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ARMS-PRODUCT AND LIMIT GRAPH OF GRAPHS

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Abstract. Graph theory have strong connections with other fields of mathematics. In this article we introduce two new concepts in graph theory by collaborating with number theory and real analysis. In the second section, a new graph product called the ARMS-product of graphs is introduced, in which the degree of each vertex and the greatest common divisor of these degrees determines the adjacency of the graph. ARMS-product of certain classes of graphs including regular graphs, paths, cycles, stars are determined. In section 3 we associate the concept of limit in real analysis with graph theory by defining a new graph called limit graph of a given graph. Limit graphs of certain standard graphs as well as graphs obtained through various graph theoretic operations are derived with suitable illustrations. Further we determine some graphs whose limit graphs are unique as well as not unique.

1. Introduction

In this paper first we define a new graph product called ARMS-product of two graphs where ARMS is the abbreviation of Akhil, Roy, Manju and Suresh. Numerous graph products exist in the literature that consider one or more parameters of the given graphs. Typically, the cartesian product of the vertex sets of the given graphs yields the vertex set of the new graph, and the edges of the new graph are determined by certain rules related to the graph's parameters or structure. Also, we define limit graph of a graph and further determine the graphs with unique and distinct limit graphs.

All graphs under our consideration are simple, connected and undirected. Let $G = (V(G), E(G))$ be a graph with order $n = |V(G)|$ and size $m = |E(G)|$. The **degree** of a vertex v_i in G is the number of edges incident on v_i and is denoted by $d(v_i)$, $deg_G(v_i)$ or simply d_i. The **distance** between two vertices u and v in G is the length of the shortest path joining them in G denoted by $d(u, v)$. For any connected graph G, nG represents a graph with n isomorphic copies of G. A graph with p vertices and q edges is denoted as a (p, q) graph. The $(1, 0)$ graph is called the **trivial** graph and the $(p, 0)$ graph is called an **empty** or **null** or **void graph**. An $n \times m$ **rook graph**, is an undirected graph that shows all possible moves that a rook chess piece may make on an $n \times m$ chessboard. A rook can travel between any two squares that share a row or column, and each square on a chessboard represents a vertex in the rook graph. Let G and H be simple graphs. A vertex function $f : V(G) \longrightarrow V(H)$ preserves adjacency if for every pair of adjacent vertices u and v in graph G, the vertices $f(u)$ and $f(v)$ are adjacent in graph H. Similarly, f preserves non-adjacency if $f(u)$ and $f(v)$ are non-adjacent whenever u and v are non-adjacent. A vertex bijection $f : V(G) \longrightarrow V(H)$ between the vertex sets of two simple graphs G and H is **structure preserving** if it preserves adjacency and non-adjacency. That is, for every pair of vertices in G, u and v are adjacent in $G \iff f(u)$ and $f(v)$ are adjacent in H. Two simple

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graphs G and H are **isomorphic**, denoted $G \cong H$, if there exists a structure preserving vertex bijection $f : V(G) \longrightarrow V(H)$. Such a function f between the vertex sets of G and H is called an isomorphism from G to H [2]. A **subgraph** of G is a graph all of whose vertices belong to $V(G)$ and all of whose edges belong to $E(G)$. Let G be a graph and S a nonempty subset of $V(G)$. A subgraph of G whose vertex set is S and all edges of G which have both their ends in S is known as the subgraph induced by S and is denoted as $\langle S\rangle$ or $G[S]$. Any subgraph induced by a set of vertices will be called a **vertex induced subgraph** or simply an **induced subgraph** [5]. A vertex u is a neighbor of v in G, if uv is an edge of G, and $u \neq v$. The set of all neighbors of v is the **open neighborhood** of v or the **neighbor set** of v, and is denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the **closed neighborhood** of v in G [1]. A tree in which one vertex say r is distinguished from others is called a **rooted tree**. The vertex r is called the root. In a rooted tree, the level (depth) of a vertex v, is the length of the unique path from the root to v. The **height** of a rooted tree is the length of the longest path from root. If a vertex u immediately precedes the vertex v on the path from root to v, then u is the parent of v and v is the child of u. A vertex v is said to be the **descendant** of a vertex u (also u is the ancestor of v) if u is on the unique path from the root to v. A leaf in a rooted tree is any vertex having no childeren and an **internal vertex** of a rooted tree is any vertex that is not a leaf. An m**-array tree** is a rooted tree in which each vertex has a maximum of m children. A **complete** m**-array tree** is an m-array tree in which each internal vertex has exactly m children and all leaves have the same depth. When $m = 2$, the corresponding complete m-array tree is called a **complete binary tree**. An m-array tree has atmost m^k vertices at level k. A **Tadpole graph** is denoted by $T_{m,n}$ we mean the graph obtained by joining a cycle graph C_m to a path graph P_n with a bridge [4]. We follow [5] for more terminologies and notations not mentioned here.

This article's goal is to investigate two new concepts in graph theory namely a novel graph product called ARMS-product by applying a well known concept in number theory viz. the greatest common divisor and the limit concept from the real analysis.

2. The ARMS-product of Graphs

In discrete mathematics, product of graphs are frequently used as tools in combinatorial constructions. Naturally, graph products can be used to study any graph invariant [6]. Graph products are helpful in the creation of numerous structural models and are also utilised in the development of various networks within the communication system domain. By approximating the structure of large-scale networks, it can also be utilised to produce some analytical estimations of their spectral features [3]. Graph products can be used to create a variety of network topologies for interconnection networks. Graph products based on adjacency have been the subject of numerous studies [7].

Definition 2.1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with order m and n respectively. The **ARMS-Product** of G_1 and G_2 denoted by $G_1 \boxtimes G_2$ is a graph G with vertex set $V = V_1 \times V_2$ and two vertices $x = (u_i, v_j)$ and $y = (u_k, v_l)$ are adjacent in G if:

(1) either $u_i = u_k$ and $\gcd(d_{G_2}(v_j), d_{G_2}(v_l)) = 1$ or
(2) $\gcd(d_{G_1}(u_i), d_{G_1}(u_k)) = 1$ and $v_j = v_l$

for all $i, k = 1, 2, \ldots, m$ with $i \neq k$ and $j, l = 1, 2, \ldots, n$ with $j \neq l$.

Remark 2.2. Following are some immediate observations from the definition.

(1) $\boxtimes$ is commutative.

(2) $\boxtimes$ doesnot preserve the connectedness. That is, even though for two connected graphs G_1 and G_2, $G_1 \boxtimes G_2$ need not be connected.

For example, consider $G_1 = C_4$ and $G_2 = P_3$. Here both C_4 and P_3 are connected, but $C_4 \boxtimes P_3$ is disconnected.

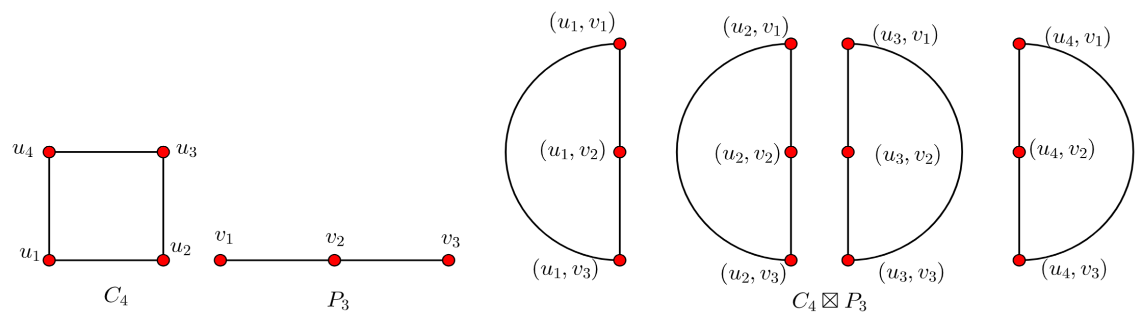

FIGURE 1. $C_4 \boxtimes P_3$

Next, the degree of an arbitrary vertex in $G_1 \boxtimes G_2$ is determined in terms of degrees of G_1 and G_2.

Let G_1 be a graph with vertex set V_1. Then for $u \in V_1$, $O(u)$ be the set defined by

$$O(u) = \{v \in V_1 : \gcd(d_{G_1}(u), d_{G_1}(v)) = 1 \text{ with } v \neq u\}.$$

Let $(u_i, v_j) \in G_1 \boxtimes G_2$, then

$$d_{G_1 \boxtimes G_2}((u_i, v_j)) = |O(u_i)| + |O(v_j)|$$

where $u_i \in V_1$ and $v_j \in V_2, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.
Next, we try to give a sufficient condition for the connectedness of the ARMS-product of two connected graphs.

Theorem 2.3. *Let G_1 and G_2 be two connected graphs. Then $G_1 \boxtimes G_2$ is connected if G_1 and G_2 has at least one pendant vertex.*

Proof. Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$ are the vertex sets of G_1 and G_2 respectively, where $m, n \geq 2$. Without loss of generality assume that $d_{G_1}(u_1) = d_{G_2}(v_1) = 1$. Consider $G_1 \boxtimes G_2$. We have to show that $G_1 \boxtimes G_2$ is connected. Consider the vertices $(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_n)$. Then (u_1, v_1) is adjacent to (u_1, v_j) $\forall\ j = 2, 3, \ldots, n$. This is due to the fact that $\gcd(d_{G_2}(v_1), d_{G_2}(v_j)) = 1$, irrespective of $d_{G_2}(v_j)$ $\forall\ j = 2, 3 \ldots, n$. Similarly it is clear that the vertex (u_2, v_1) is adjacent to (u_2, v_j) $\forall\ j = 2, 3, \ldots, n$. Continuing like this, the vertex (u_m, v_1) is adjacent to (u_m, v_j) $\forall\ j = 2, 3, \ldots, n$. These give rise to m components.

Now fix the vertex (u_1, v_1). Since $\gcd(d_{G_1}(u_1), d_{G_1}(u_k)) = 1\ \forall\ k = 2, 3, \ldots, m$ it is evident that the vertex (u_1, v_1) is adjacent to the vertices $(u_2, v_1), (u_3, v_1), \ldots, (u_m, v_1)$. We have each (u_i, v_1) is in the i^{th} component, where $i = 1, 2, \ldots, m$. That is (u_1, v_1) is in the 1^{st} component, (u_2, v_1) is in the 2^{nd} component and so on. Hence the resulting graph $G_1 \boxtimes G_2$ is a single connected component. This completes the proof. □

Remark 2.4. The condition given in Theorem 2.3 is not necessary. That is, the product $G_1 \boxtimes G_2$ is connected need not imply G_1 and G_2 having pendant vertices.

For example, in figure 2 both G_1 and G_2 have no pendant vertices. Again in figure 3, the graph $G_1 \boxtimes G_2$ is connected.

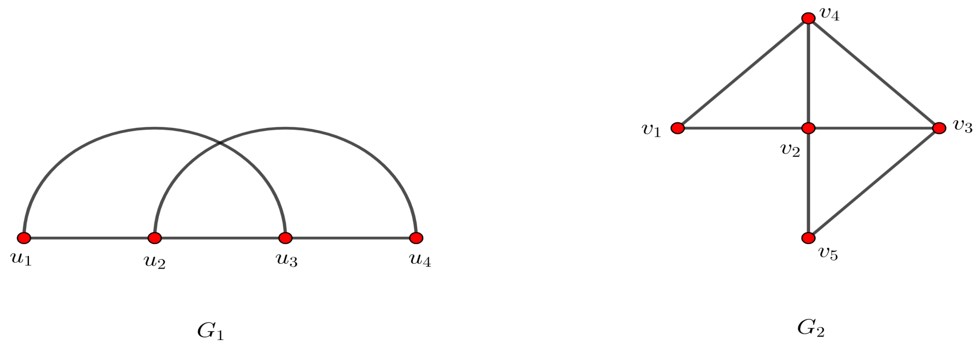

FIGURE 2. G_1 and G_2

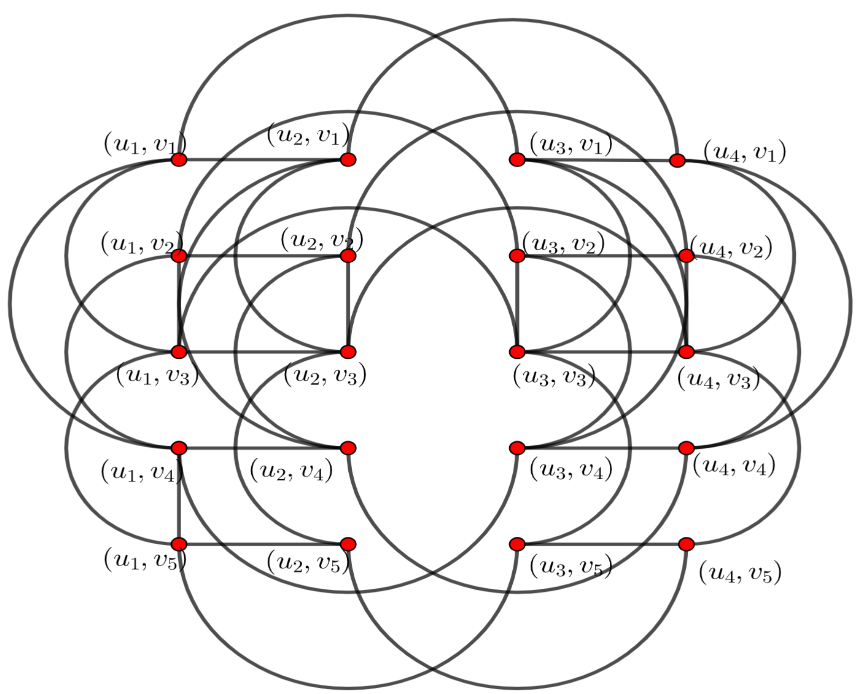

FIGURE 3. $G_1 \boxtimes G_2$

Corollary 2.5. *For $m, n \geq 2$, $P_m \boxtimes P_n$ is connected.*

Theorem 2.6. *Let $G_1 = K_{1,m}$ and $G_2 = K_{1,n}$ with $m, n \geq 2$, then $G_1 \boxtimes G_2$ is an $(m+1)\times(n+1)-$ rook graph.*

Proof. Let $V(K_{1,m}) = \{u_1, u_2, \ldots, u_{m+1}\}$ and $V(K_{1,n}) = \{v_1, v_2, \ldots, v_{n+1}\}$. Without loss of generality assume that u_1 and v_1 be the central vertices of G_1 and G_2 respectively. Consider $G_1 \boxtimes G_2$. Then (u_i, v_j) is adjacent to (u_i, v_l) $\forall$ $i = 1, 2, \ldots, m+1$ and $j, l = 1, 2, \ldots, n+1, j \neq l$. Again (u_i, v_j) is adjacent to (u_k, v_j) $\forall$ $i, k = 1, 2, \ldots, m+1, i \neq j$ and $j = 1, 2, \ldots, n+1$. The resulting graph has order $mn + m + n + 1$ and

$$\begin{aligned} d_{G_1 \boxtimes G_2}(u_i, v_j) &= |O(u_i)| + |O(v_j)| \\ &= m + n. \end{aligned}$$

This completes the proof. □

A 4×5 - rook graph is displayed in figure 4.

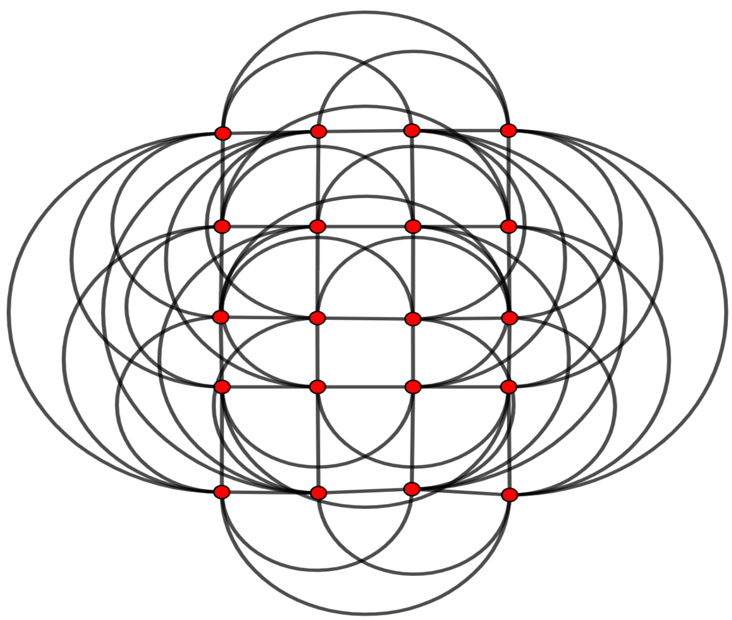

FIGURE 4. $K_{1,3} \boxtimes K_{1,4}$

Remark 2.7. From the above theorem, we can see that ARMS-product of star graphs is isomorphic to the cartesian product of two complete graphs. That is,

$$K_{1,m} \boxtimes K_{1,n} \cong K_{m+1} \square\, K_{n+1}.$$

Theorem 2.8. *Let G_1 and G_2 be two Eulerian graphs, then $G_1 \boxtimes G_2$ is totally disconnected.*

Proof. Let G_1 and G_2 be two Eulerian graphs with vertex sets $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. Euler's theorem states that a nonempty connected graph is Eulerian if and only if it has no vertices of odd degree [5]. Hence, for any two vertices (u_i, v_j) and $(u_k, v_l) \in G_1 \boxtimes G_2$, we have $\gcd\left(d_{G_1}(u_i), d_{G_1}(u_k)\right)$ and $\gcd\left(d_{G_2}(v_j), d_{G_2}(v_l)\right)$ is always an even integer which is at least 2.

Hence no two vertices in $G_1 \boxtimes G_2$ are adjacent. This results into a $(mn, 0)$ graph, which is totally disconnected. □

Theorem 2.9. *For any connected graph G_1 and a connected r-regular graph G_2 with $r \geq 2$, $G_1 \boxtimes G_2$ is disconnected.*

Proof. The proof of this theorem trivially holds. Since $\gcd(d_{G_2}(v_i), d_{G_2}(v_j)) = r \geq 2$ for all vertices $v_i, v_j \in V_2$ and $i, j = 1, 2, \ldots, n$ where V_2 is the vertex set of the r-regular graph G_2 and $n = |G_2|$. □

Theorem 2.10. *Let G_1 be an r-regular graph on m vertices and G_2 be an s-regular graph on n vertices, where $r, s \geq 2$. Then $G_1 \boxtimes G_2$ is a totally disconnected graph.*

Proof. Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$ be the vertex sets of G_1 and G_2 respectively, with $d_{G_1}(u_i) = r$, and $d_{G_2}(v_j) = s \;\forall\; i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Consider any two vertices (u_i, v_j) and (u_k, v_l) in $G_1 \boxtimes G_2$. Then (u_i, v_j) and (u_k, v_l) are adjacent if either $u_i = u_k$ and $\gcd(d_{G_2}(v_j), d_{G_2}(v_l)) = 1$ or $\gcd(d_{G_1}(u_i), d_{G_1}(u_k)) = 1$ and $v_j = v_l$. But $\gcd(d_{G_2}(v_j), d_{G_2}(v_l)) = s \neq 1$. Also $\gcd(d_{G_1}(u_i), d_{G_1}(u_k)) = r \neq 1$. So there are no edges in $G_1 \boxtimes G_2$, resulting into an empty graph on mn vertices. □

Theorem 2.11. *Let G_1 be a graph with at least one pendant vertex and G_2 be an r-regular graph on n vertices, where $r \geq 2$. Then $G_1 \boxtimes G_2$ is the union of n isomorphic components.*

Proof. Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ be the vertex set of G_1 and let u_1 be the pendant vertex. Let $V_2 = \{v_1, v_2, \ldots, v_n\}$ with $d_{G_2}(v_j) = r \;\forall\; j = 1, 2, \ldots, n$. Let (u_i, v_j) and (u_k, v_l) be the vertices of $G_1 \boxtimes G_2$. Then (u_i, v_j) is adjacent to (u_k, v_l) if either $u_i = u_k$ and $\gcd(d_{G_2}(v_j), d_{G_2}(v_l)) = 1$ or $\gcd(d_{G_1}(u_i), d_{G_1}(u_k)) = 1$ and $v_j = v_l$. But since $\gcd(d_{G_2}(v_j), d_{G_2}(v_l)) = r \neq 1$, so the only possible adjacency is from the condition that $\gcd(d_{G_1}(u_i), d_{G_1}(u_k)) = 1$ and $v_k = v_l$. Since u_1 is a pendant vertex $\gcd(deg(u_1), deg(u_k)) = 1 \;\forall\; k = 2, 3, \ldots, m$. That is, (u_i, v_1) is adjacent to (u_k, v_1). Similarly, (u_i, v_2) is adjacent to $(u_k, v_2) \;\forall\; i, k = 1, 2, \ldots, n, i \neq k$ and so on. (u_i, v_n) is adjacent to $(u_k, v_n) \;\forall\; i \neq k, i, k = 1, 2, \ldots\; n$. This results into n components. We have to show that they are isomorphic to each other. For this consider two arbitrary components $G^{(p)}$ and $G^{(q)}$ having the vertex sets

$$V(G^{(p)}) = \{(u_i, v_p) \;\forall\; i = 1, 2, \ldots, m\}$$
$$V(G^{(q)}) = \{(u_k, v_q) \;\forall\; k = 1, 2, \ldots, m\}.$$

Clearly $|V(G^{(p)})| = |V(G^{(q)})| = m$.

Consider the map defined by $f : V(G^{(p)}) \longrightarrow V(G^{(q)})$,

$$f((u_i, v_p)) = (u_i, v_q).$$

Claim: f is bijective.

For this consider $f((u_i, v_p)) = f((u_k, v_p))$. Since $f((u_i, v_p)) = (u_i, v_q)$ and $f((u_k, v_p)) = (u_k, v_q)$ this implies that $(u_i, v_q) = (u_k, v_q)$. So $u_i = u_k$. Hence, $(u_i, v_p) = (u_k, v_p)$, which shows that the map f is one-to-one. As $i = 1, 2, \ldots, m$, range of f is exactly $V(G^{(q)})$, which is nothing but the co-domain of f, which implies that f is onto. This shows that the map is bijective.

Claim: f is structure preserving.

For, we have to show that f preserves adjacency and non-adjacency. Take two vertices (u_i, v_p) and (u_k, v_p) from $G^{(p)}$. Now, we have two cases.

Case 1: (u_i, v_p) and (u_k, v_p) are adjacent
Since (u_i, v_p) is adjacent to (u_k, v_p), it is evident that $\gcd(d_{G_1}(u_i), d_{G_1}(u_k)) = 1 \; \forall \; i, k = 1, 2, \ldots, m$ and $i \neq k$. Consider $f((u_i, v_p)) = (u_i, v_q)$ and $f((u_k, v_p)) = (u_k, v_q)$. Since $\gcd(d_{G_1}(u_i), d_{G_1}(u_k)) = 1$, from the definition of the ARMS-product of graphs, the vertices $f((u_i, v_p))$ and $f((u_k, v_p))$ are adjacent. This shows that f preserves adjacency.

Case 2: (u_i, v_p) and (u_k, v_p) are non-adjacent.
Since (u_i, v_p) is not adjacent to (u_k, v_p), we can infer that $\gcd(d_{G_1}(u_i), d_{G_1}(u_k)) \neq 1 \; \forall \; i, k = 1, 2, \ldots, m$ and $i \neq k$. This implies that $f((u_i, v_p)) = (u_i, v_q)$ and $f((u_k, v_p)) = (u_k, v_q)$ are not adjacent. This shows that f preserves non-adjacency.
From both cases, we have $G^{(p)} \cong G^{(q)}$. Since p and q are arbitray, the result holds. This completes the proof. $\square$

Theorem 2.12. *For $n \geq 3$,*

$$P_m \boxtimes C_n \cong \begin{cases} nK_2, m = 2 \\ nK_{1,1,m-2}, m \geq 3. \end{cases}$$

Proof. Let $V(P_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex sets of P_m and C_n respectively.
Case 1: $m = 2$
Here $P_2 \boxtimes C_n$ has vertex set $V(P_2 \boxtimes C_n) = \{(u_i, v_j) \; \forall \; i = 1, 2, j = 1, 2, \ldots, n\}$. Since C_n is 2-regular, two vertices (u_i, v_j) and (u_k, v_l) in $P_2 \boxtimes C_n$ are adjacent if $\gcd(d_{P_2}(u_i), d_{P_2}(u_k)) = 1$ and $v_j = v_l$. Since $d_{P_2}(u_1) = d_{P_2}(u_2) = 1$, vertices (u_1, v_j) and (u_2, v_j) are adjacent $\forall \; j = 1, 2, \ldots, n$. This gives n isomorphic copies of P_2 in which (u_1, v_j) and (u_2, v_j) are the end vertices, $\forall \; j = 1, 2, \ldots, n$. Again since $d_{C_n}(v_j) = 2 \; \forall \; j = 1, 2, \ldots, n$, vertices (u_1, v_j) is not adjacent to (u_1, v_k) for any j, where $j = 1, 2, \ldots, n$. A similar result holds for the vertex u_2. Hence $P_2 \boxtimes C_n$ is the union of n copies of P_2.
Case 2: $m \geq 3$
As in the case 1, $V(P_m \boxtimes C_n) = \{(u_i, v_j) \; \forall \; i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$. Assume that for the path P_m, $d_{P_m}(u_1) = d_{P_m}(u_m) = 1$ and $d_{P_m}(u_i) = 2 \; \forall \; i = 2, 3, \ldots, m-1$. In this case two vertices (u_i, v_j) and (u_k, v_l) are adjacent if $\gcd(d_{P_m}(u_i), d_{P_m}(u_k)) = 1$ and $v_j = v_l \; \forall \; i, k = 1, 2, \ldots, m, j, l = 1, 2, \ldots, n$.
Consider the following partitions of $V(P_m \boxtimes C_n)$.
For all $j = 1, 2, \ldots, n$,

$$\begin{aligned} U_1 &= \{(u_1, v_j)\} \\ U_2 &= \{(u_m, v_j)\} \\ U_3 &= \{(u_i, v_j) \; \forall \; i = 2, 3, \ldots, m-1\}. \end{aligned}$$

Since u_1 and u_2 are the pendant vertices, the vertex in U_1 is adjacent to the vertex in U_2 $\forall \; j = 1, 2, \ldots, n$. Again using the same arguement the vertices in U_3 is adjacent to U_1 and U_2 $\forall \; i = 2, 3, \ldots, m-1$ and $j = 1, 2, \ldots, n$. Clearly for a fixed j, the graph having vertices U_1, U_2 and U_3 is the complete tripartite graph $K_{1,1,m-2}$. Since j varies from 1 to n, there are n such isomorphic components. Hence $P_m \boxtimes C_n$ is the union of n isomorphic copies of $K_{1,1,m-2}$. Hence the proof. $\square$

Theorem 2.13. *Let $G_1 = K_{m,n}$ with $\gcd(m, n) = 1$ and G_2 be an r-regular graph on p vertices, $r \geq 2$, then*

$$G_1 \boxtimes G_2 \cong pK_{m,n}.$$

Proof. Proof immediately follows from theorem 2.3. □

3. The Limit Graph of a Graph

In many different fields of mathematics, graph theory is used. Among these fields one is Real Analysis. In Real Analysis, the concept of limit plays an important role. We aim to define relatively a new concept in graph theory called limit graph of a given graph. Further we categorise graphs, based on whether or not they have a unique limit graph. In this section we introduce the concept of limit graph of a graph and determine the limit graphs of certain standard graphs.

Definition 3.1. Let $G = (V, E)$ be a connected graph on n vertices, $n \geq 2$ and $S \subset V(G)$. Let $H = \langle S \rangle$ be a connected subgraph of G with minimal order and size such that $N(S) = V(G)$. This H is said to be the **limit graph** of G and is denoted by $\lim(G)$.

Remark 3.2. (1) Every graph has at least one limit graph.
(2) Limit graph of G need not be unique.

Remark 3.3. Let G be a graph and $H = \langle S \rangle$ be a limit graph of G. This H is unique if for any S' satisfying the properites of S, and $H' = \langle S' \rangle$, then $H' \cong H$.

Figure 5 is an example for a graph with unique limit graphs.
For this consider the subsets $S_1 = \{v_1, v_6\}$, and $S_2 = \{v_4, v_6\}$. Here $\langle S_1 \rangle \cong \langle S_2 \rangle \cong K_2$. Hence $\lim(G)$ is K_2.

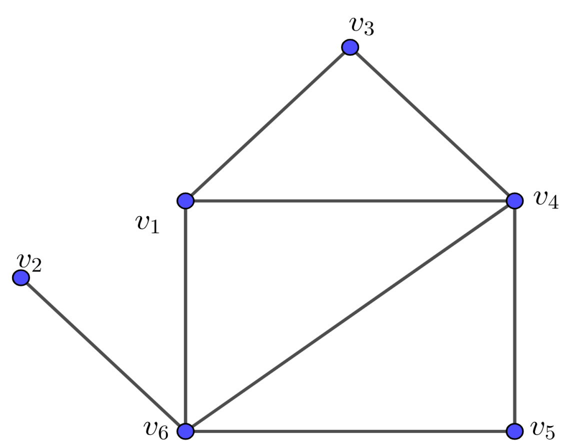

FIGURE 5. G

Figure 6 is an example of a graph without a unique limit graph.
In this graph, we can find two sets $S_1, S_2 \subset V(G)$ with the same cardinality with $N(S_1) = N(S_2) = V(G)$ whose induced subgraphs have the same order and size which are non-isomorphic. Here $S_1 = \{v_3, v_4, v_5, v_6\}$ and $S_2 = \{v_1, v_4, v_5, v_6\}$.

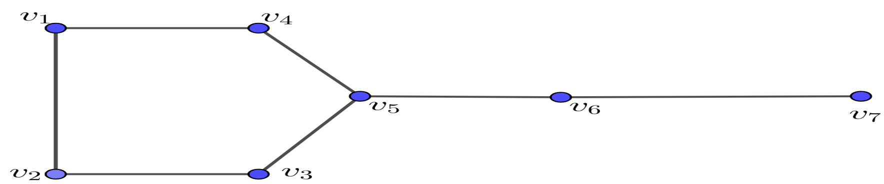

FIGURE 6. G

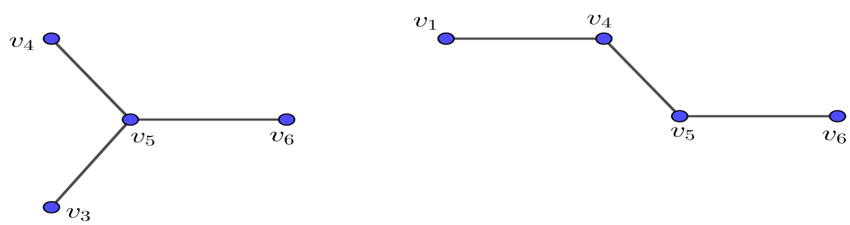

FIGURE 7. $\langle S_1 \rangle$ and $\langle S_2 \rangle$, Two non-isomorphic subgraphs of G having least order and size with $N(S_1) = N(S_2) = V(G)$

3.1. Some Standard Graphs with unique Limit Graphs. In this section, we determine some classes of graphs which posses a unique limit graph and determine the correseponding limit graphs.

Proposition 3.4. *For* $n \geq 3$, $\lim(P_n)$ *is* P_{n-2}.

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and assume that each v_i is adjacent to $v_{i+1}, i = 1, 2, \ldots, n-1$. Take all the internal vertices of P_n. That is, $S = \{v_i,\ \forall\ i = 2, 3, \ldots, n-1\}$. Then $N(S) = V(P_n)$ also $\langle S \rangle$ have minimal order and size. Otherwise, the induced subgraph of a set without taking at least one internal vertex leads to a disconnected graph. □

Proposition 3.5. *For* $n \geq 3$, $\lim(C_n)$ *is* P_{n-2}.

Proof. It is similar to the proof of proposition 3.4. □

Proposition 3.6. *For* $n \geq 2$, $\lim(K_n)$ *is* K_2.

Proof. The proof is trivial. Since all the induced subraphs of K_n with vertex sets consisting of two of its vertices are isomorphic to each other. □

Proposition 3.7. *For* $m, n \geq 1$, $\lim(K_{m,n})$ *is* K_2.

Proof. Let V_1 and V_2 be the two partite sets of $K_{m,n}$ where $m, n \geq 1$. Let $\{u_1, u_2, \ldots, u_m\}$ be the vertex set of V_1 and $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of V_2. Choose one vertex from V_1 and one from V_2. The induced subgraph of G with these two vertices have minimal order and size and whose one neighhbourhood is $V(K_{m,n})$. All such induced subgraphs are isomorphic to K_2. Hence $lim(K_{m,n})$ is K_2. □

Theorem 3.8. *For the Petersen graph G,* $\lim(G)$ *is* $K_{1,3}$.

Proof. Let $G = (V, E)$ be the Petersen graph, where the vertex set and edge set is given by $V(G) = \{a_i, b_i, 1 \leq i \leq 5\}$ and $E(G) = \{a_i a_{i+1}, a_i b_i, b_i b_{i+2}, 1 \leq i \leq 5\}$, where the subscripts are expressed as integer modulo 5. Now we are considering sets which satisfies the conditions given in the definition of the limit graph. Sets with cardinalities 1, 2 and 3 never satisfies the requirements. Hence the minimal cardinality of a set S, which satisfies $N(S) = V(G)$ is 4. There are $\binom{10}{4}$ possibilities. But among these, only 5 sets $S_i, i = 1, 2, 3, 4, 5$ will induces a subgraph $\langle S_i \rangle$ which is connected and having minimal order and size with $N(S_i) \cong V(G)$. Those sets are given by

$$S_i = \{a_i, b_i, b_{i+2}, b_{i+3}\}, i = 1, 2, 3, 4, 5$$

where the subscripts are taken integer *modulo* 5. The subgraphs induced by the sets $S_i, i = 1, 2, 3, 4, 5$ are all isomorphic to each other and no other set with same cardinality produces a subgraph which satisfies all the conditions of limit graph which is not isomorphic to $\langle S_i \rangle$. Here $\langle S_i \rangle = K_{1,3}$. Hence $\lim(G)$ is $K_{1,3}$. □

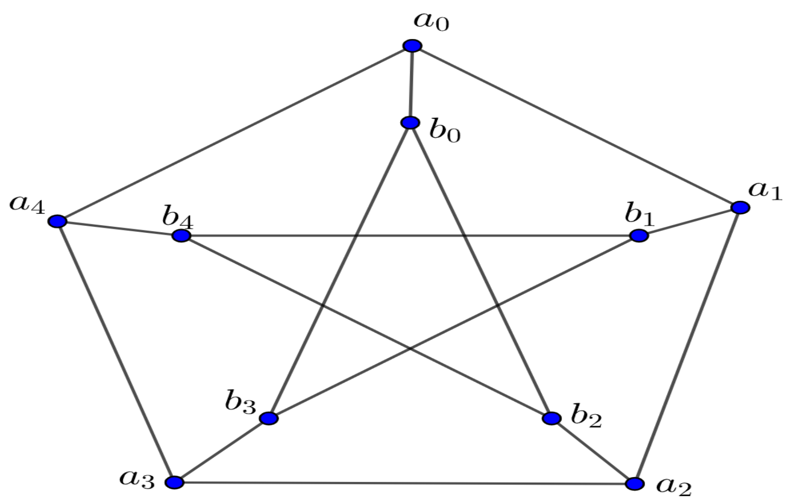

FIGURE 8. Petersen graph

Theorem 3.9. *For any tree G, limit graph of G is unique.*

Proof. Let G be a tree with n vertices. Let $v_1, v_2, \ldots, v_k$ be the pendant vertices and $u_1, u_2, \ldots, u_l$ be the internal vertices, so that $k + l = n$.

Case 1: When $k = 1$

Consider $S_i = \{v_1, u_i\}, 1 \leq i \leq l$. All the $\langle S_i \rangle$ are isomorphic to each other and is the graph with minimal order and size such that $N(S_i) = V(G)$. In this case the resulting graph is a star graph. Hence $\lim(G)$ is K_2.

Case 2: When $k \geq 2$

Consider the subgraph of G obtained by removing all the pendant vertices from G. We will show that this subgraph is the limit graph of G. The removal of all the pendant vertices doesn't change the connectivity of the subgraph, since the path between all the internal vertices are unique even after the removal of pendant vertices. Also the subgraph so formed is unique (since we are removing all the pendant vertices from G). So in this case we can take $S = \{u_i \ \forall\ 1 \leq i \leq l\}$. The resulting graph $\langle S \rangle$ meets all the requirements for the limit graph of G. Hence this subgraph obtained by deleting all the pendent vertices is unique. □

Example 3.10. The following example shows that removal of all pendant vertices from G results into the limit graph of G.

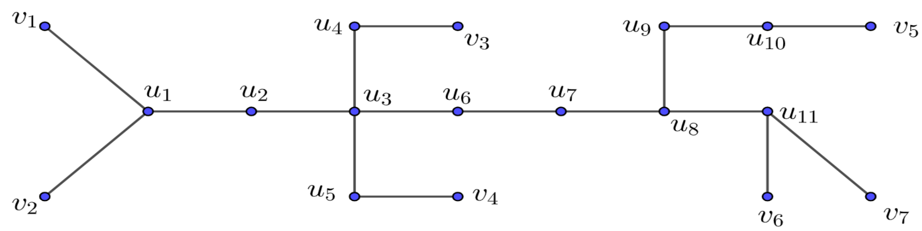

FIGURE 9. A tree G

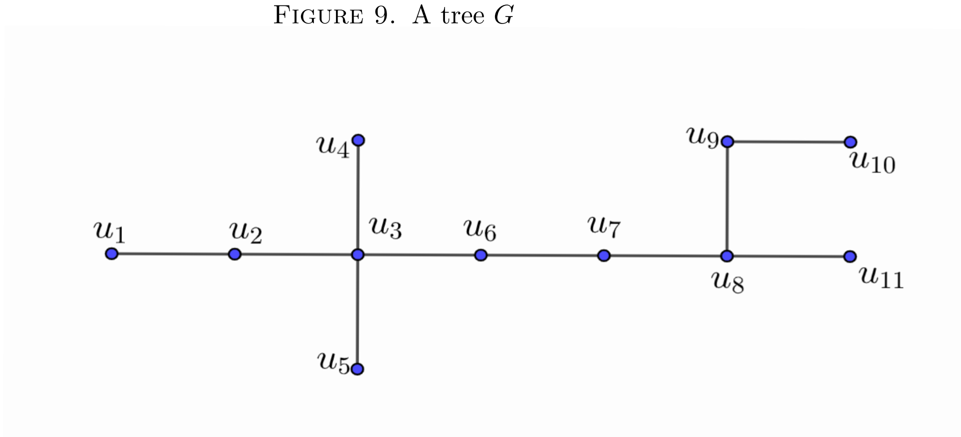

FIGURE 10. $\lim(G)$

Corollary 3.11. *Let T be a complete binay tree of height h, then the limit graph of T is again a complete binary tree with height $h-1$.*

Next, we discuss some graph theoretical operations on graphs in which the resulting graph has a unique limit graph.

Theorem 3.12. *Let G be a connected graph with at least 2 vertices and H be any simple graph, then $\lim(G \odot H)$ is G.*

Proof. Let G be a connected graph with $|G|= n$ and $|H|= r \geq 1$. Let $v_1, v_2, \ldots, v_n$ be the vertices of G. Then $G \odot H$ is a connected graph with $|G \odot H|= n(r+1)$. Take $S = V(G)$. Then

$N(S) = V(G \odot H)$. Also by assumption $\langle S \rangle$ is connected and no proper subset of S cannot satisfy the requirements of limit graph of $G \odot H$. Hence $\lim(G \odot H)$ is G. □

Theorem 3.13. *For any two graphs G, H, $\lim(G \vee H) = K_2$.*

Proof. Let G be a graph with vetex set $\{v_1, v_2, \ldots, v_m\}$ and the vertex set of graph H is $\{u_1, u_2, \ldots, u_n\}$, where $m, n \geq 1$. Then $G \vee H$ is a connected graph with $m+n$ vertices and each v_i in G is adjacent to u_j in H where $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$. Take $S = \{v_i, u_j\}$, where $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$. Then S is a connected minimal ordered set with $N(S) = V(G \vee H)$. We can see that the graph induced by S is K_2. Also, the graph induced by the vertex set $S' = \{v_k, u_l\}$, where $i \neq k$ and $j \neq l$ is $\langle S' \rangle \cong \langle S \rangle$. Hence $\lim(G \vee H)$ is K_2. □

3.2. Graphs with more than one Limit Graph. In this section we determine some graphs having different non-isomorphic limit graphs.

Theorem 3.14. *For $m \geq 5, n \geq 2$ limit graph of $T_{m,n}$ is not unique.*

Proof. Let the vertices of $C_m, m \geq 5$ be labeled as $v_1, v_2, \ldots, v_m$ and that of $P_n, n \geq 2$ is $u_1, u_2, \ldots, u_n$. Let the bridge connecting C_m and P_n is $v_1 u_1$. Consider the sets, for $1 \leq k \leq m-1$

$$S_k = \{u_i, v_j, \forall\ 1 \leq i \leq n-1, 1 \leq j \leq m, j \neq k+1, k+2\}.$$

Clearly $N(S_k) = V(T_{m,n})$ and all the induced graphs, $\langle S_k \rangle$ have order $n+m-2$ and size $n+m-3$. But the induced graph of S_1, S_{m-1} is not isomorphic with that of S_k where $2 \leq k \leq m-2$. Hence the limit graph of $T_{m,n}$ is not unique. □

Definition 3.15. Consider $P_m, m \geq 2$ and $C_n, n \geq 3$. Adjoin C_n to each vertices of P_m. The resulting graph is denoted by $G_{m,n}$.

An example is given below.

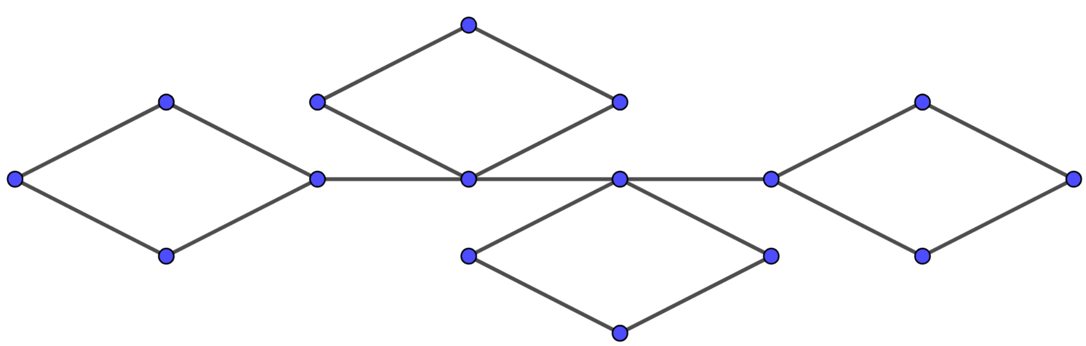

FIGURE 11. $G_{4,4}$

Theorem 3.16. *For $m \geq 2, n \geq 5$ limit graph of $G_{m,n}$ is not unique.*

Proof. Let $v_1, v_2, \ldots, v_m$ be the vertices of P_m and $\{v_i, u_{ij}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n-1\}$, be the vertices of each C_n adjoined to the i^{th} vertex of P_n. Consider the two sets $S_1, S_2 \subset V(G_{m,n})$:

$$S_1 = \{v_i, u_{ij},\ \forall i = 1, 2, \ldots, m, j = 1, 2, \ldots, n-3\}$$
$$S_2 = \{v_i, u_{ij},\ \forall i = 1, 2, \ldots, m, j = 1, 4, \ldots, n-1, j \neq 2, 3\}\,.$$

Clearly $|S_1|= |S_2|= m(n-2)$ and $\langle S_1\rangle$, $\langle S_2\rangle$ have the same size given by $mn - 2m - 1$. Also $\langle S_1\rangle$, and $\langle S_2\rangle$ are connected with $N(S_1) = N(S_2) = V(G_{m,n})$. But $\langle S_1\rangle \ncong \langle S_2\rangle$. Since P_m is a path in $G_{m,n}$ every vertices of P_m should be included in the set and $\lim(C_n)$ is P_{n-2}, so $n-2$ vertices must be chosen fom the vertices of each C_n. Hence no other sets with fewer cardinality can satisfy affirmatively the conditions for the limit graph of $G_{m,n}$. Hence limit graph of $G_{m,n}$ is not unique. □

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