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HYPER ZAGREB AND FORGOTTEN INDICES FOR GENERALIZED GRAPHS UNDER LEXICOGRAPHIC PRODUCT

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Abstract - **Quantitative structures activity bond and quantitative structures property bond models are investigated using graph-theoretic modeling or methodologies. Topological Indices (TIs) are the basic mechanism which are often used to study the several structure and chemical propertiesof the chemical graphs such as boiling point, solubility, surface tension, density, heat of evaporation, melting, freezing point, heat of formation and symmetry. Assume that S_k and R_k are two generalized subdivision and semi total point operation respectively. Then the generalized F-sum graphs $G_1[G_2]_{F_k}$ are obtained by Lexicographic product of connected graphs $F_k(G_1)$ and G_2. In this work, the Hyper-Zagreb index and F-index are calculated for the F_k-sum graphs by using Lexicographic product in terms of their elementary graphs. For the illustration of the obtained results, the Hyper and Forgotten indices are also computed on the generalized F-sum graphs of paths for $k \in \{1, 2, 3, 4\}$. Moreover, a comparison is also calculated to show that Hyper Zagreb index remains larger than Forgotten index for all the values of k.**

Index Terms - Molecular graph, Lexicographic product, Generalized F-sum graph.

Mathematics Subject Classification - 15A18, 05C50, 05C40, 05D05.

INTRODUCTION

A molecular descriptor known as a topological index (TI) is a mathematical formula that ties a molecular network to a real number and predicts its chemical, biological, and structural features. Trinjastic, Gutman (1972) [9] and Wiener (1947) [7] employed molecular descriptors to determine paraffin's boiling point as well as the total π-electron energy of the molecules. In the field of cheminformatics, [24, 26, 27] TIs are also used to categories molecules based on their quantitative structure behavior and property connections. All TI's remains constant for the isomorphic shapes, see [2, 13, 21–23].

Many TIs exist in the literature of chemical graph theory. The three most common typesof TIs are degree, distance, and polynomial-based TIs. The degrees-based TIs are more well-known than the others [8, 25, 28, 29]. Graph operations are frequently used in chemical graph theory to identify differentsorts of graphs. Yan et al. [14] showed how to do subdivision and semi-total point operations on a molecular graph G and got the Wiener indices of the resulting graphs $S_k(G)$, $S_1(G)$ and $R_1(G)$. The F-sum graph $G_1 +_F G_2$ was then explained by Eliasi and Taeri [12] using Cartesian product of G_1 and $S(G_2)$ where, $F \in \{S_1, R_1\}$. Deng et al. [4] used lexicographic product to explain the F-sum graph ($G_1[G_2]$) and constructed the first and second Zagreb indices and the forgotten index of F-sum graphs was also determined by Akhter et al. [1].

Lately, Liu et al. [10, 11], generalized these subdivision and semi total point operations of graphs by using cartesian product i.e $G_1 +_{F_k} G_2$ for $F_k \in \{S_k, R_k\}$, where k

≥ 1 is an integral value. They also calculated the 1st and 2nd Zagreb indices for F-sum graphs [5, 6, 15–20].
The hyper-Zagreb index was first introduced by Shirdel et al. [3]. The mathematical expression $HM(G) = \sum_{(u;v)\in E(G)}(d(u) + d(v))^2$ is hyper-Zagreb index and $F(G) = \sum_{u\in V(G)} d^3G(u) = \sum_{uv\in E(G)}(d^2G(u) + d^2G(v))$ is Forgotten index of a graph G.
In this paper the Hyper Zagreb and Forgotten indices for generalized F-sum graphs are solved using generalized subdivision, generalized semi-total point, and lexicographic product operations. Section I contains some prior knowledge linked to our work, Section II contains some essential definitions, Section III contains the significant findings, and the conclusion and applications of these indexes are found in Section IV.

PRELIMINARIES

Here all the considered graphs are simple and connected. If $G = (V(G), E(G))$ presents a graph then $\boldsymbol{V(G) = |n|}$ and $\boldsymbol{E(G) = |m|}$ are called order and size respectively. In particular, for the case of a molecular graph vertices are called atoms, while the edges indicate the atoms' bonding.

Moreover, the degree of each vertex refers to the number of incident edges. A path is a graph in which two values are from path and all other are of degree 2.

Definition 2.1 Consider two graphs G_1 and G_2, where $V(G_1) = \{y_1, y_2, ..., y_n\}$ and $V(G_2) = \{z_1, z_2, ..., z_n\}$ respectively. The Cartesian product $G_1\times G_2$ of these graphs is defined as follows:

$$V(G_1 \times G_2) = V(G_1) \times V(G_2)$$

$$E(G_1 \times G_2) = \{(y_1, z_1)(y_2, z_2) : (y_1, z_1), (y_2, z_2) \in V(G_1 \times G_2)\}$$

with conditions either
$y_1 = y_2$ in $V(G_1)$ and $z_1z_2 \in E(G_2)$ or $y_1y_2 \in E(G_1)$ and $z_1 = z_2$ in $V(G_2)$.

Definition 2.2 Consider two graphs G_1 and G_2, where $V(G_1) = \{y_1, y_2, ..., y_n\}$ and $V(G_2) = \{z_1, z_2, ..., z_n\}$ respectively. Then the lexicographic product $G_1[G_2]$ of these graphs is defined as:

$$V(G_1[G_2]) = V(G_1) \times V(G_2)$$

$$E(G_1[G_2]) = \{(y_1, z_1)(y_2, z_2): (y_1, z_1), (y_2, z_2) \in V(G_1[G_2])\}$$

with conditions either
$y_1 = y_2$ in $V(G_1)$ and $z_1z_2 \in E(G_2)$ or $y_1y_2 \in E(G_1)$ and z_1, z_2 in $V(G_2)$.
Under the process of two linked graphs G_1 and G_2 based on the Cartesian product, Taeri and Eliasi proposed the graphs, [12]. Deng et al. [4] established the F-sum graphs, based on the Lexicographic product, under the process of two linked graphs G_1 and G_2.

Definition 2.3 Consider two graphs G_1 and G_2, $F_k \in \{S_k, R_k\}$ and $F_k(G_1)$ be a graph with vertex set $V(F_k(G_1))$ and edge set $E(F_k(G_1))$. Then the generalized F-sum graph under lexicographic product is a $G_1[G_2]_{F_k}$ is a graph having vertex set

$$V(G_1[G_2])_{Fk} = V(F_k(G_1)) \times V(G_2)$$

$$E(G_1[G_2])_{Fk} = \{(y_1, z_1)(y_2, z_2) : (y_1, z_1), (y_2, z_2) \in V(G_1[G_2]_{Fk})\}$$

with conditions either
$y_1 = y_2$ in $V(F_k(G_1))$ and $z_1z_2 \in E(G_2)$ or $y_1y_2 \in E(F_k(G_1))$ and z_1, z_2 in $V(G_2)$.
For integer $\boldsymbol{k \geq 1}$, Liu et al. [11] explained the generalized graphs by using generalized subdivision and semi total point operations as follows:

- $S_k(G)$ is obtained by replacing each edge of G with the path of length k+1.
- $R_k(G)$ is extended to $S_k(G)$ by joining the old vertices which are adjacent in G.

For more explanation, see Figure 1, Figure 2, Figure 3 and Figure 4.

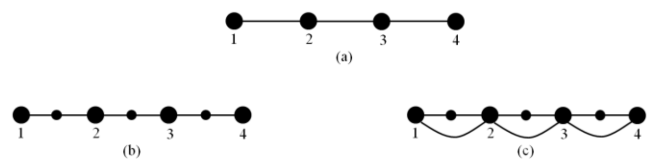

FIGURE 1
(a) $G \cong P_4$, (b) $S_1(G)$, (c) $R_1(G)$

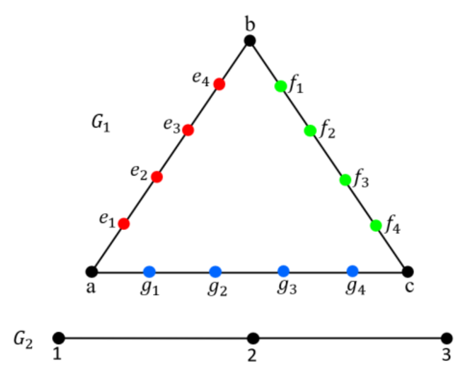

FIGURE 2
(a) $G_1 \cong S_4(C_3)$, (b) $G_2 \cong P_3$

MAIN RESULTS

The hyper Zagreb index and F-index of generalized F-sum graphs for generalized subdivision and semi total point operations connected to lexicographic product determined in this section.

Theorem 3.1 For $\boldsymbol{k} \geq \mathbf{1}$, the hyper Zagreb index of generalized S-sum graph under the lexicographic product is

$$HM\big(G_1[G_2]_{s_k}\big) = 8n_2^2e_2\, M_1(G_1) + 8n_2e_1M_1(G_2) + n_1HM(G_2) + n_2^4F(G_1) + 2n_2e_1M_1(G_2) + 8n_2^4e_1 + 16n_2^2e_1e_2 + 4n_2^4M_1(G_1) + 16e_1n_2^3(k-1) + 32n_2^2e_1(k-1)\sum_{i=1}^{n_2-1}(n_2-i)$$

Proof:

Let $d(u,v) = d_{G_1[G_2]_{s_k}}(u,v)$ be the graph of the degree's vertex (u,v) in the graph $G_1\,[G_2]_{s_k.}$

$$HM\big(G_1[G_2]_{s_k}\big) = \sum_{(u,v)\epsilon E(G_1[G_2])_{S_k}} [d(u)+d(v)]^2$$

$$= \sum_{(u_1,v_1)(u_2,v_2)\epsilon E(G_1[G_2])_{S_k}} [d(u_1,v_1) + d(u_2,v_2)]^2$$

$$= \sum_{u_1=u_2=u\varepsilon V(G_1)} \sum_{v_1v_2\,\varepsilon\, E(G_2)} [d(u,v_1)+d(u,v_2)]^2 + \sum_{v_1\varepsilon\, V(G_2)} \sum_{v_2\varepsilon\, V(G_2)} \sum_{u_1u_2\varepsilon E(s_k(G_1))} [d(u_1,v_1) + d(u_2,v_2)]^2 + \sum_{v_1=v_2=v\varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(G_1)\\ u_1u_2\varepsilon V(s_k(G_1))-V(G_1)}} [d(u_1,v_1) + d(u_2,v_2)]^2 + \sum_{v_1\varepsilon\, V(G_2)} \sum_{v_2\varepsilon\, V(G_2)} \sum_{u_1u_2\varepsilon E(s_k(G_2))} [d(u_1,v_1) + d(u_2,v_2)]^2$$

$$= \sum 1 + \sum 2 + \sum 3 + \sum 4$$

Now first we calculate

$$\sum 1 = \sum_{u_1=u_2=u\varepsilon V(G_1)} \sum_{v_1v_2\,\varepsilon\, E(G_2)} [d(u,v_1)+d(u,v_2)]^2$$

$$= \sum_{u\varepsilon V(G_1)} \sum_{v_1v_2\,\varepsilon\, E(G_2)} [n_2d_{G_1}(u) + d_{G_2}(v_1) + n_2d_{G_1}(u) + d_{G_2}(v_2)]^2$$

$$= \sum_{u\varepsilon V(G_1)} \sum_{v_1v_2\,\varepsilon\, E(G_2)} [2n_2d_{G_1}(u) + d_{G_2}(v_1) + d_{G_2}(v_2)]^2$$

$$= \sum_{u\varepsilon V(G_1)} \sum_{v_1v_2\,\varepsilon\, E(G_2)} 4n^2{}_2d^2{}_{G_1}(u) + d^2{}_{G_2}(v_1) + d^2{}_{G_2}(v_2) + 4n_2d_{G_1}(u)d_{G_2}(v_1) + 2d_{G_2}(v_1)d_{G_2}(v_2) + 4n_2d_{G_1}(u)d_{G_2}(v_2)$$

$$= \sum_{u\varepsilon V(G_1)} \sum_{v_1v_2\,\varepsilon\, E(G_2)} 4n^2{}_2d^2{}_{G_1}(u) + \Big(d_{G_2}(v_1)+d_{G_2}(v_2)\Big)^2 + 4n_2d_{G_1}(u)\,(d_{G_2}(v_1)+d_{G_2}(v_2))$$

$$= \sum_{u\varepsilon V(G_1)} e_2 4n^2{}_2\big|d_{G_1}(u)\big|^2 + \sum_{u\varepsilon V(G_1)} 4n_2d_{G_1}(u)\,M_1(G_2) + \sum_{u\varepsilon V(G_1)} HM(G_2)$$

$$= e_2 4n^2{}_2M_1(G_1) + 4n_2(2e_1)M_1(G_2) + n_1HM(G_2)$$

$$= 4n^2{}_2e_2M_1(G_1) + 8n_2e_1M_1(G_2) + n_1HM(G_2)$$

$$\sum 2 = \sum_{v_1\varepsilon\, V(G_2)} \sum_{v_2\varepsilon\, V(G_2)} \sum_{u_1u_2\varepsilon E(s_k(G_1))} [d(u_1,v_1) + d(u_2,v_2)]^2$$

$$= \sum_{v_1\varepsilon\, V(G_2)} \sum_{v_2\varepsilon\, V(G_2)} \sum_{\substack{u\varepsilon v(G_1), a\varepsilon E(G_1)\\ u \text{ and } a \text{ are incident}}} [d(u,v_1) + d(u,v_2)]^2$$

$$= \sum_{v_1\varepsilon\, V(G_2)} \sum_{v_2\varepsilon\, V(G_2)} \sum_{\substack{u\varepsilon v(G_1), a\varepsilon E(G_1)\\ u \text{ and } a \text{ are incident}}} \big[n_2d_{G_1}(u) + d_{G_2}(v_1) + 2n_2\big]^2$$

$$= \sum_{v_1\varepsilon\, V(G_2)} \sum_{v_2\varepsilon\, V(G_2)} \sum_{u\varepsilon v(G_1)} \Big(d_{G_1}(u)\Big)\big[n_2d_{G_1}(u) + d_{G_2}(v_1) + 2n_2\big]^2$$

$$= \sum_{v_1\varepsilon\, V(G_2)} \sum_{v_2\varepsilon\, V(G_2)} \sum_{u\varepsilon v(G_1)} \Big(d_{G_1}(u)\Big)\,\big[n^2{}_2d^2{}_{G_1}(u) + d^2{}_{G_2}(v_1) + 4n^2{}_2 + 2n_2d_{G_1}(u)d_{G_2}(v_1) + 4n_2d_{G_2}(v_1) + 4n^2{}_2d_{G_1}(u)\big]$$

$$= \sum_{v_1\varepsilon\, V(G_2)} \sum_{v_2\varepsilon\, V(G_2)} \sum_{u\varepsilon v(G_1)} \big[n^2{}_2d^3{}_{G_1}(u) + d_{G_1}(u)d^2{}_{G_2}(v_1) + 4n^2{}_2d_{G_1}(u) + 2n_2d^2{}_{G_1}(u)d_{G_2}(v_1) + 4n_2d_{G_2}(v_1)d_{G_1}(u) + 4n^2{}_2d^2{}_{G_1}(u)\big]$$

$$= n_2{}^4F(G_1) + 2n_2e_1M_1(G_2) + 8n_2{}^4e_1 + 4n^2{}_2e_2M_1(G_1) + 16n^2{}_2e_1e_2 + 4n_2{}^4M_1(G_1)$$

$$\sum 3 = \sum_{v_1=v_2=v\epsilon V(G_2)} \sum_{\substack{u_1u_2\epsilon E(G_1)\\ u_1u_2\epsilon V(s_k(G_1))-V(G_1)}} [d(u_1,v_1) + d(u_2,v_2)]^2$$

$$= n_2(k-1)e_1[2n_2+2n_2]^2 = 16e_1n^3{}_2(k-1)$$

$$\sum 4 = \sum_{v_1 \epsilon V(G_2)} \sum_{v_2 \epsilon V(G_2)} \sum_{u_1 u_2 \epsilon E(s_k(G_2))} [d(u_1, v_1) + d(u_2, v_2)]^2$$

$$= [2n_2 + 2n_2]^2 \sum_{i=1}^{n_2-1} (2n_2 - 2i)(k-2)e_1 + [2n_2 + 2n_2]^2 \sum_{i=1}^{n_2-1} (2n_2 - 2i)e_1$$

$$= 32{n_2}^2 e_1 (k-1) \sum_{i=1}^{n_2-1} (n_2 - i)$$

So, the result is

$$= 8n^2{}_2 e_2 M_1(G_1) + 8n_2 e_1 M_1(G_2) + n_1 HM(G_2) + {n_2}^4 F(G_1) + 2n_2 e_1 M_1(G_2) + 8{n_2}^4 e_1 + 16n^2{}_2 e_1 e_2 + 4{n_2}^4 M_1(G_1) + 16 e_1 n^3{}_2 (k-1) + 32{n_2}^2 e_1 (k-1) \sum_{i=1}^{n_2-1} (n_2 - i)$$

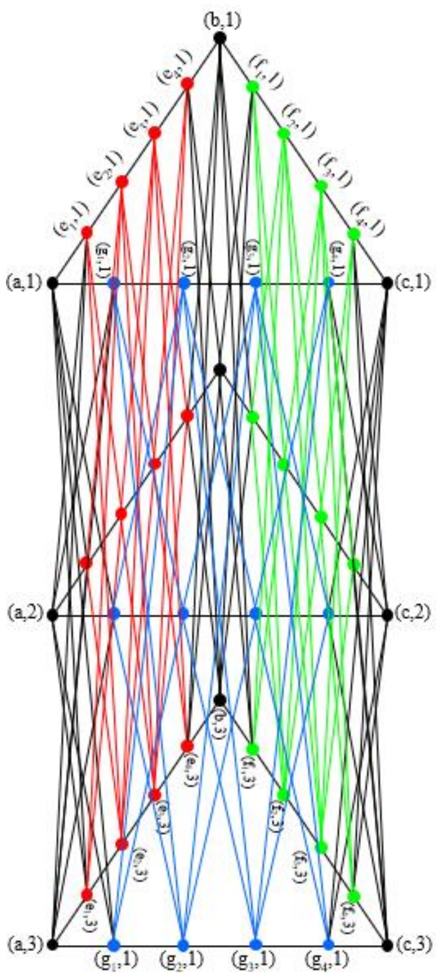

FIGURE 3
G ≅ $C_3[P_3]_{S4}$

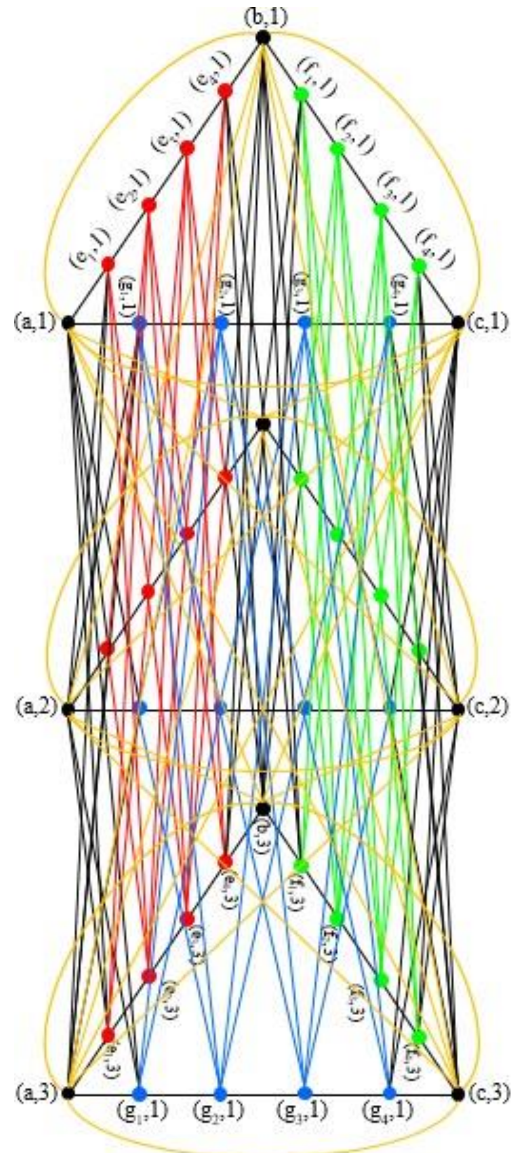

FIGURE 4
G ≅ $C_3[P_3]_{R4}$

Theorem 3.2. For $k \geq 1$, the hyper Zagreb index of generalized R-sum graph under the lexicographic product is

$$HM(G_1[G_2]_{R_k}) = 8{n_2}^4 F(G_1) + 40{n_2}^2 e_2 M_1(G_1) + 20n_2 e_1 M_1(G_2) + n_1 HM(G_2) + 8{n_2}^4 [M_1(G_1) + M_2(G_1) + e_1] + 8e_1 e_2 [e_2 + 2{n_2}^2] + 16 e_1 n^3{}_2 (k-1) + 32{n_2}^2 e_1 (k-1) \sum_{i=1}^{n_2-1} (n_2 - i)$$

Proof:

Let $d(u,v) = d_{G_1[G_2]_{R_k}}(u,v)$ be the graph of the degree's vertex (u,v) in the graph $G_1[G_2]_{R_k}$.

$$HM\left(G_1[G_2]_{R_k}\right) = \sum_{(u,v)\epsilon E(G_1[G_2])_{R_k}} [d(u)+d(v)]^2$$
$$= \sum_{(u_1,v_1),(u_2,v_2)\epsilon E(G_1[G_2])_{R_k}} [d(u_1,v_1) + d(u_2,v_2)]^2$$
$$= \sum_{u_1=u_2=u\epsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [d(u,v_1)+d(u,v_2)]^2$$
$$+ \sum_{v_1\varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(R_k(G_1))} [d(u_1,v_1)+d(u_2,v_2)]^2$$
$$+ \sum_{v_1=v_2=v\varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(G_1)\\ u_1u_2 \varepsilon V(R_k(G_1))-V(G_1)}} [d(u_1,v_1) + d(u_2,v_2)]^2$$
$$+ \sum_{v_1\varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(R_k(G_2))} [d(u_1,v_1)+d(u_2,v_2)]^2$$
$$= \sum 1 + \sum 2 + \sum 3 + \sum 4$$

Now first we calculate

$$\sum 1 = \sum_{u_1=u_2=u\epsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [d(u,v_1) + d(u,v_2)]^2$$
$$= \sum_{u\epsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [n_2 d_{R_kG_1}(u) + d_{G_2}(v_1) + n_2 d_{R_kG_1}(u) + d_{G_2}(v_2)]^2$$
$$= \sum_{u\epsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [2n_2 d_{R_kG_1}(u) + d_{G_2}(v_1) + d_{G_2}(v_2)]^2$$
$$= \sum_{u\epsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [4n_2 d_{G_1}(u) + d_{G_2}(v_1) + d_{G_2}(v_2)]^2$$
$$= \sum_{u\epsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} 16{n^2}_2 d^2{}_{G_1}(u) + d^2{}_{G_2}(v_1) + d^2{}_{G_2}(v_2) + 8n_2 d_{G_1}(u) d_{G_2}(v_1) + 2d_{G_2}(v_1)d_{G_2}(v_2) + 8n_2 d_{G_1}(u)d_{G_2}(v_2)$$
$$= 16{n^2}_2 e_2 M_1(G_1) + 16n_2 e_1 M_1(G_2) + n_1 HM(G_2)$$

$$\sum 2 = \sum_{v_1\varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E((G_1))\\ u_1 \varepsilon V(G_1) u_2 \varepsilon V(R_k(G_1)-V(G_1))}} [d(u_1,v_1) + d(u_2,v_2)]^2$$
$$+ \sum_{v_1\varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(R_k(G_1))\\ u_1 \varepsilon V(G_1) u_2 \varepsilon V(R_k(G_1)-V(G_1))}} [d(u_1,v_1) + d(u_2,v_2)]^2$$
$$\sum 2 = \sum 2' + \sum 2''$$

$$\sum 2' = \sum_{v_1\varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(G_1)\\ u_1 \varepsilon V(G_1) u_2 \varepsilon V(R_k(G_1)-V(G_1))}} [d(u_1,v_1) + d(u_2,v_2)]^2$$
$$= \sum_{v_1\varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(G_1)} \left[n_2 d_{R_kG_1}(u_1) + d_{G_2}(v_1) + n_2 d_{R_kG_1}(u_2) + d_{G_2}(v_2)\right]^2$$
$$= \sum_{v_1\varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(G_1)} \left[2n_2 d_{G_1}(u_1) + d_{G_2}(v_1) + 2n_2 d_{G_1}(u_2) + d_{G_2}(v_2)\right]^2$$
$$= \sum_{v_1\varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(G_1)} \left[2n_2 d_{G_1}(u_1) + d_{G_2}(v_1)\right]^2 + \left[2n_2 d_{G_1}(u_2) + d_{G_2}(v_2)\right]^2 + 2\left[2n_2 d_{G_1}(u_1) + d_{G_2}(v_1)\right]\left[2n_2 d_{G_1}(u_2) + d_{G_2}(v_2)\right]$$
$$= \sum_{v_1\varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(G_1)} \{[4{n^2}_2\, {d_{G_1}}^2(u_1) + {d_{G_2}}^2(v_1) + 4n_2 d_{G_1}(u_1)d_{G_2}(v_1) + 4{n^2}_2\, {d_{G_1}}^2(u_2) + {d_{G_2}}^2(v_2) + 4n_2 d_{G_1}(u_2)d_{G_2}(v_2)] + 2[4{n^2}_2 d_{G_1}(u_1)d_{G_1}(u_2) + 2n_2 d_{G_1}(u_1)d_{G_2}(v_2) + 2n_2 d_{G_1}(u_2)d_{G_2}(v_1) + d_{G_2}(v_1)d_{G_2}(v_2)]\}$$
$$= \sum_{v_1\varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(G_1)} \{[4{n^2}_2\,({d_{G_1}}^2(u_1) + {d_{G_1}}^2(u_2)) + ({d_{G_2}}^2(v_1) + {d_{G_2}}^2(v_2)) + 4n_2 d_{G_1}(u_1)d_{G_2}(v_1) + 4n_2 d_{G_1}(u_2)d_{G_2}(v_2)]\} + 8{n^4}_2 M_2(G_2) + 8e_2{n^2}_2 \sum_{u_1u_2\varepsilon E(G_1)} [d_{G_1}(u_1) + d_{G_1}(u_2)] + 8e_1{e^2}_2$$
$$= 4{n^4}_2 F(G_1) + 2n_2 e_1 M_1(G_2) + 8{n^2}_2 e_2 M_1(G_1) + 8{n^4}_2 M_2(G_1) + 8{n^2}_2 e_2 M_1(G_1) + 8e_1{e^2}_2$$
$$= 4{n^4}_2 F(G_1) + 2n_2 e_1 M_1(G_2) + 16{n^2}_2 e_2 M_1(G_1) + 8{n^4}_2 M_2(G_1) + 8e_1{e^2}_2$$

$$\sum 2'' = \sum_{v_1\varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(R_k(G_1))\\ u_1 \varepsilon V(G_1) u_2 \varepsilon V(R_k(G_1)-V(G_1))}} [d(u_1,v_1) + d(u_2,v_2)]^2$$

$$= \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(R_k(G_1)) \\ u_1 \varepsilon V(G_1) u_2 \varepsilon V(R_k(G_1)-V(G_1))}} \Big[n_2 d_{R_{k(G_1)}}(u_1) + d_{G_2}(v_1) + n_2 d_{R_{k(G_1)}}(u_2)\Big]^2$$

$$= \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(R_k(G_1)) \\ u_1 \varepsilon V(G_1) u_2 \varepsilon V(R_k(G_1)-V(G_1))}} \big[2n_2 d_{G_1}(u_1) + d_{G_2}(v_1) + 2n_2\big]^2$$

$$= \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(R_k(G_1)) \\ u_1 \varepsilon V(G_1) u_2 \varepsilon V(R_k(G_1)-V(G_1))}} [4n^2{}_2 d^2{}_{G_1}(u_1) + d^2{}_{G_2}(v_1) + 4n^2{}_2 + 4n_2 d_{G_1}(u_1) d_{G_2}(v_1) + 4n_2 d_{G_2}(v_1) + 8n^2{}_2 d_{G_1}(u_1)]$$

$$= 4n^4{}_2 F(G_1) + 2n_2 e_1 M_1(G_2) + 8e_1 n^4{}_2 + 8n^2{}_2 e_2 M_1(G_1) + 16 e_1 e_2 n^2{}_2 + 8n^4{}_2 M_1(G_1)$$

$$\sum 3 = \sum_{v_1=v_2=v\varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(G_1) \\ u_1u_2 \varepsilon V(R_k(G_1))-V(G_1)}} [d(u_1,v_1) + d(u_2,v_2)]^2$$

$$= n_2(k-1)e_1(2n_2+2n_2)^2$$

$$= 16 e_1 n^3{}_2 (k-1)$$

$$\sum 4 = \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(R_k(G_2))} [d(u_1,v_1) + d(u_2,v_2)]^2$$

$$= [2n_2+2n_2]^2 \sum_{i=1}^{n_2-1} (2n_2-2i)(k-2)e_1 + [2n_2+2n_2]^2 \sum_{i=1}^{n_2-1} (2n_2-2i)e_1$$

$$= 32n^2{}_2 e_1(k-1) \sum_{i=1}^{n_2-1} (n_2-i)$$

So, the result is

$$= 8n^4{}_2 F(G_1) + 40n^2{}_2 e_2 M_1(G_1) + 20 n_2 e_1 M_1(G_2) + n_1 HM(G_2) + 8n^4{}_2[M_1(G_1) + M_2(G_1) + e_1] + 8e_1e_2[e_2 + 2n^2{}_2] + 16 e_1 n^3{}_2(k-1) + 32n^2{}_2 e_1 (k-1) \sum_{i=1}^{n_2-1}(n_2-i).$$

Theorem 3.3. For $\boldsymbol{k} \geq \mathbf{1}$, the Forgotten index of generalized S-sum graph under the lexicographic product is

$$F(G_1[G_2]_{s_k}) = 6n_2{}^2 e_2 M_1(G_1) + n_1 F(G_2) + 6n_2 e_1 M_1(G_2) + n_2{}^4 F(G_1) + 8n_2{}^4 e_1 + 8n^3{}_2 e_1(k-1) + 16 n_2{}^2 e_1 (k-1) \sum_{i=1}^{n_2-1}(n_2-i)$$

Proof:

Let $d(u,v) = d_{G_1[G_2]_{s_k}}(u,v)$ be the graph of the degree's vertex (u,v) in the graph $G_1[G_2]_{s_k}$.

$$F(G_1[G_2]_{s_k}) = \sum_{(u,v)\epsilon E(G_1[G_2])_{s_k}} [d(u)+d(v)]^3$$

$$= \sum_{(u_1,v_1),(u_2,v_2)\epsilon E(G_1[G_2])_{s_k}} [d(u_1,v_1)]^2 + [d(u_2,v_2)]^2$$

$$= \sum_{u_1=u_2=u\varepsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [d(u_1,v_1)]^2 + [d(u_2,v_2)]^2 + \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(S_k(G_1))} [d(u_1,v_1)]^2 + [d(u_2,v_2)]^2$$

$$+ \sum_{v_1=v_2=v\varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(G_1) \\ u_1u_2\varepsilon V(S_k(G_1))-V(G_1)}} [d(u_1,v_1)]^2 + [d(u_2,v_2)]^2 + \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(S_k(G_2))} [d(u_1,v_1)]^2 + [d(u_2,v_2)]^2$$

$$= \sum 1 + \sum 2 + \sum 3 + \sum 4$$

Now first we calculate

$$\sum 1 = \sum_{u_1=u_2=u\varepsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [d(u_1,v_1)]^2 + [d(u_2,v_2)]^2$$

$$= \sum_{u\varepsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [n_2 d_{G_1}(u) + d_{G_2}(v_1)]^2 + [n_2 d_{G_1}(u) + d_{G_2}(v_2)]^2$$

$$= \sum_{u\varepsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [n^2{}_2 d^2{}_{G_1}(u) + d^2{}_{G_2}(v_1) + 2n_2 d_{G_1}(u) d_{G_2}(v_1) + n^2{}_2 d^2{}_{G_1}(u) + d^2{}_{G_2}(v_2) + 2n_2 d_{G_1}(u) d_{G_2}(v_2)$$

$$= \sum_{u\varepsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [\, 2n^2{}_2 d^2{}_{G_1}(u) + (d^2{}_{G_2}(v_1) + d^2{}_{G_2}(v_2)) + 2n_2 d_{G_1}(u)\,(d_{G_2}(v_1) + d_{G_2}(v_2))]$$

$$= 2n^2{}_2e_2M_1(G_1) + n_1F(G_2) + 4n_2e_1M_1(G_2)$$

$$\sum 2 = \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u_1u_2 \varepsilon E(S_k(G_1))} [d(u_1, v_1)]^2 + [d(u_2, v_2)]^2$$

$$= \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{\substack{u \varepsilon v(G_1), a \varepsilon E(G_1) \\ u \text{ and } a \text{ are incident}}} [d(u, v_1)]^2 + [d(a, v_2)]^2$$

$$= \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{\substack{u \varepsilon v(G_1), a \varepsilon E(G_1) \\ u \text{ and } a \text{ are incident}}} [n_2 d_{G_1}(u) + d_{G_2}(v_1)]^2 + [2n_2]^2$$

$$= \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u \varepsilon v(G_1)} \left(d_{G_1}(u)\right) \{[n_2 d_{G_1}(u) + d_{G_2}(v_1)]^2 + [2n_2]^2\}$$

$$= \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u \varepsilon v(G_1)} \left(d_{G_1}(u)\right) \left[n^2{}_2 d^2{}_{G_1}(u) + d^2{}_{G_2}(v_1) + 2n_2 d_{G_1}(u) d_{G_2}(v_1) + 4n^2{}_2\right]$$

$$= \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u \varepsilon v(G_1)} \left[n^2{}_2 d^3{}_{G_1}(u) + d_{G_1}(u)\, d^2{}_{G_2}(v_1) + 2n_2 d^2{}_{G_1}(u) d_{G_2}(v_1) + 4n^2{}_2 d_{G_1}(u)\right]$$

$$= n_2{}^4F(G_1) + 8n_2{}^4e_1 + 2n_2e_1M_1(G_2) + 4n^2{}_2e_2M_1(G_1)$$

$$\sum 3 = \sum_{v_1 = v_2 = v \epsilon V(G_2)} \sum_{\substack{u_1u_2 \epsilon E(G_1) \\ u_1u_2 \epsilon V(S_k(G_1)) - V(G_1)}} [d(u_1, v_1)]^2 + [d(u_2, v_2)]^2$$

$$= n_2(k-1)e_1([2n_2]^2 + [2n_2]^2) = 8e_1n^3{}_2(k-1)$$

$$\sum 4 = \sum_{v_1 \epsilon V(G_2)} \sum_{v_2 \epsilon V(G_2)} \sum_{u_1u_2 \epsilon E(S_k(G_2))} [d(u_1, v_1)]^2 + [d(u_2, v_2)]^2$$

$$= ([2n_2]^2 + [2n_2]^2) \sum_{i=1}^{n_2-1} (2n_2 - 2i)(k-2)e_1 + ([2n_2]^2 + [2n_2]^2) \sum_{i=1}^{n_2-1} (2n_2 - 2i)e_1$$

$$= 16\, n_2{}^2e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i)$$

So, the result is

$$= 6n^2{}_2e_2M_1(G_1) + n_1F(G_2) + 6n_2e_1M_1(G_2) + n_2{}^4F(G_1) + 8n_2{}^4e_1 + 8e_1n^3{}_2(k-1) + 16n_2{}^2e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i)$$

Theorem 3.4. For $\boldsymbol{k \geq 1}$, the Forgotten index of generalized R-sum graph under the lexicographic product is

$$F\left(G_1[G_2]_{R_k}\right) = 8n_2{}^4F(G_1) + 12n_2e_1M_1(G_2) + 24n_2{}^2e_2\, M_1(G_1) + n_1F(G_2) + 8e_1n_2{}^4 + 8e_1n^3{}_2(k-1) + 16n_2{}^2e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i)$$

Proof:

Let $d(u, v) = d_{G_1[G_2]_{R_k}}(u, v)$ be the graph of the degree's vertex (u, v) in the graph $G_1[G_2]_{R_k}$.

$$FM\left(G_1[G_2]_{R_k}\right) = \sum_{(u,v) \epsilon E(G_1[G_2])_{R_k}} [d(u_1, v_1)]^2 + [d(u_2, v_2)]^2$$

$$= \sum_{(u_1,v_1),(u_2,v_2) \epsilon E(G_1[G_2])_{R_k}} [d(u_1, v_1)]^2 + [d(u_2, v_2)]^2$$

$$= \sum_{u_1 = u_2 = u \varepsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [d(u, v_1)]^2 + [d(u, v_2)]^2 + \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u_1u_2 \varepsilon E(R_k(G_1))} [d(u_1, v_1)]^2 + [d(u_2, v_2)]^2$$

$$+ \sum_{v_1 = v_2 = v \varepsilon V(G_2)} \sum_{\substack{u_1u_2 \varepsilon E(G_1) \\ u_1u_2 \varepsilon V(R_k(G_1)) - V(G_1)}} [d(u_1, v_1)]^2 + [d(u_2, v_2)]^2 + \sum_{v_1 \varepsilon V(G_2)} \sum_{v_2 \varepsilon V(G_2)} \sum_{u_1u_2 \varepsilon E(R_k(G_2))} [d(u_1, v_1)]^2 + [d(u_2, v_2)]^2$$

$$= \sum 1 + \sum 2 + \sum 3 + \sum 4$$

Now first we calculate

$$\sum 1 = \sum_{u_1 = u_2 = u \varepsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [d(u, v_1)]^2 + [d(u, v_2)]^2$$

$$= \sum_{u \varepsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [n_2 d_{R_kG_1}(u) + d_{G_2}(v_1)]^2 + [n_2 d_{R_kG_1}(u) + d_{G_2}(v_2)]^2$$

$$= \sum_{u \varepsilon V(G_1)} \sum_{v_1v_2 \varepsilon E(G_2)} [2n_2 d_{G_1}(u) + d_{G_2}(v_1)]^2 + [2n_2 d_{G_1}(u) + d_{G_2}(v_2)]^2$$

$$= \sum_{u\varepsilon V(G_1)} \sum_{v_1v_2\varepsilon E(G_2)} 4n^2{}_2 d^2{}_{G_1}(u) + d^2{}_{G_2}(v_1) + 4n_2 d_{G_1}(u) d_{G_2}(v_1) + 4n^2{}_2 d^2{}_{G_1}(u) + d^2{}_{G_2}(v_2) + 4n_2 d_{G_1}(u) d_{G_2}(v_2)$$

$$= \sum_{u\varepsilon V(G_1)} \sum_{v_1v_2\varepsilon E(G_2)} 8n^2{}_2 d^2{}_{G_1}(u) + (d^2{}_{G_2}(v_1) + d^2{}_{G_2}(v_2)) + 4n_2 d_{G_1}(u) d_{G_2}(v_1) + 4n_2 d_{G_1}(u)(d_{G_2}(v_1) + d_{G_2}(v_2))$$

$$= 8n^2{}_2 e_2 M_1(G_1) + n_1 F(G_2) + 8n_2 e_1 M_1(G_2).$$

$$\sum 2 = \sum_{v_1\varepsilon V(G_2)} \sum_{v_2\varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E((G_1)) \\ u_1\varepsilon V(G_1) u_2\varepsilon V(R_k(G_1)-V(G_1))}} [d(u_1,v_1)]^2 + [d(u_2,v_2)]^2 + \sum_{v_1\varepsilon V(G_2)} \sum_{v_2\varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(R_k(G_1)) \\ u_1\varepsilon V(G_1) u_2\varepsilon V(R_k(G_1)-V(G_1))}} [d(u_1,v_1)]^2 + [d(u_2,v_2)]^2$$

$$\sum 2 = \sum 2' + \sum 2''$$

$$\sum 2' = \sum_{v_1\varepsilon V(G_2)} \sum_{v_2\varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(G_1) \\ u_1\varepsilon V(G_1) u_2\varepsilon V(R_k(G_1)-V(G_1))}} [d(u_1,v_1)]^2 + [d(u_2,v_2)]^2$$

$$= \sum_{v_1\varepsilon V(G_2)} \sum_{v_2\varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(G_1)} [n_2 d_{R_kG_1}(u_1) + d_{G_2}(v_1)]^2 + [n_2 d_{R_kG_1}(u_2) + d_{G_2}(v_2)]^2$$

$$= \sum_{v_1\varepsilon V(G_2)} \sum_{v_2\varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(G_1)} [2n_2 d_{G_1}(u_1) + d_{G_2}(v_1)]^2 + [2n_2 d_{G_1}(u_2) + d_{G_2}(v_2)]^2$$

$$= \sum_{v_1\varepsilon V(G_2)} \sum_{v_2\varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(G_1)} [4n^2{}_2\, d_{G_1}{}^2(u_1) + d_{G_2}{}^2(v_1) + 4n_2 d_{G_1}(u_1) d_{G_2}(v_1) + 4n^2{}_2\, d_{G_1}{}^2(u_2) + d_{G_2}{}^2(v_2) + 4n_2 d_{G_1}(u_2) d_{G_2}(v_2)]$$

$$= \sum_{v_1\varepsilon V(G_2)} \sum_{v_2\varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(G_1)} [4n^2{}_2\,(d_{G_1}{}^2(u_1) + d_{G_1}{}^2(u_2)) + (d_{G_2}{}^2(v_1) + d_{G_2}{}^2(v_2)) + 4n_2 d_{G_1}(u_1) d_{G_2}(v_1) + 4n_2 d_{G_1}(u_2) d_{G_2}(v_2)]$$

$$= 4n^4{}_2 F(G_1) + 2n_2 e_1 M_1(G_2) + 8n^2{}_2 e_2 M_1(G_1)$$

$$\sum 2'' = \sum_{v_1\varepsilon V(G_2)} \sum_{v_2\varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(R_k(G_1)) \\ u_1\varepsilon V(G_1) u_2\varepsilon V(R_k(G_1)-V(G_1))}} [d(u_1,v_1)]^2 + [d(u_2,v_2)]^2$$

$$= \sum_{v_1\varepsilon V(G_2)} \sum_{v_2\varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(R_k(G_1)) \\ u_1\varepsilon V(G_1) u_2\varepsilon V(R_k(G_1)-V(G_1))}} \left[n_2 d_{R_{k(G_1)}}(u_1) + d_{G_2}(v_1)\right]^2 + \left[n_2 d_{R_{k(G_1)}}(u_2)\right]^2$$

$$= \sum_{v_1\varepsilon V(G_2)} \sum_{v_2\varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(R_k(G_1)) \\ u_1\varepsilon V(G_1) u_2\varepsilon V(R_k(G_1)-V(G_1))}} [2n_2 d_{G_1}(u_1) + d_{G_2}(v_1)]^2 + [2n_2]^2$$

$$= \sum_{v_1\varepsilon V(G_2)} \sum_{v_2\varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(R_k(G_1)) \\ u_1\varepsilon V(G_1) u_2\varepsilon V(R_k(G_1)-V(G_1))}} [4n^2{}_2 d^2{}_{G_1}(u_1) + d^2{}_{G_2}(v_1) + 4n_2 d_{G_1}(u_1) d_{G_2}(v_1) + 4n^2{}_2]$$

$$= 4n^4{}_2 F(G_1) + 2n_2 e_1 M_1(G_2) + 8e_1 n^4{}_2 + 8n^2{}_2 e_2 M_1(G_1).$$

$$\sum 3 = \sum_{v_1=v_2=v\varepsilon V(G_2)} \sum_{\substack{u_1u_2\varepsilon E(G_1) \\ u_1u_2\varepsilon V(R_k(G_1))-V(G_1)}} [d(u_1,v_1)]^2 + [d(u_2,v_2)]^2$$

$$n_2(k-1)e_1[(2n_2)^2 + (2n_2)^2] = 8e_1 n^3{}_2(k-1)$$

$$\sum 4 = \sum_{v_1\varepsilon V(G_2)} \sum_{v_2\varepsilon V(G_2)} \sum_{u_1u_2\varepsilon E(R_k(G_2))} [d(u_1,v_1)]^2 + [d(u_2,v_2)]^2$$

$$= [(2n_2)^2 + (2n_2)^2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)(k-2)e_1 + [(2n_2)^2 + (2n_2)^2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)e_1$$

$$= 16n^2{}_2 e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i)$$

So, the result is

$$= 8n^4{}_2 F(G_1) + 12n_2 e_1 M_1(G_2) + 24n^2{}_2 e_2 M_1(G_1) + n_1 F(G_2) + 8e_1 n^4{}_2 + 8e_1 n^3{}_2(k-1) + 16n^2{}_2 e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i).$$

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APPLICATIONS

Here In this section, we compute the hyper and forgotten index of the generalized F-sum Graphs for $k = 1$ as follow:

1. $HM\left(G_1[G_2]_{s_1}\right) = 8n^2{}_2 e_2 M_1(G_1) + 8n_2 e_1 M_1(G_2) +$

$n_1HM(G_2) + n^4{}_2F(G_1) + 2n_2e_1M_1(G_2) + 8n^4{}_2e_1 + 16n^2{}_2e_1e_2 + 4n^4{}_2M_1(G_1)$.

2. $HM(G_1[G_2]_{R_1}) = 8n^4{}_2F(G_1) + 40n^2{}_2e_2M_1(G_1) + 20n_2e_1M_1(G_2) + n_1HM(G_2) + 8n^4{}_2[M_1(G_1)+M_2(G_1) + e_1] + 8e_1e_2[e_2 + 2n^2{}_2]$
3. $F(G_1[G_2]_{s_1}) = 6n^2{}_2e_2M_1(G_1) + n_1F(G_2) + 6n_2e_1M_1(G_2) + n^4{}_2F(G_1)+8e_1n^4{}_2$
4. $F(G_1[G_2]_{R_1}) = 8n^4{}_2F(G_1) + 12n_2e_1M_1(G_2) + 24n^2{}_2e_2M_1(G_1) + n_1F(G_2) + +8e_1n^4{}_2$

These results are computed by Anandkumar et. al [IJPAM: 112(2017); 239-252] and S, Akhter et. al. [AKCE: 14(2017); 70-79] who only worked for k = 1.

COMPARISON AND CONCLUSION

Let $G_1 = p_n$ and $G_2 = p_m$ be two graphs, then we have $n_1 = n, n_2 = m, e_1 = n-1, e_2 = m-1,\ M_1(G_1) = 4n-6, M_1(G_2) = 4m-6,\ M_2(G_1) = 4(n-2), M_2(G_2) = 4(m-2),\ F(G_1) = 8n-14, F(G_2) = 8m-14, HM(G_1) = 16n-30,\ HM(G_2) = 16m-30$.
We produce Table 1, Table 2, Table 3 and Table 4 by applying these values to the above deduced outcomes in Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4 and Figure 5, Figure 6, Figure 7 and Figure 8 offer graphical representations of the above-mentioned results in tables. The results of generalized F-Sum graphs with $k = \{1, 2, 3, 4\}$ are summarized in Figure 9.

For k = 1

(a) $HM(p_n[p_m]_{s_1}) = 32m^4n - 46m^4 + 48m^3n - 64m^3 - 8m^2n + 24m^2 - 44mn + 60m - 30n$,
(b) $HM(p_n[p_m]_{R_1}) = 136m^4n - 232m^4 + 176m^3n - 256m^3 - 88m^2n + 168m^2 - 24mn + 136m - 118n - 8$,
(c) $F(p_n[p_m]_{s_1}) = 16m^4n - 22m^4 + 24m^3n - 36m^3 + 12m^2 - 16mn + 36m - 14n$,
(d) $F(p_n[p_m]_{R_1}) = 72m^4n - 120m^4 + 96m^3n - 144m^3 - 48m^2n + 96m^2 - 64mn + 72m - 14n$.

For k = 2

(a) $HM(p_n[p_m]_{s_2}) = 32m^4n - 46m^4 + 64m^3n - 80m^3 - 8m^2n + 24m^2 - 44mn + 60m - 30n + (32m^2n - 32m^2)\sum_{i=1}^{m-1}(m-i)$,
(b) $HM(p_n[p_m]_{R_2}) = 136m^4n - 232m^4 + 192m^3n - 272m^3 - 88m^2n + 168m^2 - 24mn + 136m - 118n - 8 + (32m^2n - 32m^2)\sum_{i=1}^{m-1}(m-i)$,
(c) $F(p_n[p_m]_{s_2}) = 16m^4n - 22m^4 + 32m^3n - 44m^3 + 12m^2 - 16mn + 36m - 14n + (16m^2n - 16m^2)\sum_{i=1}^{m-1}(m-i)$,
(d) $F(p_n[p_m]_{R_2}) = 72m^4n - 120m^4 + 104m^3n - 152m^3 - 48m^2n + 96m^2 - 64mn + 72m - 14n + (16m^2n - 16m^2)\sum_{i=1}^{m-1}(m-i)$,

For k=3

(a) $HM(p_n[p_m]_{s_3}) = 32m^4n - 46m^4 + 80m^3n - 96m^3 - 8m^2n + 24m^2 - 44mn + 60m - 30n + (64m^2n - 16m^2)\sum_{i=1}^{m-1}(m-i)$,
(b) $HM(p_n[p_m]_{R_3}) = 136m^4n - 232m^4 + 208m^3n - 288m^3 - 88m^2n + 168m^2 - 24mn + 136m - 118n - 8 + (64m^2n - 64m^2)\sum_{i=1}^{m-1}(m-i)$,
(c) $F(p_n[p_m]_{s_3}) = 16m^4n - 22m^4 + 40m^3n - 52m^3 + 12m^2 - 16mn + 36m - 14n + (32m^2n - 32m^2)\sum_{i=1}^{m-1}(m-i)$,
(d) $F(p_n[p_m]_{R_3}) = 72m^4n - 120m^4 + 112m^3n - 160m^3 - 48m^2n + 96m^2 - 64mn + 72m - 14n + (32m^2n - 32m^2)\sum_{i=1}^{m-1}(m-i)$.

For k=4

(a) $HM(p_n[p_m]_{s_4}) = 32m^4n - 46m^4 + 96m^3n - 112m^3 - 8m^2n + 24m^2 - 44mn + 60m - 30n + (96m^2n - 96m^2)\sum_{i=1}^{m-1}(m-i)$,
(b) $HM(p_n[p_m]_{R_4}) = 136m^4n - 232m^4 + 224m^3n - 304m^3 - 88m^2n + 168m^2 - 24mn + 136m - 118n - 8 + (964m^2n - 96m^2)\sum_{i=1}^{m-1}(m-i)$,
(c) $F(p_n[p_m]_{s_4}) = 16m^4n - 22m^4 + 48m^3n - 60m^3 + 12m^2 - 16mn + 36m - 14n + (48m^2n - 48m^2)\sum_{i=1}^{m-1}(m-i)$,
(d) $F(p_n[p_m]_{R_4}) = 72m^4n - 120m^4 + 120m^3n - 168m^3 - 48m^2n + 96m^2 - 64mn + 72m - 14n + (48m^2n - 48m^2)\sum_{i=1}^{m-1}(m-i)$.

TABLE 1

Numerical Comparison for generalized graph for Hyper Zagreb index and F-index $G(P_n[P_m]_{F_k})$ for $k=1$

[m,n]	$HM(G)(P_n[P_m]_{S_k})$	$HM(G)(P_n[P_m]_{R_k})$	$F(G)(P_n[P_m]_{S_k})$	$F(G)(P_n[P_m]_{R_k})$
(5, 5)	91900	350682	46760	182290
(5, 6)	117450	455244	59666	237756
(5, 7)	143000	559806	72572	293222
(5, 8)	168550	664368	85478	348688
(5, 9)	194100	768930	98384	404154
(5, 10)	219650	873492	111290	459620
(6, 5)	184074	705098	93410	369962
(6, 6)	235332	915940	119220	481884
(6, 7)	286590	1126782	145030	593806
(6, 8)	337848	1337624	170840	705728
(6, 9)	389106	1548466	196650	817650
(6, 10)	440364	1759308	222460	929572
(7, 5)	332028	1275866	168280	673218
(7, 6)	424594	1658172	214802	876204
(7, 7)	517160	2040478	261324	1079190
(7, 8)	609726	2422784	307846	1282176
(7, 9)	702292	2805090	354368	1485162
(7, 10)	794858	3187396	400890	1688148
(8, 5)	554602	2136618	280922	1131514
(8, 6)	709356	2777844	358604	1471980
(8, 7)	864110	3419070	436286	1812446
(8, 8)	1018864	4060296	513968	2152912
(8, 9)	1173618	4701522	591650	2493378
(8, 10)	1328372	5342748	669332	2833844
(9, 5)	873372	3371738	442280	1790066
(9, 6)	1117242	4384876	564594	2327964
(9, 7)	1361112	5398014	686908	2865862
(9, 8)	1604982	6411152	809222	3403760
(9, 9)	1848852	7424290	931536	3941658
(9, 10)	2092722	8437428	1053850	4479556
(10, 5)	1312650	5076362	664690	2699850
(10, 6)	1679380	6603204	848516	3510396
(10, 7)	2046110	8130046	1032342	4320942
(10, 8)	2412840	9656888	1216168	5131488
(10, 9)	2779570	11183730	1399994	5942034
(10,10)	3146300	12710572	1583820	6752580

TABLE 2

Numerical Comparison of generalized graphs for Hyper Zagreb index and F-index $G(P_n[P_m]_{F_k})$ for $k=2$

[m,n]	$HM(G)(P_n[P_m]_{S_k})$	$HM(G)(P_n[P_m]_{R_k})$	$F(G)(P_n[P_m]_{S_k})$	$F(G)(P_n[P_m]_{R_k})$
(5, 5)	131900	390682	66760	207090
(5, 6)	167450	505244	84666	267556
(5, 7)	203000	619806	102572	328022
(5, 8)	238550	734368	120478	388488
(5, 9)	274100	848930	138384	448954
(5, 10)	309650	963492	156290	509420
(6, 5)	267018	788042	134882	418346
(6, 6)	339012	1019620	171060	540636
(6, 7)	411006	1251198	207238	662926
(6, 8)	483000	1482776	243416	785216
(6, 9)	554994	1714354	279594	907506
(6, 10)	626988	1945932	315772	1029796
(7, 5)	485692	1429530	245112	759458
(7, 6)	616674	1850252	310842	981652
(7, 7)	747656	2270974	376572	1203846

(7, 8)	878638	2691696	442302	1426040
(7, 9)	1009620	3112418	508032	1648234
(7, 10)	1140602	3533140	573762	1870428
(8, 5)	816746	2398762	411994	1274874
(8, 6)	1037036	3105524	522444	1648108
(8, 7)	1257326	3812286	632894	2021342
(8, 8)	1477616	4519048	743344	2394576
(8, 9)	1697906	5225810	853794	2767810
(8, 10)	1918196	5932572	964244	3141044
(9, 5)	1293276	3791642	652232	2015570
(9, 6)	1642122	4909756	827034	2605956
(9, 7)	1990968	6027870	1001836	3196342
(9, 8)	2339814	7145984	1176638	3786728
(9, 9)	2688660	8264098	1351440	4377114
(9, 10)	3037506	9382212	1526242	4967500
(10, 5)	1952650	5716362	984690	3039050
(10, 6)	2479380	7403204	1248516	3929596
(10, 7)	3006110	9090046	1512342	4820142
(10, 8)	3532840	10776888	1776168	5710688
(10, 9)	4059570	12463730	2039994	6601234
(10, 10)	4586300	14150572	2303820	7491780

TABLE 3

Numerical Comparison of generalized graphs for Hyper Zagreb and F-index $G(P_n[P_m]_{F_k})$ for $k = 3$

[m,n]	$HM(G)(P_n[P_m]_{S_k})$	$HM(G)(P_n[P_m]_{R_k})$	$F(G)(P_n[P_m]_{S_k})$	$F(G)(P_n[P_m]_{R}k)$
(5, 5)	171900	430682	86760	227090
(5, 6)	217450	555244	109666	292556
(5, 7)	263000	679806	132572	358022
(5, 8)	308550	804368	155478	423488
(5, 9)	354100	928930	178384	488954
(5, 10)	399650	1053492	201290	554420
(6, 5)	349962	870986	176354	459818
(6, 6)	442692	1123300	222900	592476
(6, 7)	535422	1375614	269446	725134
(6, 8)	628152	1627928	315992	857792
(6, 9)	720882	1880242	362538	990450
(6, 10)	813612	2132556	409084	1123108
(7, 5)	639356	1583194	321944	836290
(7, 6)	808754	2042332	406882	1077692
(7, 7)	978152	2501470	491820	1319094
(7, 8)	1147550	2960608	576758	1560496
(7, 9)	1316948	3419746	661696	1801898
(7, 10)	1486346	3878884	746634	2043300
(8, 5)	1078890	2660906	543066	1405946
(8, 6)	1364716	3433204	686284	1811948
(8, 7)	1650542	4205502	829502	2217950
(8, 8)	1936368	4977800	972720	2623952
(8, 9)	2222194	5750098	1115938	3029954
(8, 10)	2508020	6522396	1259156	3435956
(9, 5)	1713180	4211546	862184	2225522
(9, 6)	2167002	5434636	1089474	2868396
(9, 7)	2620824	6657726	1316764	3511270
(9, 8)	3074646	7880816	1544054	4154144
(9, 9)	3528468	9103906	1771344	4797018
(9, 10)	3982290	10326996	1998634	5439892
(10, 5)	2592650	6356362	1304690	3359050
(10, 6)	3279380	8203204	1648516	4329596
(10, 7)	3966110	10050046	1992342	5300142

(10, 8)	4652840	11896888	2336168	6270688
(10, 9)	5339570	13743730	2679994	7241234
(10, 10)	6026300	15590572	3023820	8211780

TABLE 4.
Numerical Comparison of generalized graphs for Hyper Zagreb and F-index $G(P_n[P_m]_{F_k})$ for $k = 4$

[m,n]	$HM(G)(P_n[P_m]_{S_k})$	$HM(G)(P_n[P_m]_{R_k})$	$F(G)(P_n[P_m]_{S_k})$	$F(G)(P_n[P_m]_{R_k})$
(5, 5)	211900	470682	106760	247090
(5,6)	267450	605244	134666	317556
(5, 7)	323000	739806	162572	388022
(5, 8)	378550	874368	190478	458488
(5, 9)	434100	1008930	218384	528954
(5, 10)	489650	1143492	246290	599420
(6, 5)	432906	953930	217826	501290
(6, 6)	546372	1226980	274740	644316
(6, 7)	659838	1500030	331654	787342
(6, 8)	773304	1773080	388568	930368
(6, 9)	886770	2046130	445482	1073394
(6, 10)	1000236	2319180	502396	1216420
(7, 5)	793020	1736858	398776	913122
(7, 6)	1000834	2234412	502922	1173732
(7, 7)	1208648	2731966	607068	1434342
(7, 8)	1416462	3229520	711214	1694952
(7, 9)	1624276	3727074	815360	1955562
(7, 10)	1832090	4224628	919506	2216172
(8, 5)	1341034	2923050	674138	1537018
(8, 6)	1692396	3760884	850124	1975788
(8, 7)	2043758	4598718	1026110	2414558
(8, 8)	2395120	5436552	1202096	2853328
(8, 9)	2746482	6274386	1378082	3292098
(8, 10)	3097844	7112220	1554068	3730868
(9, 5)	2133084	4631450	1072136	2435474
(9, 6)	2691882	5959516	1351914	3130836
(9, 7)	3250680	7287582	1631692	3826198
(9, 8)	3809478	8615648	1911470	4521560
(9, 9)	4368276	9943714	2191248	5216922
(9, 10)	4927074	11271780	2471026	5912284
(10, 5)	3232650	6996362	1624690	3679050
(10, 6)	4079380	9003204	2048516	4729596
(10, 7)	4926110	11010046	2472342	5780142
(10, 8)	5772840	13016888	2896168	6830688
(10, 9)	6619570	15023730	3319994	7881234
(10, 10)	7466300	17030572	3743820	8931780

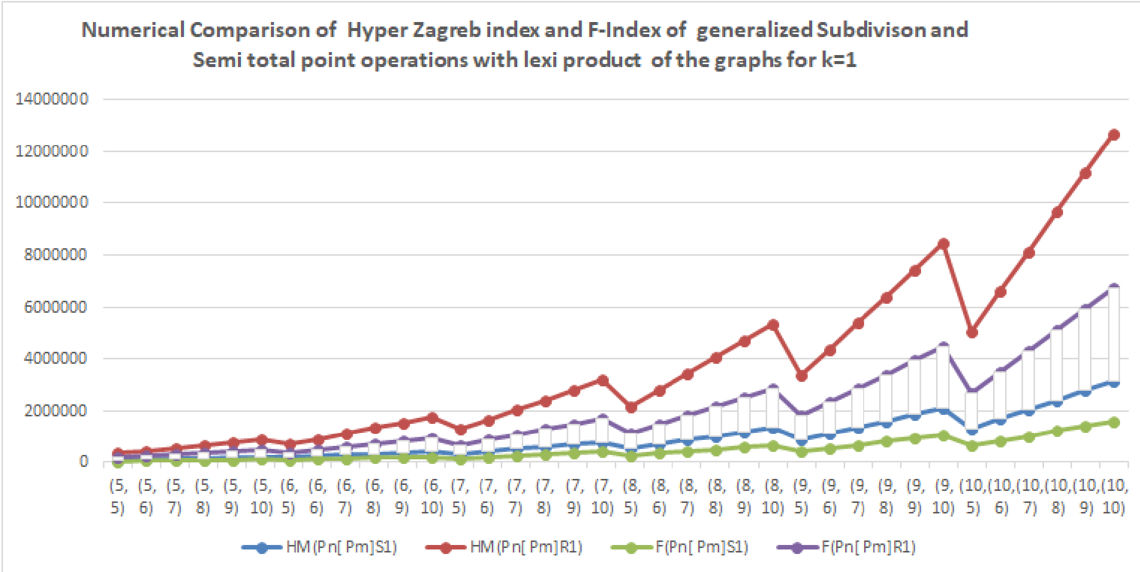

FIGURE 5.

For $k = 1$, a numerical comparison of the Hyper Zagreb index and the F-index of generalized subdivisionand semi-total point operations with the lexicographic product of the graphs

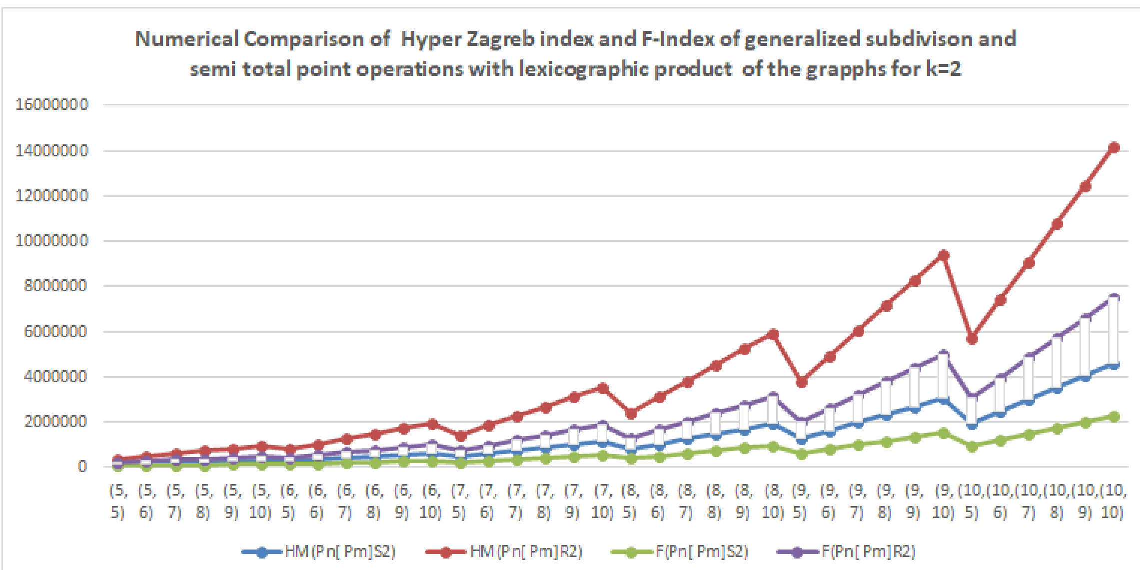

FIGURE. 6.

For $k = 2$, a numerical comparison of the Hyper Zagreb index and the F-index of generalized subdivisionand semi-total point operations with the lexicographic product of the graphs

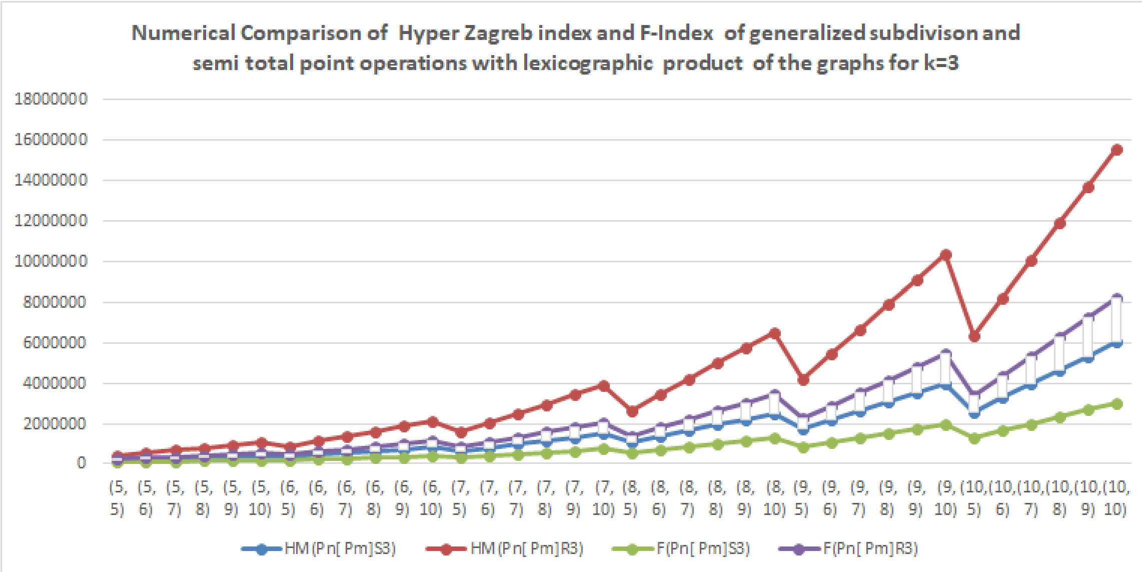

FIGURE. 7.
For $k = 3$, a numerical comparison of the Hyper Zagreb index and the F-index of generalized subdivisionand semi-total point operations with the lexicographic product of the graphs is performed.

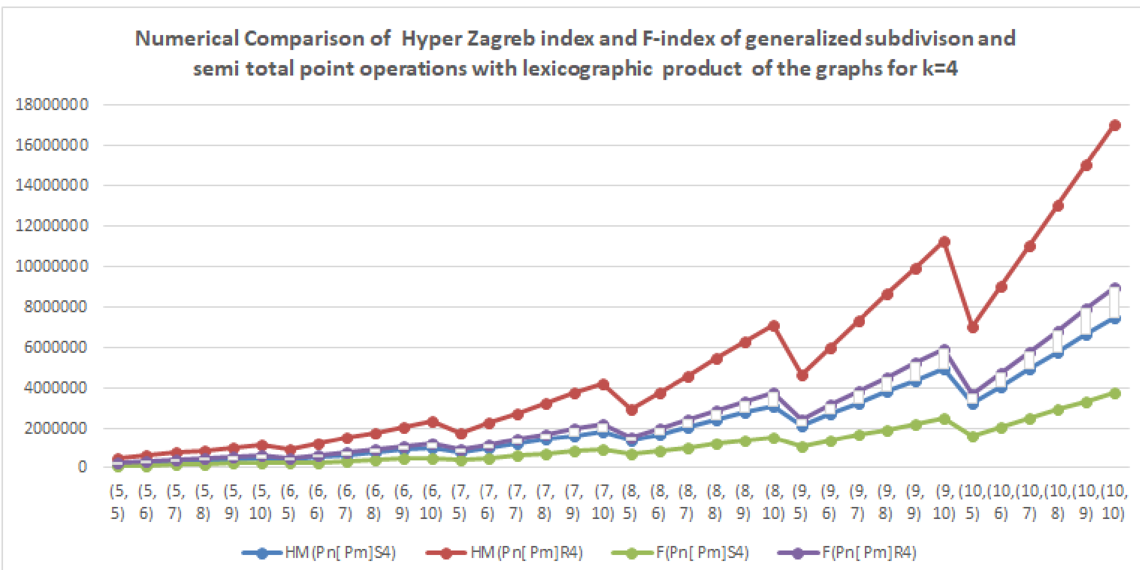

FIGURE. 8.
For $k = 4$, a numerical comparison of the Hyper Zagreb index and the F-index of generalized subdivisionand semi total point operations with the lexicographic product of the graphs

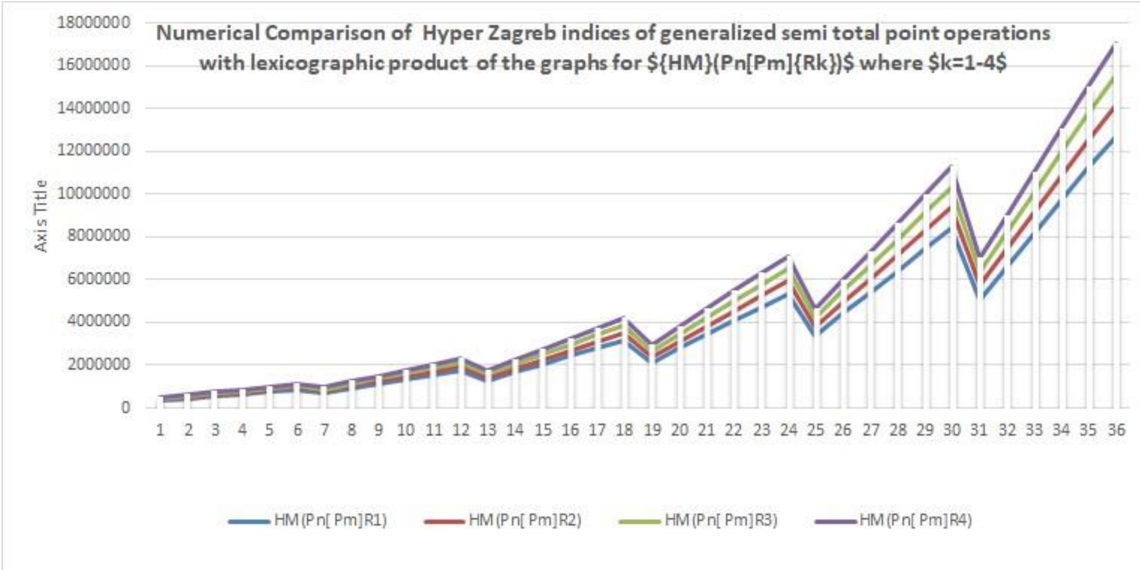

FIGURE. 9.
For k = {1, 2, 3, 4}, a numerical comparison of the Hyper Zagreb index of generalized semi total point operations with the graphs' lexicographic product.

We used generalized subdivision, generalized semi total point operation, and lexicographicproduct to show the result of the Hyper Zagreb index and Forgotten index of graphs $G_1[G_2]_{F_k}$. The Forgotten index and the Hyper Zagreb index in the form $(M_i(G_1[G_2]_{F_k})$, where i = (1, 2). To compare these results, we plot the graphs of each derived values individually, then plot a combined graph for $k = \{1, 2, 3, 4\}$. It is obvious from this result that the Hyper Zagreb index for generalized semi total point $(M_2(G_1[G_2]_{F_R})$, performs better than other values for $k = \{1, 2, 3, 4\}$.

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