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MULTI-POINT BOUNDARY VALUE PROBLEM FOR SECOND ORDER DIFFERENTIAL EQUATIONS ON MANIFOLDS

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Abstract. We investigate the multi-point boundary value problem for second order differential equation of the form

$$\frac{D}{dt}\dot{m}(t) = F(t, m(t), \dot{m}(t)),$$

$$m(0) = m_0; \quad \dot{m}(1) = \sum_{i=1}^{q} \beta_i \Gamma_{m(1)} \dot{m}(t_i),$$

on a complete Riemannian manifold, where $\frac{D}{dt}$ is the covariant derivative of Levi-Civita connection and $F(t, m(t), \dot{m}(t))$ is a vector field (it has either less than linear, linear or quadratic growth in velocity). A generalization to the problem of the same sort subjected to a non-holonomic constraint, is also presented.

1. Introduction

Let M be a finite-dimensional complete Riemannian manifold and TM be its tangent bundle with the natural projection $\pi : TM \to M$. Consider a map $F : R \times TM \to TM$ such that for any point (m, X) (this means that $X \in T_m M$, i.e., X is a tangent vector to M at the point $m \in M$) the relation $\pi F(t, m, X) = \pi(m, X) = m$ holds. Take $\beta_i \in R$, $t_i \in (0, 1)$, $i = 1, ..., q$. By $\Gamma_{m(1)}$ denote the operator of parallel translation along the curve $m(\cdot)$ to the point $m(1)$ and by $\Gamma_{m(0)}$ denote the operator of parallel translation along the curve $m(\cdot)$ to the point $m(0)$. This paper is concerned with the problem of existence of a solution for the multipoint boundary value problem:

$$\frac{D}{dt}\dot{m}(t) = F(t, m(t), \dot{m}(t)), \tag{1.1}$$

$$m(0) = m_0; \quad \dot{m}(1) = \sum_{i=1}^{q} \beta_i \Gamma_{m(1)} \dot{m}(t_i). \tag{1.2}$$

We suppose that F have less than linear, linear or quadratic growth in X. The main aim of the paper is to find conditions that guarantee the solvability for the boundary value problem (1.1) – (1.2) with right-hand sides as mentioned above, i.e., to find a C^1 – curve $m(t)$, $t \in [0, 1]$, satisfying (1.1) – (1.2) on $[0, 1]$. In

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the last section of the article we explore multi-point boundary value problem for systems with constraints. Similar boundary value problem for equation on R^n with right-hand side satisfying Caratheodorys conditions was investigated in [1] and [2].

2. Mathematical machinery

In this section we modify some constructions from [2], [3] and [4] for the problem under consideration. Take $m_0 \in M$, and let $v : [0,1] \to T_{m_0}M$ be a continuous curve. It is shown in [3] that there exists a unique C^1–curve $m : [0,1] \to M$ such that $m(0) = m_0$ and the vector $\dot{m}(t)$ is parallel along $m(\cdot)$ to the vector $v(t) \in T_{m_0}M$ at any $t \in [0,1]$.

Denote the curve $m(t)$ constructed above from the curve $v(t)$, by the symbol $Sv(t)$. Thus we have defined a continuous operator S that sends the Banach space $C^0([0,1], T_{m_0}M)$ of continuous maps (curves) from $[0,1]$ to $T_{m_0}M$ into the Banach manifold $C^1([0,1], M)$ of C^1- maps from $[0,1]$ to M. Denote by Γ_{m^*} the operator of parallel translation of vectors along $m(\cdot)$ at the point m^*. Suppose that $F(t, m, X)$ is a continuous force field. Let $1 \neq \sum_{i=1}^{q} \beta_i$. Define the operator

$$B : C^0([0,1], T_{m_0}M) \to C^0([0,1], T_{m_0}M)$$

by the formula:

$$B(v) = \frac{1}{1-\sum_{i=1}^{q}\beta_i}\Big[\sum_{i=1}^{q}\beta_i\int_0^{t_i}\Gamma_{m_0}F(s, Sv(s), \frac{d}{ds}Sv(s))ds - \\ - \int_0^1 \Gamma_{m_0}F(s, m(s), \frac{d}{ds}Sv(s))ds\Big] + \int_0^t (\Gamma_{m_0}F(s, m(s), \frac{d}{ds}Sv(s))ds, \tag{2.1}$$

where $t \in [0,1]$. It is shown in [3] that operator ΓS is compact. Since ΓS is compact and F is continious it is easy to see that B is compact.

Let $u(t) \in C^0([0,1], T_{m_0}M)$ be a fixed point of operator B. Let us show that $m(t) = S(u(t))$ is the desired solution of BVP (1.1)–(1.2). Firstly, the equality $\dot{u}(t) = \Gamma_{m_0}F(t, S(u), \frac{d}{dt}S(u))$ holds for all points t at which the derivative exists. Using the properties of the covariant derivative and the definition of u, one can show that $\dot{u}(t)$ is parallel to $\frac{D}{dt}\dot{m}(t)$ along $m(\cdot)$ and $\Gamma F(t, S(u), \frac{d}{dt}S(u))$ is parallel to $F(t, m(t), \dot{m}(t))$. Hence, $\frac{D}{dt}\dot{m}(t) = F(t, m(t), \dot{m}(t))$ and curve $m(t) = S(u)$ satisfies (1.1). Secondly, by direct calculations we get: $\Gamma_{m(1)}\beta_1 B'(u)(t_1) + \Gamma_{m(1)}\beta_2 B'(u)(t_2) + ... + \Gamma_{m(1)}\beta_q B'(u)(t_q) = B'(u)(1)$, hence $S(u(t))$ satisfies (1.2).

Following lemmas can be easily proven by simple algebraic transformations.

Lemma 2.1. *Suppose* $1 - \sum_{i=1}^{q}\beta_i > 0$ *and*

$$c < \frac{(1-\sum_{i=1}^{q}\beta_i)}{(2+\sum_{i=1}^{q}(|\beta_i|t_i - \beta_i))}, \tag{2.2}$$

then the following inequality holds:

$$\frac{1}{1-\sum_{i=1}^{q}\beta_i}\Big(\sum_{i=1}^{q}|\beta_i|c\varepsilon t_i + c\varepsilon\Big) + c\varepsilon < \varepsilon$$

Lemma 2.2. *Suppose* $1-\sum_{i=1}^{q}\beta_i<0$ *and*

$$c<\frac{|1-\sum_{i=1}^{q}\beta_i|}{(\sum_{i=1}^{q}(|\beta_i|t_i+\beta_i))}, \tag{2.3}$$

then the following inequality holds:

$$\frac{1}{|1-\sum_{i=1}^{q}\beta_i|}\Big(\sum_{i=1}^{q}|\beta_i|c\varepsilon t_i+c\varepsilon\Big)+c\varepsilon<\varepsilon$$

Lemma 2.3. *Suppose* $1-\sum_{i=1}^{q}\beta_i>0$ *and*

$$a<\frac{(1-\sum_{i=1}^{q}\beta_i)}{(2+\sum_{i=1}^{q}(|\beta_i|t_i-\beta_i))\varepsilon}, \tag{2.4}$$

then the following inequality holds:

$$\frac{1}{1-\sum_{i=1}^{q}\beta_i}\Big(\sum_{i=1}^{q}|\beta_i|a\varepsilon^2 t_i+a\varepsilon^2\Big)+a\varepsilon^2<\varepsilon$$

Lemma 2.4. *Suppose* $1-\sum_{i=1}^{q}\beta_i<0$ *and*

$$a<\frac{|1-\sum_{i=1}^{q}\beta_i|}{(\sum_{i=1}^{q}(|\beta_i|t_i+\beta_i))\varepsilon}, \tag{2.5}$$

then the following inequality holds:

$$\frac{1}{|1-\sum_{i=1}^{q}\beta_i|}\Big(\sum_{i=1}^{q}|\beta_i|a\varepsilon^2 t_i+a\varepsilon^2\Big)+a\varepsilon^2<\varepsilon$$

Definition 2.5. The force field $F(t,m,X)$ is said to have less than linear growth in X if for any compact $\Theta\subset M$ and interval $[0,1]$ the relation:

$$\lim_{\|X\|\to\infty}\frac{\|F(t,m,X)\|}{\|X\|}=0 \tag{2.6}$$

holds uniformly in $t\in[0,1]$ and $m\in\Theta$.

Remark 2.6. Let force field F satisfy definition 2.5. For every real $c>0$, there exists a real $K>0$ such that for all X, $\|X\|\geq K$ implies that $\|F(t,m,Y)\|<c\|X\|$ if $\|Y\|\leq\|X\|$.

Definition 2.7. The force field $F(t,m,X)$ is said to have linear growth in X if for any compact $\Theta\subset M$ and interval $[0,1]$ the relation:

$$\lim_{\|X\|\to\infty}\frac{\|F(t,m,X)\|}{\|X\|}=c(t,m) \tag{2.7}$$

holds uniformly in $t\in[0,1]$ and $m\in\Theta$, if $c(t,m)\geq 0$ - is a continuous real-valued function on $[0,1]\times\Theta$, that is not identically equal to zero.

Remark 2.8. From the definition (2.7) it follows that there exists a number L such that for any $\|X\|\geq L$ at any point of M and any $t\in[0,1]$ we have $\|F(t,m,Y)\|\leq c(t,m)\|X\|$ if $\|Y\|\leq\|X\|$.

Remark 2.9. By Δ denote a set of a curves $S(v)$ such that $v \in U_L \subset C^0(I, T_{m_0}M)$. It is easy to see that there exists a compact $\Xi \subset M$ such that $\Delta \subset \Xi$.

Remark 2.10. Let Ξ be the compact from remark 2.9. Then there exists real number c such that on $[0,1] \times \Xi$ the following inequality holds : $c(t,m) \leq c$.

Definition 2.11. The force field $F(t,m,X)$ is said to have quadratic growth in X if for any compact $\Theta \subset M$ and any finite interval $[0,1]$ the relation:

$$\lim_{\|X\| \to \infty} \frac{\|F(t,m,X)\|}{\|X\|^2} = a(t,m) \tag{2.8}$$

holds uniformly in $t \in [0,1]$ and $m \in \Theta$, if $a(t,m) \geq 0$ - is a continuous real-valued function on $[0,1] \times \Theta$, that is not identically equal to zero.

Remark 2.12. From the definition (2.11) it follows that there exists a number Q such that for any $\|X\| \geq Q$ at any point of M and any $t \in [0,1]$ we have $\|F(t,m,Y)\| \leq a(t,m)\|X\|^2$ if $\|Y\| \leq \|X\|$.

Remark 2.13. By Δ denote a set of a curves $S(v)$ such that $v \in U_Q \subset C^0(I, T_{m_0}M)$. It is easy to see that there exists a compact $\Phi \subset M$ such that $\Delta \subset \Phi$.

Remark 2.14. Let Φ be the compact from remark 2.13. Then there exists a real number a such that on $[0,1] \times \Xi$ the following inequality holds : $a(t,m) \leq a$.

3. Main statements

In this section we investigate the existence of solutions of problem 1.1-1.2 with right-hand side satisfing conditions 2.5, 2.7 or 2.11.

Theorem 3.1. *Let the force field $F(t,m,X)$ have less than linear growth in X (See Definition (2.5)). Then for the multi-point boundary value problem (1.1)–(1.2) there exists a solution $m(t)$.*

Proof. Part 1. Suppose $1 - \sum_{i=1}^q \beta_i > 0$. Take $c < \frac{(1-\sum_{i=1}^q \beta_i)}{(2+\sum_{i=1}^q (|\beta_i| t_i - \beta_i))}$ from Lemma 2.1. Take K from Remark 2.6 Then the following inequality holds:

$$\|B(v)\| \leq \frac{1}{1 - \sum_{i=1}^q \beta_i} \Big(\sum_{i=1}^q \beta_i cKt_i + cK \Big) + cK \leq K. \tag{3.1}$$

Thus B sends the ball U_Q into itself and from the Schauder principle it follows that it has a fixed point $u^* \in U_Q$. Then $m(t) = S(u^*(t))$ is the desired solution.

Part 2 Suppose $1 - \sum_{i=1}^q \beta_i < 0$ and $c < \frac{|1-\sum_{i=1}^q \beta_i|}{(\sum_{i=1}^q (|\beta_i| t_i + \beta_i))}$. The proof of this case follows from the same scheme of arguments as that for part 1.

$\square$

Theorem 3.2. *Let the force field $F(t,m,X)$ have a linear growth in X (See Definition (2.7)) and for number the c from 2.10 the following inequality holds: $c < \frac{(1-\sum_{i=1}^q \beta_i)}{(2+\sum_{i=1}^q (|\beta_i| t_i - \beta_i))}$ (in case $1 - \sum_{i=1}^q \beta_i > 0$) or $c < \frac{|1-\sum_{i=1}^q \beta_i|}{(\sum_{i=1}^q (|\beta_i| t_i + \beta_i))}$ (in case $1 - \sum_{i=1}^q \beta_i < 0$) Then for the multi-point boundary value problem (1.1)–(1.2) there exists a solution $m(t)$.*

Proof. Let L and c be from Remarks 2.8 and 2.10 respectively.

Part 1. Suppose conditions of the Lemma 2.1 holds. For the operator B defined on $U_L \subset C^0([0,1], T_{m_0}M)$ the following inequality holds:

$$\|B(v)\| \leq \frac{1}{1-\sum_{i=1}^{q}\beta_i}\Big(\sum_{i=1}^{q}\beta_i cLt_i + cL\Big) + cL \leq L. \tag{3.2}$$

Thus B sends the ball U_Q into itself and from the Schauder principle it follows that it has a fixed point $u^* \in U_Q$. Then $m(t) = S(u^*(t))$ is the desired solution.

Part 2. Suppose conditions of the Lemma 2.2 holds. The proof of this case follows from the same scheme of arguments as that for part 1. □

Theorem 3.3. *Let the force field $F(t,m,X)$ have a quadratic growth in X (See Definition (2.11)) and for number a and Q from Remarks 2.14 and 2.12, respectively, the following inequality holds: $a < \frac{(1-\sum_{i=1}^{q}\beta_i)}{(2+\sum_{i=1}^{q}(|\beta_i|t_i-\beta_i))Q}$ (in case $1-\sum_{i=1}^{q}\beta_i > 0$) or $a < \frac{|1-\sum_{i=1}^{q}\beta_i|}{(\sum_{i=1}^{q}(|\beta_i|t_i+\beta_i))Q}$ (in case $1-\sum_{i=1}^{q}\beta_i < 0$) Then for the multi-point boundary value problem (1.1)–(1.2) there exists a solution $m(t)$.*

Proof. Let Q and a be from Remarks 2.12 and 2.14 accordingly.

Part 1. Suppose conditions of the Lemma 2.3 holds. For the operator B defined on $U_Q \subset C^0([0,1], T_{m_0}M)$ the following inequality holds:

$$\|B(v)\| \leq \frac{1}{1-\sum_{i=1}^{q}\beta_i}\Big(\sum_{i=1}^{q}\beta_i aQ^2 t_i + aQ^2\Big) + aQ^2 \leq Q. \tag{3.3}$$

Thus B sends the ball U_Q into itself and from Schauders principle it follows that it has a fixed point $u^* \in U_Q$. Then $m(t) = S(u^*(t))$ is the desired solution.

Part 2. Suppose conditions of the Lemma 2.4 holds. The proof of this case follows from the same scheme of arguments as that for part 1.

□

4. Systems with linear constraints

In this section, we show how to generalize existence theorems of previous section to systems with constraints. We refer the reader, say, to [3] for preliminary material about systems with constraints. Here we introduce only some notions necessary for understanding the constructions.

Definition 4.1. A linear constraint in the system is a smooth distribution (i.e., a subbundle of the tangent bundle) β on M.

If the distribution β is integrable, the constraint is called holonomic and non-holonomic in the other case.

Definition 4.2. A tangent vector is called admissible if it lies in the distribution β. A curve in M is admissible if all its tangent vectors are admissible.

A constraint β imposes a restriction on the motion of the system. Namely, all its solutions must be admissible.

Let $Q : TM \to \beta$ be the operator of orthogonal projection (with respect to the Riemannian metric on M) of the tangent spaces on their subspaces β, i.e., we have $P_m : T_mM \to \beta_m$ for every $m \in M$. Introduce the so-called reduced covariant derivative along a curve by the formula $\frac{\bar{D}}{dt} = P\frac{D}{dt}$. In fact it is generated by the so-called *reduced connection* (see [3]). Below in this section we use the parallel translation of admissible vectors along admissible curves generated by the reduced connection. Let M be a complete Riemannian manifold equipped with a constraint β. Instead of operators S and Γ we will use their constraint analogs S^β and Γ^β(see [3]). We investigate the BVP of the form:

$$\frac{\bar{D}}{dt}\dot{m}(t) = PF(t, m(t), \dot{m}(t)), \tag{4.1}$$

$$m(0) = m_0; \quad \dot{m}(1) = \sum_{i=1}^{q} \alpha_i \Gamma^{\beta}_{m(1)} \dot{m}(t_i), \tag{4.2}$$

Suppose $1 \neq \sum_{i=1}^{q} \beta_i$. Let us define the operator $B^\beta : C^0([0,1], \beta_{m_0}) \to C^0([0,1], \beta_{m_0}M)$ by the formula:

$$B^\beta(v) = \frac{1}{1 - \sum_{i=1}^{q} \alpha_i}\Big[\sum_{i=1}^{q} \alpha_i \int_0^{t_i} \Gamma^\beta_{m_0} F(s, S^\beta v(s), \frac{d}{ds}S^\beta v(s))ds -$$
$$- \int_0^1 \Gamma^\beta_{m_0} F(s, m(s), \frac{d}{ds}S^\beta v(s))ds\Big] + \int_0^t (\Gamma^\beta_{m_0} F(s, m(s), \frac{d}{ds}S^\beta v(s))ds \tag{4.3}$$

where $t \in [0,1]$. Similarly to the operator B, defined by 2.1, one can show that B^β is compact and fixed point of B^β is a solution to boundary value problem 4.1 - 4.2. Notions of less than linear, linear, quadratic growth can be defined for the case of systems with constraints in the same way as in 2.5, 2.7, 2.11.

Theorem 4.3. *Let the force field $PF(t, m, X)$ have less than linear growth in X (See Definition (2.5)). Then for the multi-point boundary value problem (4.1)–(4.2) there exists a solution $m(t)$.*

Theorem 4.4. *Let the force field $PF(t, m, X)$ have a linear growth in X (See Definition (2.7)) and for number c from Remark 2.10 the following inequality holds: $c < \frac{(1-\sum_{i=1}^{q}\alpha_i)}{(2+\sum_{i=1}^{q}(|\alpha_i|t_i-\alpha_i))}$ (in case $1 - \sum_{i=1}^{q}\alpha_i > 0$) or $c < \frac{|1-\sum_{i=1}^{q}\alpha_i|}{(\sum_{i=1}^{q}(|\alpha_i|t_i+\alpha_i))}$ (in case $1 - \sum_{i=1}^{q}\alpha_i < 0$) Then for the multi-point boundary value problem (4.1)–(4.2) there exists a solution $m(t)$.*

Theorem 4.5. *Let the force field $PF(t, m, X)$ have a quadratic growth in X (See Definition (2.11)) and for number a and Q from Remarks 2.14 and 2.12, respectively, the following inequality holds: $a < \frac{(1-\sum_{i=1}^{q}\alpha_i)}{(2+\sum_{i=1}^{q}(|\alpha_i|t_i-\alpha_i))Q}$ (in case $1 - \sum_{i=1}^{q}\alpha_i > 0$) or $a < \frac{|1-\sum_{i=1}^{q}\alpha_i|}{(\sum_{i=1}^{q}(|\alpha_i|t_i+\alpha_i))Q}$ (in case $1 - \sum_{i=1}^{q}\alpha_i < 0$) Then for the multi-point boundary value problem (4.1)–(4.2) there exists a solution $m(t)$.*

The proof of these theorems follows from the same scheme of arguments as that for theorems 3.1, 3.2, 3.3.

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