $(\epsilon < 0)$.

Now, we perform numerical simulations taking $x(t)$ as an exact solution coming from some of the cases i,ii,ii or iv and $y(t)$ a solution corresponding to a scenario where $b \neq c$. In the simulations we plot the first coordinate of the solutions.

Notice that according to the theorem 4.1, the exact periodic solution give an approximation for the almost periodic solutions (see Theorem 1 in[9]) corresponding to the case where $b \neq c$. Also, counter intuitively we observe that the approximations is better when b and c are not too close. We believe that is related with the estimate (4.5), when ϵ smaller then exists a more tight bound for the error of the approximation, which is proportional to the integral in (4.5).

5. Concluding remarks

For the kind of system considered we have exhibited a catalog of exact periodic solutions and it has been shown how to use this to obtain approximately solutions,

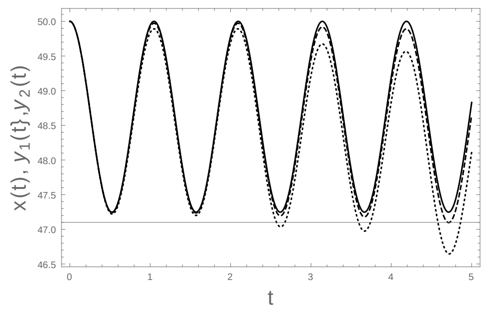

FIGURE 2. In the figure, continuous line represents the exact periodic solution in the case (ii). Dashed and dotted line represent the almost periodic solution in the case $a = 0$, $b = 1$ and $c = \epsilon$, with $\epsilon = 0.01$ and $\epsilon = 0.1$ respectively.

in some scenarios, when the parameters in our system do not correspond with the relations in the catalog.
Our results correspond to some non trivial cases and it is worth to note that the framework presented could be applied to provide a wider catalog of exact periodic solutions. Also, in a future research more general linear systems, periodic or not, with antisymmetric matrix could be considered and again the first challenge is to construct a catalog of exact solutions of some nature.

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A. Acosta: School of Mathematical Sciences and Information Technology, Department of Mathematics, Yachay Tech, Ecuador.
Email address: aacosta@yachaytech.edu.ec

P. García: Laboratorio de Sistemas Complejos, Departamento de Física Aplicada, Universidad Central de Venezuela, Venezuela and Red Iberoamericana de Investigadores en Matematicas Aplicadas a Datos, AUIP.
Email address: pedro@fisica.ciens.ucv.ve

R. Velastegui: School of Mathematical Sciences and Information Technology, Department of Mathematics, Yachay Tech, Ecuador.
Email address: ronny.velastegui@yachaytech.edu.ec

A. López: Facultad de Ciencias Naturales y Matemáticas, Escuela Superior Politécnica del Litoral, Guayaquil, Ecuador.
Email address: alexlop@espol.edu.ec