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HAMMING INDEX OF DERIVED GRAPHS

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Abstract. Let G be a simple and undirected graph with n vertices. The row entries corresponding to the vertex v in the adjacency matrix of G are denoted by $s(v)$. The number of positions at which the elements of the strings $s(u)$ and $s(v)$ differ is the Hamming distance between them. The sum of Hamming distances between all the pairs of vertices is the Hamming index. The proposed study finds various bounds for Hamming index. It also computes the Hamming index generated by the adjacency matrix of a few derived graphs.

1. Introduction

Let G be a simple graph of size m and order n. The degree of a vertex $v \in V(G)$ denoted as $\deg_G(v)$, is the number of edges incident with v. An $r-$regular graph G is a graph with $\deg_G(v_i) = r, \ \forall \ v_i \in V(G)$. The distance between two vertices u, v of G, denoted by $d_G(u, v)$ is the number of edges in the shortest $u - v$ path. We denote vertex u adjacent to vertex v by $u \sim v$ and u not adjacent to v by $u \nsim v$. Neighbors of a vertex $v \in V(G)$, are the adjacent vertices of v, denoted by $N_G(v)$. A common neighbor of the vertices $u, v \in V(G)$ is the vertex which is adjacent to both the vertices u and v and the set of common neighbors of u, v is represented by $N_G(u, v)$. The set of non-common neighbors of the vertices $u, v \in V(G)$ are the vertices which are non-adjacent to any one of the vertices u and v. The adjacency matrix of G of order n is an $n \times n$ matrix $A(G)$, whose elements are a_{ij}, where a_{ij} is 1 if $a_i \sim a_j$ in G and it is 0 if $a_i \nsim a_j$ in G.

For the strings $u = u_1, u_2, \ldots, u_n$ and $v = v_1, v_2, \ldots, v_n$ of equal length n, the Hamming distance between them is the number of positions at which the values of u and v differ. A string $v = v_1, v_2, \ldots, v_n$ is binary if each v_i is either 0 or 1 for every i. The weight of a string is defined as the number of $1's$ in it. For a graph, each vertex v can be labeled by a binary string $s(v)$ which is the row elements of the adjacency matrix. The Hamming distance between $s(u)$ and $s(v)$ is denoted by $H_d(s(u), s(v))$. A graph G with pair of vertices (u, v) is called Hamming graph [1, 2] if $H_d(s(u), s(v)) = d_G(u, v), \ \forall u, v \in V(G)$. For the vertices u, v of a graph, $H_d(u, v) = H_d(s(u), s(v))$.

The derived graphs of a graph G are the graphs obtained after performing unary operation on G such as addition or deletion of vertices or edges. Consider the graph $G(n, m)$. The line graph $L(G)$ of G is a graph with m vertices and two

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vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. The total graph $T(G)$ of G is a graph with $m+n$ vertices and two vertices of $T(G)$ are adjacent if and only if the corresponding vertices or edges adjacent. The subdivision graph $S(G)$ is obtained by inserting a new vertex into each edge of G. The vertex semi-total $T_1(G)$ can be obtained by adding a vertex e'_i into each edge e_i of G and e'_i is adjacent to the vertices which are incident by e_i. The edge semi-total graph $T_2(G)$ can be obtained by inserting a vertex into each edge of G and the new vertices are adjacent if the corresponding edges are adjacent in G. The splitting graph $S'(G)$ is a graph obtained from adding a new vertex v'_i into each vertex v_i of G and the new vertex v'_i is adjacent to the vertices which are adjacent to v_i.

2. Preliminaries

Definition 2.1. The first Zagreb index of a graph G,

$$M_1(G) = \sum_{v_iv_j \in E(G)} (\deg_G(v_i) + \deg_G(v_j)) = \sum_{i=1}^{n} (\deg_G(v_i))^2.$$

Definition 2.2. The second Zagreb index of a graph G,

$$M_2(G) = \sum_{v_iv_j \in E(G)} \deg_G(v_i)\deg_G(v_j).$$

Definition 2.3. [12] The first general Zagreb index of a graph G,

$$M^{\alpha}(G) = \sum_{i=1}^{n} (\deg_G(v_i))^{\alpha},$$

for any real number α.

Proposition 2.4. [4] *For two vertices v_i, v_j of a graph G,*
$H_d(v_i, v_j : G) = \deg_G(v_i) + \deg_G(v_j) - 2|N_G(v_i, v_j)|$, *where* $|N_G(v_i, v_j)|$ *is the number vertices adjacent to both v_i and v_j.*

Definition 2.5. [3] The Hamming index $H_A(G)$ of a graph G of order n with respect to adjacency matrix is,
$H_A(G) = \sum_{1 \le i < j \le n} H_d(v_i, v_j)$.

Theorem 2.6. [5] *The Hamming index of an $r-$regular graph G of order n is* $H_A(G) = nr(n-r)$.

Hamming indices of various graphs are found in [2, 3, 4, 6, 7]. For more information on Hamming graphs, one can refer [8, 9]. In this paper, the authors study few properties and bounds for Hamming Index. This paper also gives the Hamming index of few derived graphs of a graph such as line graph, total graph, subdivision graph, vertex semi-total graph, edge semi-total graph, and splitting graph. The Hamming indices of these derived graphs are in terms of Hamming index, number of vertices, number of edges, the first general Zagreb index and the second Zagreb index of the original graph. The proposed study is purely mathematical and the Hamming index of classes of graph found in this paper may find some applications in computer science and coding theory.

3. Bounds for Hamming index

Theorem 3.1. *Let G be a graph with n vertices. Then $H_A(G) \leq \frac{n^3}{4}$. Equality holds for $\frac{n}{2}-$ regular graph with n vertices.*

Proof.

$$H_A(G) = \sum_{1 \leq i < j \leq n} (\deg_G(v_i) + \deg_G(v_j) - 2|N_G(v_i, v_j)|).$$

But, $|N_G(v_i, v_j)| = \sum_{i=1}^{n} \binom{\deg_G(v_i)}{2}$.
Therefore,

$$H_A(G) = \sum_{i=1}^{n} \deg_G(v_i)(n - \deg_G(v_i)).$$

Since, $\deg_G(v_i)(n - \deg_G(v_i))$ is maximum if $\deg_G(v_i) = \frac{n}{2}$,

$$H_A(G) \leq \frac{n^3}{4}.$$

If G is a $\frac{n}{2}-$ regular graph of order n, then $H_A(G) = \frac{n^3}{4}$. □

We also obtain sharp bounds for the Hamming index of a connected regular graph.

Theorem 3.2. *Let G be a connected regular graph with n vertices. Then,*

$$n(n-1) \leq H_A(G) \leq \begin{cases} \dfrac{n(n^2-1)}{4}, & \text{if } n \text{ is odd} \\ \dfrac{n^3}{4}, & \text{if } n \text{ is even.} \end{cases}$$

The first inequality holds for $G = K_n$ and right inequality holds for $\frac{n}{2}-$regular graph with n vertices if n is even, $\frac{n\pm1}{2}-$regular graph with n vertices if n is odd.

Proof. Let G be a connected regular graph having n vertices with regularity s. From Theorem 2.6, we have $H_A(G) = ns(n-s)$ where $1 \leq s \leq n-1$. Let $s(n-s) = f(s)$.
Among the values $[1, n-1]$ of s, the expression $f(s)$ is minimum when $s = (n-1)$ for a connected graph.

$$\implies H_A(G) \geq n(n-1). \tag{3.1}$$

The lower bound sharpness occurs for a complete graph since $H_A(K_n) = n(n-1)$ [2].
To achieve the upper bound inequality, we will consider two cases.
Case (i): When n is odd.
Suppose $n = 4t+1,\ t = 0, 1, 2, \ldots$, then $f(s)$ attains maximum at $s = \dfrac{n \pm 1}{2}$. But $\dfrac{n+1}{2}$ is odd, if n is of the form $4t+1$ and there doesn't exist a regular graph of odd degree with odd number of vertices.

Therefore $f(s)$ attains maximum at $s = \frac{n-1}{2}$.

$$\implies H_A(G) \leq \frac{n(n^2-1)}{4}. \tag{3.2}$$

Similarly, for $n = 4t+3,\ t = 0, 1, 2, \ldots$, we note that the function $f(s)$ attains maximum at $s = \dfrac{n+1}{2}$.

$$\implies H_A(G) \leq \frac{n(n^2-1)}{4}. \tag{3.3}$$

Case (ii): When n is even, the function $f(s)$ attains maximum at $s = \dfrac{n}{2}$.

$$\implies H_A(G) \leq \frac{n^3}{4}. \tag{3.4}$$

On combining equations (3.1), (3.2), (3.3) and (3.4), we get

$$n(n-1) \leq H_A(G) \leq \begin{cases} \dfrac{n(n^2-1)}{4}, & \text{if n is odd} \\ \dfrac{n^3}{4}, & \text{if n is even.} \end{cases}$$

If G is a $\frac{n}{2}-$regular graph with n vertices and n is even, then $H_A(G) = \frac{n^3}{4}$. If G is $\frac{n\pm1}{2}-$regular graph with n vertices and n is odd, then $H_A(G) = \frac{n(n^2-1)}{4}$. □

Theorem 3.3. *Let e be an edge of a graph G of size m and order n. Then,*

$$H_A(G-e) - 2n + 6 \leq H_A(G) \leq H_A(G+e) + 2n - 6.$$

Lower bound equality holds if the edge removed is adjacent to the vertices of degree $n-1$. Upper bound equality holds if the edge is added to the vertices of degree $n-2$.

Proof. Hamming index of a graph G is given by,

$$H_A(G) = \sum_{1 \leq i < j \leq n} (\deg_G(v_i) + \deg_G(v_j) - 2|N_G(v_i, v_j)|).$$

But,

$$\sum_{1 \leq i < i \leq n} (\deg_G(v_i) + \deg_G(v_j)) = 2m(n-1)$$

and

$$\sum_{1 \leq i < j \leq n} |N_G(v_i, v_j)| = \sum_{1 \leq i \leq n} \binom{\deg_G(v_i)}{2}.$$

Then,

$$H_A(G) = 2m(n-1) - 2\sum_{1 \leq i \leq n} \binom{\deg_G(v_i)}{2}. \tag{3.5}$$

Let $e = (v_x, v_y)$ be the edge that is removed from G. Then,

$$H_A(G-e) = 2(m-1)(n-1) - 2\sum_{1\le i\le n}\binom{\deg_{(G-e)}(v_i)}{2}. \tag{3.6}$$

Note that,

$$\binom{\deg_{(G-e)}(v_x)}{2} = \binom{\deg_G(v_x)}{2} - \deg_G(v_x) + 1$$

and

$$\binom{\deg_{(G-e)}(v_y)}{2} = \binom{\deg_G(v_y)}{2} - \deg_G(v_y) + 1.$$

Then equation (3.6) implies,

$$\begin{aligned} H_A(G-e) &= 2m(n-1) - 2n + 2 - 2\sum_{1\le i\le n}\binom{\deg_G(v_i)}{2} + 2\deg_G(v_x) - 2 \\ &+ 2\deg_G(v_y) - 2 \\ &\le 2m(n-1) - 2n + 2 - 2\sum_{1\le i\le n}\binom{\deg_G(v_i)}{2} + 2(n-1) - 2 \\ &+ 2(n-1) - 2 \\ &= H_A(G) + 2n - 6. \end{aligned}$$

This proves the lower bound inequality. If the edge removed is adjacent to the vertices of degree $n-1$. Then,

$$\begin{aligned} H_A(G-e) &= 2m(n-1) - 2n + 2 - 2\sum_{1\le i\le n}\binom{\deg_G(v_i)}{2} + 2(n-1) - 2 \\ &+ 2(n-1) - 2 \\ &= H_A(G) + 2n - 6. \end{aligned}$$

To prove upper bound inequality, consider equation (3.5),

$$H_A(G) = 2m(n-1) - 2\sum_{1\le i\le n}\binom{\deg_G(v_i)}{2}.$$

Let $e = (v_x, v_y)$ be the edge that is added to G. Then,

$$H_A(G+e) = 2(m+1)(n-1) - 2\sum_{1\le i\le n}\binom{\deg_{(G-e)}(v_i)}{2}. \tag{3.7}$$

Note that,

$$\binom{\deg_{(G+e)}(v_x)}{2} = \binom{\deg_G(v_x)}{2} + \deg_G(v_x)$$

and

$$\binom{\deg_{(G+e)}(v_y)}{2} = \binom{\deg_G(v_y)}{2} + \deg_G(v_y).$$

Then equation (3.7) implies,

$$\begin{aligned} H_A(G+e) &= 2(m+1)(n-1) - 2\sum_{1\leq i\leq n}\binom{\deg_G(v_i)}{2} - 2\deg_G(v_x) - 2\deg_G(v_y) \\ &\geq 2m(n-1) + 2n - 2 - 2\sum_{1\leq i\leq n}\binom{\deg_G(v_i)}{2} - 2(n-2) - 2(n-2) \\ &= H_A(G) - 2n + 6. \end{aligned}$$

If the edge is added to the vertices of degree $n-2$. Then,

$$\begin{aligned} H_A(G+e) &= 2m(n-1) + 2n - 2 - 2\sum_{1\leq i\leq n}\binom{\deg_G(v_i)}{2} - 2(n-2) - 2(n-2) \\ &= H_A(G) - 2n + 6. \end{aligned}$$

□

Theorem 3.4. *Let w be a vertex of a graph G of size m and order n. Then,*

(1) $H_A(G-w) = H_A(G) + d(d+1) - 2(m+nd) + 2\sum_d \deg_G(v_i)$, *where d is the degree of w and $v_i's$ are the vertices which are adjacent to the vertex w.*

(2) $H_A(G+w) = H_A(G) + n^2 + n - 2m$.

Proof. Let G be a graph of size m and order n

(1) Let w be any vertex of G with degree d that is removed from G. Then

$$\begin{aligned} \sum_{1\leq i<j\leq n-1}(\deg_{(G-w)}(v_i) + \deg_{(G-w)}(v_j)) = &\sum_{1\leq i<j\leq n-1}(\deg_G(v_i) + \deg_G(v_j)) \\ &- 2m - 2nd + 4d. \end{aligned}$$

and

$$|N_{(G-w)}(v_i, v_j)| = |N_G(v_i, v_j)| - \frac{d^2}{2} + \frac{3d}{2} - \sum_d \deg_G(v_i).$$

Here $v_i's$ are the vertices which are adjacent to the vertex w. Therefore,

$$H_A(G-w) = H_A(G) + d(d+1) - 2(m+nd) + 2\sum_d \deg_G(v_i).$$

(2) Let w be any vertex that is added to G. Then

$$\begin{aligned} \sum_{1\leq i<j\leq n+1}(\deg_{(G+w)}(v_i) + \deg_{(G+w)}(v_j)) = &\sum_{1\leq i<j\leq n-1}(\deg_G(v_i) + \deg_G(v_j)) \\ &+ 2\binom{n}{2} + n\deg_{(G+w)}(w) \\ &+ \sum_{i=1}^{n}(\deg_G(v_i) + 1). \end{aligned}$$

and

$$|N_{(G+w)}(v_i, v_j)| = |N_G(v_i, v_j)| + \binom{n}{2} + \sum_{i=1}^{n} deg_G(v_i).$$

But, $\deg_{(G+w)}(w) = n$. Therefore,

$$H_A(G + w) = H_A(G) + n^2 + n - 2m.$$

□

4. Hamming index of some derived graphs

Theorem 4.1. *The Hamming index of line graph $L(G)$ of a graph $G(n, m)$ is given by*

$$H_A(L(G)) = (m + 4)M_1(G) - 2M_2(G) - M^3(G) - 2m^2 - 4m$$

Proof. Let $\{e'_1, e'_2, \ldots, e'_m\}$ be the set of vertices of $L(G)$ corresponding to the edges $\{e_1, e_2, \ldots, e_m\}$ of G respectively.

Consider $e'_i, e'_j \in V(L(G))$. Let v_i, v_j be the end vertices of edge e_i and v_k, v_l be the end vertices of edge e_j in G. Then

$$\deg_{L(G)}(e_i)' = \deg_G(v_i) + \deg_G(v_j) - 2.$$

From Theorem 2.4, we have

$$\begin{aligned} H_d(e'_i, e'_j : T(G)) = (&\deg_G(v_i) + \deg_G(v_j)) + (\deg_G(v_k) + \deg_G(v_l)) - 4 \\ &- 2\,|N_G(e_i, e_j)|\,. \end{aligned}$$

Where,

$$\deg_G(v_i) + \deg_G(v_j) + \deg_G(v_k) + \deg_G(v_l) = (m - 1)\sum_{i=1}^{n}(\deg_G(v_i))^2.$$

$$\begin{aligned} |N_G(e_i, e_j)| &= \sum_{i=1}^{n}(\deg_G(v_i))^3 + 2\sum_{v_iv_j\in E(G)} \deg_G(v_i)\deg_G(v_j) \\ &\quad - 4\sum_{i=1}^{n}(\deg_G(v_i))^2 + 4m \\ &= M^3(G) + 2M_2(G) - 4M_1(G) + 4m. \end{aligned}$$

Therefore,

$$H_A(L(G)) = (m + 4)M_1(G) - 2M_2(G) - M^3(G) - 2m^2 - 4m. \tag{4.1}$$

□

Theorem 4.2. *Let $G(n, m)$ be a graph of order n and size m. The Hamming index of total graph of order $n + m$ is given by*

$$H_A(T(G)) = H_A(G) + 4m^2 + 2mn + (m + n - 3)M_1(G) - M^3(G) - 2M_2(G).$$

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ and $\{e'_1, e'_2, \ldots, e'_m\}$ be the set of vertices of $T(G)$ corresponding to the vertices $\{v_1, v_2, \ldots, v_n\}$ and edges $\{e_1, e_2, \ldots, e_m\}$ of G respectively.

Case (i): Consider $v_i, v_j \in V(T(G))$. Then $\deg_{T(G)}(v_i) = 2\deg_G(v_i)$ and

$$N_{T(G)}(v_i, v_j) = \begin{cases} N_G(v_i, v_j) + 1, & \text{if } v_i \sim v_j \text{ in } T(G) \\ N_G(v_i, v_j), & \text{if } v_i \sim v_j \text{ in } T(G). \end{cases}$$

Noting that the number of pairs of adjacent vertices is equal to m,

$$\begin{aligned}\sum_{v_i, v_j \in V(G)} H_d(v_i, v_j : T(G)) &= m(2\deg_G(v_i) + 2\deg_G(v_j)) - 2(|N_G(v_i, v_j)| + 1) \\ &+ \left(\binom{n}{2} - m\right)(2\deg_G(v_i) + 2\deg_G(v_j)) \\ &- 2\left(|N_G(v_i, v_j)|\right) \\ &= H_d\left(v_i, v_j : G\right) - 2m \\ &+ \binom{n}{2}(\deg_G(v_i) + \deg_G(v_j)) \\ &= H_A(G) + (n-2)2m. \qquad (4.2)\end{aligned}$$

Case (ii): Consider $e'_i, e'_j \in V(T(G))$. Let v_i, v_j be the end vertices of edge e_i and v_k, v_l be the end vertices of edge e_j in G. Then

$$\begin{aligned}\deg_{T(G)}(e'_i) + \deg_{T(G)}(e'_j) &= \deg_G(v_i) + \deg_G(v_j) + \deg_G(v_k) + \deg_G(v_l) \\ &= (m-1)\sum_{i=1}^{n}(\deg_G(v_i))^2.\end{aligned}$$

and

$$\begin{aligned}|N_G(e_i, e_j)| &= \sum_{i=1}^{n}(\deg_G(v_i))^3 + 2\sum_{v_i v_j \in E(G)}(\deg_G(v_i)\deg_G(v_j)) \\ &- 4\sum_{i=1}^{n}(\deg_G(v_i))^2 + 4m \\ &= \frac{1}{2}(M^3(G) + 2M_2(G) - 4M_1(G) + 4m).\end{aligned}$$

$$\sum_{e'_i, e'_j \in V(T(G))} H_d(e_i{}', e_j{}' : T(G)) = (m+3)M_1 - M^3(G) - 2M_2(G) - 4m. \qquad (4.3)$$

Case (iii): Consider $v_i, e_j{}' \in V(T(G))$. Let v_k and v_l be end vertices of the edge e_j in G. Then,

$$\begin{aligned}\deg_{T(G)}(v_i) + \deg_{T(G)}(e'_j) &= 2\deg_G(v_i) + \deg_G(v_k) + \deg_G(v_l) \\ &= 2m\sum_{i=1}^{n}\deg_G(v_i) + n\sum_{i=1}^{n}(\deg_G(v_i))^2 \\ &= 4m^2 + nM_1(G)\end{aligned}$$

and

$$|N_G(v_i, e_j{}')| = 3\sum_{i=1}^{n}(\deg_G(v_i))^2 - 4m = 3M_1(G) - 4m.$$

This implies,

$$\sum_{v_i, e'_j \in V(T(G))} H_d(v_i, e_j{}') = 4m(m+2) + (n-6)M_1(G). \tag{4.4}$$

On combining equations (4.2), (4.3) and (4.4), we get

$$H_A(T(G)) = H_A(G) + 4m^2 + 2mn + (m+n-3)M_1(G) - M^3(G) - 2M_2(G).$$

□

Theorem 4.3. *For a subdivision graph $S(G)$ of $G(n,m)$,*

$$H_A(S(G)) = H_A(G) + 4m^2 - 4m + 2mn$$

.

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of graph G. For every edge e_j, let $e_j{}'$ be the corresponding vertex in $S(G)$, $j = 1, 2, \ldots, m$.
Case (i): Consider $v_i, v_j \in V(S(G))$. Then, $\deg_{S(G)}(v_i) = \deg_G(v_i)$, for $1 \leq i \leq n$ and
$|N_{S(G)}(v_i, v_j)| = \begin{cases} 1, & \text{if } v_i \sim v_j \text{ in } G \\ 0, & \text{if } v_i \not\sim v_j \text{ in } G. \end{cases}$
Thus,

$$\begin{aligned}\sum_{v_i, v_j \in V(S(G))} H_d(v_i, v_j : S(G)) &= \deg_G(v_i) + \deg_G(v_j) - 2 \\ &\quad - 2|N_G(v_i, v_j)| + 2|N_G(v_i, v_j)| \\ &= H_A(G) - 2m + 2\sum_{i=1}^{n}\binom{\deg_G(v_i)}{2} \\ &= H_A(G) + M_1(G) - 4m. \end{aligned} \tag{4.5}$$

Case (ii): Consider $e_i{}', e_j{}' \in V(S(G))$. Then, $\deg_{S(G)}(e_i{}') = 2$, for $1 \leq i \leq m$ and $\sum\limits_{e_i{}', e_j{}' \in V(S(G))} |N_{S(G)}(e_i{}', e_j{}')| = \sum\limits_{x=1}^{n} \binom{deg_G(v_x)}{2} = \frac{1}{2}(M_1(G) - 2m)$.
Thus,

$$\sum_{e_i{}', e_j{}' \in V(S(G))} H_d(e_i{}', e_j{}' : S(G)) = 2m^2 - M_1(G). \tag{4.6}$$

Case (iii): Let $v_i, e_j{}' \in V(S(G))$. Then $\deg_{S(G)}(v_i) = \deg_G(v_i)$, $\deg_{S(G)}(e_j{}') = 2$ and $|N_{S(G)}(v_i, e_j{}')| = 0$. Thus,

$$\begin{aligned}\sum_{v_i, e_j{}' \in V(S(G))} H_d(v_i, e_j{}' : S(G)) &= \sum_{i=1}^{n} (m \deg_G(v_i) + 2m) \\ &= m(2m) + 2mn \\ &= 2m^2 + 2mn. \qquad (4.7)\end{aligned}$$

On combining equations (4.5), (4.6) and (4.7), we get
$H_A(S(G)) = H_A(G) + 4m^2 - 4m + 2mn.$ □

Theorem 4.4. *For a splitting graph $S'(G)$ of $G(n, m)$,*

$$H_A(S'(G)) = 5H_A(G) + 2nm.$$

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of graph G. For every vertex v_i, let v_i' be the corresponding vertex in $S'(G)$, $i = 1, 2, \ldots, n$.
Case (i): Consider $v_i, v_j \in V(S'(G))$. Then, $\deg_{S'(G)}(v_i) = 2\deg_G(v_i)$, for $1 \leq i \leq n$ and $|N_{S'(G)}(v_i, v_j)| = 2|N_G(v_i, v_j)|$. Therefore,

$$\sum_{v_i, v_j \in S'(V(G))} H_d(v_i, v_j : S'(G)) = 2H_A(G). \qquad (4.8)$$

Case (ii): Consider $v_i{}', v_j{}' \in V(S'(G))$. Then, $\deg_{S'(G)}(v_i{}') = \deg_G(v_i)$, for $1 \leq i \leq n$ and $N_{S'(G)}(v_i{}', v_j{}') = N_G(v_i, v_j)$. Therefore,

$$\sum_{v_i{}', v_j{}' \in S'(V(G))} H_d(v_i{}', v_j{}' : S'(G)) = H_A(G). \qquad (4.9)$$

Case (iii): Consider $v_i, v_j{}' \in V(S'(G))$. Then,
$\deg_{S'(G)}(v_i) = 2\deg_G(v_i)$, $\deg_{S'(G)}(v_j{}') = \deg_G(v_j)$ and

$$|N_{S'(G)}(v_i, v_j{}')| = \begin{cases} \deg_G(v_i), & \text{if } v_i = v_j{}' \\ |N_G(v_i, v_j)|, & \text{if } v_i \neq v_j{}'. \end{cases}$$

Therefore,

$$\sum_{v_i, v_j{}' \in V(S'(G))} H_d(v_i, v_j{}' : S'(G)) = 2H_A(G) + 2mn. \qquad (4.10)$$

On combining equations (4.8), (4.9) and (4.10), we get

$$H_A(S'(G)) = 5H_A(G) + 2mn.$$

□

Theorem 4.5. *For a vertex semi-total graph $T_1(G)$ of $G(n, m)$,*

$$H_A(T_1(G)) = H_A(G) - 3M_1(G) + 6m^2 + 4mn - 4m.$$

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of graph G. For every edge e_j, let $e_j{}'$ be the corresponding vertex in $T_1(G)$, $j = 1, 2, \ldots, m$.
Case (i): Consider $v_i, v_j \in V(T_1(G))$. Then, $\deg_{T_1(G)}(v_i) = 2\deg_G(v_i)$, for $1 \leq i \leq n$ and

$$|N_{T_1(G)}(v_i,v_j)| = \begin{cases} |N_G(v_i,v_j)+1|, & \text{if } v_i \sim v_j \text{ in } G \\ |N_G(v_i,v_j)|, & \text{if } v_i \nsim v_j \text{ in } G. \end{cases}$$

Thus,

$$\begin{aligned}\sum_{v_i,v_j \in V(T_1(G))} H_d(v_i,v_j : T_1(G)) &= \sum_{v_i,v_j \in V(T_1(G))} (2\deg_G(v_i) + 2\deg_G(v_j) \\ &- 2|N_G(v_i,v_j)|) - 2m \\ &= H_A(G) + 2mn - 4m. \end{aligned} \tag{4.11}$$

Case (ii): Consider $e_i', e_j' \in V(T_1(G))$. Then, $\deg_{T_1(G)}(e_i') = 2$, for $1 \leq i \leq m$ and

$\sum\limits_{e_i',e_j' \in V(T_1(G))} |N_{T_1(G)}(e_i',e_j')| = \sum\limits_{i=1}^{n} \binom{\deg_G(v_i)}{2} = \frac{1}{2}(M_1(G) - 2m).$

Thus,

$$\sum_{e_i',e_j' \in V(T_1(G))} H_d(e_i',e_j' : T_1(G)) = 2m^2 - M_1(G). \tag{4.12}$$

Case (iii): Let $v_i, e_j' \in V(T_1(G))$. Then $\deg_{T_1(G)}(v_i) = 2\deg_G(v_i)$, $\deg_{T_1(G)}(e_j') = 2$ and

$\sum\limits_{e_i',e_j' \in V(T_1(G))} |N_{T_1(G)}(v_i,e_j')| = \sum\limits_{i=1}^{n} (\deg_{T_1(G)}(i))^2$. Thus,

$$\sum_{v_i,e_j' \in V(T_1(G))} H_d(v_i,e_j' : T_1(G)) = 4m^2 + 2mn - 2M_1(G). \tag{4.13}$$

On combining equations (4.11), (4.12) and (4.13), we get

$H_A(T_1(G)) = H_A(G) - 3M_1(G) + 6m^2 + 4mn - 4m.$ □

Theorem 4.6. *For an edge semi-total graph $T_2(G)$ of $G(n,m)$,*

$$H_A(T_2(G)) = H_A(G) + (m+n)M_1(G) - M^3(G) - 2M_2(G) + 2m^2.$$

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of graph G. For every edge e_j, let e_j' be the corresponding vertex in $T_1(G)$, $j = 1, 2, \ldots, m$.

Case (i): Consider $v_i, v_j \in V(T_2(G))$. Then, $\deg_{T_2(G)}(v_i) = \deg_G(v_i)$, for $1 \leq i \leq n$ and

$$|N_{T_2(G)}(v_i,v_j)| = \begin{cases} 1, & \text{if } v_i \sim v_j \text{ in } G \\ 0, & \text{if } v_i \nsim v_j \text{ in } G. \end{cases}$$

Thus,

$$\begin{aligned}\sum_{v_i,v_j \in V(T_2(G))} H_d(v_i,v_j : T_2(G)) &= \sum_{v_i,v_j \in V(T_2(G))} (\deg_G(v_i) + \deg_G(v_j) \\ &- 2|N_G(v_i,v_j)|) \\ &+ \sum_{v_i,v_j \in V(T_2(G))} 2|N_G(v_i,v_j)| - 2m \\ &= H_A(G) + M_1(G) - 4m. \end{aligned} \tag{4.14}$$

Case (ii): Consider ${e_i}', {e_j}' \in V(T_1(G))$. Let v_i, v_j be the end vertices of edge e_i and v_k, v_l be the end vertices of edge e_j in G. Then, $\deg_{T_2(G)}({e_i}') = \deg_G(v_i) + \deg_G(v_j)$, for $1 \le i \le m$ and $\sum\limits_{{e_i}', {e_j}' \in V(T_2(G))} |N_{T_2(G)}({e_i}', {e_j}')| = \frac{1}{2}(M^3(G) + 2M_2(G) - 4M_1(G) + 4m)$.
Thus,

$$\sum_{{e_i}', {e_j}' \in V(T_2(G))} H_d({e_i}', {e_j}' : T_2(G)) = (m+3)M_1(G) - M^3(G) - 2M_2(G) - 4m. \tag{4.15}$$

Case (iii): Let $v_i, {e_j}' \in V(T_2(G))$. Let v_i, v_j be the end vertices of edge e_j in G. Then $\deg_{T_2(G)}(v_i) = \deg_G(v_i)$, $\deg_{T_2(G)}({e_j}') = \deg_G(v_i) + \deg_G(v_j)$ and $\sum\limits_{{e_i}', {e_j}' \in V(T_2(G))} |N_{T_2(G)}(v_i, {e_j}')| = 3M_1(G) - 4m$. Thus,

$$\sum_{v_i, {e_j}' \in V(T_1(G))} H_d(v_i, {e_j}' : T_1(G)) = (n-4)M_1(G) + 2m^2 + 8m. \tag{4.16}$$

On combining equations (4.14), (4.15) and (4.16), we get
$H_A(T_1(G)) = H_A(G) + (m+n)M_1(G) - M^3(G) - 2M_2(G) + 2m^2$. □

5. Conclusion

In this article, the authors have obtained an upper bound for Hamming index of a graph and the bounds for Hamming index of a regular graph in terms of number of vertices. The authors compared the Hamming index of a graph and Hamming index of a graph when an edge is removed or added. The authors also compared the Hamming index of a graph and Hamming index of a graph when a vertex is removed or added. Also, the authors have obtained Hamming index of line graph, total graph, subdivision graph, vertex semi-total graph, edge semi-total graph, and splitting graph, and subdivision graph by noting the degree of all the vertices and number of common neighbors between all the pairs of vertices.

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