Journal of Mathematical Control Science & Applications, Vol. 8 No. 2 (July-December, 2022)
ISSN : 0974-0570

EVALUATION OF C- SERVERS QUEUEING SYSTEMS WITH MULTIPLE DIFFERENTIATED VACATIONS AND COMPLETE VACATION INTERRUPTIONS

VYSHNA UNNI AND K. JULIA ROSE MARY

Abstract. In this paper we discuss an $M/M/C$ queueing system with differentiated server vacations and vacation interruptions. We offer two types of vacation strategies for the C servers of the system with distinct durations, which are namely type 1 and type 2 vacations where servers can go for a vacation of type 1 only after serving at least one customer who entered into the system and they can take type 2 vacations if all the C-queues are empty when they returned from the vacations. We also assume that both types of vacations can be interrupted when the number of customers in the system reaches some predefined thresholds where each type of vacation has a threshold different from the other. We derive the expressions for the steady-state probabilities of various states and the average waiting time generated in the system. Further, we analyse the impacts of complete vacation interruptions on the mean delay of the system with various choices of parameters. Also numerically examine the relationship between thresholds, number of servers, durations of vacations and the average waiting time generated in the system.

1. Introduction

Queueing system subject to server vacations is the area of interest for many upcoming researchers due to its immense applications in real-life situations. The current generation has to compete for more to get favorable outcomes. Nowadays, the main aim of service providers in all streams is to offer better service for the customers within a limited time due to the tough competitions occur in the field of service providing. Queueing models possess a vital role in designing the service structure and for deciding the procedures for treating the customers to get better outputs.

Queueing system with server vacation is one in which a server may become unavailable for a certain period of time from the primary service center and the time that the server is away from the service area is termed as a vacation or a break in a queueing system. There are many factors affect a server to take a vacation which includes power-saving mode, machine break downs, the insufficient workload in human behaviour, the secondary task assigned for the servers, post service processing, refreshment times, etc. The concept of vacations into the traditional model of queueing systems was first introduced by the researchers Levi and

2000 *Mathematics Subject Classification.* 60K25; 68M20; 90B22.

Key words and phrases. multiple servers, vacation queueing systems, differentiated vacations, average waiting time, complete vacation interruptions.

Yenchiali[11]. An exhaustive survey on vacation models was done by Doshi[4, 5] and a detailed interpretation of the model was discussed in the book of Takagi[17]. Several researchers like Tian and Zhang[18] had shown interest in vacation policies in queueing systems and extend this notion a while.

There are three major types of vacation policies in queueing systems which are namely single vacation, multiple vacations and working vacations. In single vacation scheme, the server takes a vacation of random duration when the queue is empty. At the end of his vacation, he returns to the queue and if he finds at least one customer is waiting in the queue he immediately starts to serve them by exhaustive, gated or limited service policy. Whereas in multiple vacations scheme, the server immediately takes another vacation if he finds no customer in the queue when he returned from a vacation. In working vacation strategies the server has to work with a different rate rather than completely stopping the service during a vacation. This concept was first discussed by Servi and Finn[15] while modeling a wavelength division multiplexing optical access network with an $M/M/1$ queue. Many researchers attracted into the working vacation policies in queueing systems and provided a lot of fruitful theories related to that concept. Baba[2] approached a $GI/M/1$ queue with working vacations by using matrix analytical method. Further Wu and Takagi[21] generalized the model described by Servi and Finn to an $M/G/1$ queue with general working vacation. Tian et al.[19] analyzed the discrete-time $Geo/Geo/1$ queue with multiple working vacations.

Server vacations in a queueing system can be applied to handle the situations where the servers wish to utilize their idle time for different purposes which may or may not be related to the assigned jobs for them. Jain and Upadhyaya[9] obtained the steady-state probabilities of the number of failed machines in the system together with some performance measures by using matrix recursive method for multiple vacations policies. Some researchers elaborated the working vacation strategies into batch arrival queues, Xu et al.[22] discussed a batch arrival $M^X/M/1$ queue with single working vacation. But Baba[3] studied $M^X/M/1$ with multiple working vacations. In both the papers, they derived the probability generating function of the stationary system length distribution corresponding to vacations under consideration. For a batch arrival queueing system Jain and Agrawal[8] presented the queue size distribution and other performance measures with modified Bernoulli vacation under N- policy using the generating function methodology.

Ibe and Isijola[6] introduced new type of vacations called differentiated vacations which distiguish the durations of consecutive vacations offered for the server in multiple vacations queueing systems. Recently Vyshna Unni and Julia Rose Mary[20] studied about queueing systems with multiple servers under differentiated working vacations. In majority of service sectors occur situations like the server has to return from a vaction when the wave of customers in the system crosses the limit. Li and Tian[12] analyzed this scenario and termed it as vacation interruptions in queueing systems and they studied $M/M/1$ queues with vacation interruption policies. They also discussed about discrete time $GI/Geo/1$ queue under working vacations and vacation interruptions[13]. Further lot of comprehensive studies realted to vacation interruption policies emerged. Zhang and

Hou[23] discussed about the $M/G/1$ queue with working vacations and vacation interruptions. By using Erlang-k type distribution Ayyappan et al.[1] analyzed $M/M/1$ retrail queueing system with vacation interruptions. Krishnamoorthy and Sreenivasan[10] studied $M/M/2$ queueing system with heterogeneous servers where one server is always available and the other goes on a working vacation if there are no customers in the system. Sreenivasan et al.[16] introduced the concept of threshold for vacations interruptions while analysing a single server queueing model in which the customers arrive according to a Markovian arrival process. Ibe and Isijola(2014)[7] analyzed $M/M/1$ queueing systems with differentiated vacations and vacation interruptions.

Queueing systems with vacation interruption under differentiated vacations can be widely used to model many physical situations, for example in a hospital the vacation taken by the doctor must be interrupted in an emergency situation like accident, fire attack and so on. The corresponding doctor will come back to the normal working level without any hesitation if we provide the opportunity of differentiated vacations for him and hence fast and better treatment will obtain for the patient. Similarly, the leave sanctioned for a soldier may suddenly get cancelled to handle some public defense like terrorist attack. Like these scenarios, if we model the system with more service providers the customer's impatient rate will get decreased tremendously. As a result, the rating and the profit of the service sectors will improve. Thus, queueing system with more than one server under differentiated vacation and vacation interruption will open a vast platform for the new explorations in the theory of queue. Hence in this paper we introduced the vacation interruption policies into multiple servers queueing system with differentiated vacations.

We assume that the system consists of C servers and for each of them has the opportunity for taking two types of vacations with different durations, namely type 1 and type 2 vacations. The servers can opt for the type 1 of vacation only after serving atleast one customer. Hence type 1 vacation can be started by the servers if each of them has completed a busy period of nonzero duration. But the servers can go for a vacation of type 2 if there are no customers are waiting in any of the C- queues for the service when they return from a vacation. We consider the model with vacation interruption strategies and so the C- servers of the recommended model are forced to return from the vacations when the number of customers in the system reaches some predefined thresholds. Thus depending upon the flow of customers the servers may come back to the normal working level before the period of vacations end. By analysing the characteristics of a differentiated vacation queueing system its quite trivial that when an interruption of vacation in such system is desired interrupt type 2 vacation first. Thus we assume that in the proposed queueing model both types of vacations can be interrupted by ensuring that the type 2 vacation of servers will be interrupted first, because it takes after a busy period of zero duration. We refer to this as a complete vacation interruption policy in a multiple server queueing systems under differentiated vacations.

The rest of the paper is organized as follows. Section 2 describes the recommended model with its state transition diagram. Section 3 analyses the steady-state probabilities and the average waiting time generated in the system. Numerical results are discussed in section 4 with graphical interpretations and the obained results are concluded in section 5.

2. System model

We consider a multiple vacations queueing system with C- servers where the customers arrive according to a Poisson process with rate λ and the service take place according to an exponential distribution with mean $\frac{1}{\mu}$, where $\mu > \lambda$. Depending upon the number of customers have treated by each server during their busy schedule we offer two types of vacation policies for the servers of the system, which are namely type 1 and type 2 vacations. The former vacation is taken by the servers after completing a non zero busy period of each of them, i.e., type 1 vacations can be started by the servers if each of them has serverd atleast one customer entered into the system. But type 2 vacations can be taken after a busy period of zero duration. In other words servers can go for a vacation of type 2 if no customers are waiting in any of the C-queues for service when they retured from a vacation. Note that the servers can repeat type 2 vacations till they find atleast one non empty queue among the C- queues of the system. We assume that the durations of type 1 and type 2 vacations are exponentially distributed with means $\frac{1}{\omega_1}$ and $\frac{1}{\omega_2}$ respectively. We define the state of the system as (k, s), where k is the number of customers in the system and s is defined as follows

$$s = \begin{cases} 0, & \text{if all the servers are in active mode} \\ 1, & \text{if all the servers are in type 1 vacation} \\ 2, & \text{if all the servers are in type 2 vacation} \end{cases}$$

We formulate the model by also introducing the complete vacation interruption strategies among the servers of the system; in which both types of vacation can be interrupted but while considering the nature of the vacations that we allowed in the system make sense to interrupt a type 2 vacation before interrupting a type 1 vacation. Suppose that a type 2 vacation can be interrupted when the number of customers in the system reaches the threshold value k_2 and a type 1 vacation can be interrupted when the number of customers in the system reaches the threshold k_1. Since the considered queueing model is occupied with C- servers, we assume that the vacation interruptions will take place only after the number of customers exceeds the number of servers in the system, i.e., $k_1 \geq k_2 > C$. Thus the system can be modelled by a continuous time Markov chain with state transition diagram drawn as in Fig. 1. While look into the diagram its obvious that when the system is in state $(k_2 - 1, 2)$, the next customer arrival forces the type 2 vacations to end and a transition to the state $(k_2, 0)$ take place. Simillarly when the system is in the state $(k_1 - 1, 1)$ the next customer arrival forces the type 1 vacation to end and a transition to the state $(k_1, 0)$ take place.

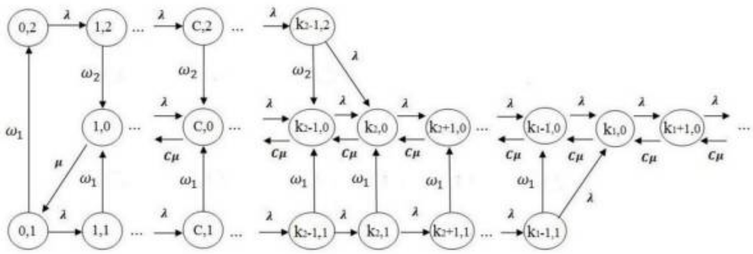

FIGURE 1. State Transition-rate Diagram

3. Steady-state Analysis

Let $P_{k,s}$ denote the steady-state probability that the system is in the state (k, s) where $k \in \mathbb{N}$ and $s = 0, 1, 2$. By overviewing the state transition diagram of the recommended model given in Fig. 1 we derive the following theorem regarding the steady-state probabilities and average queue length of the model.

Theorem 3.1. *Under complete vacation interruption policies the steady-state probabilities $P_{k,s}$ for different values of s are given by*

(i)

$$P_{k,0} = \begin{cases} \left(\frac{\rho^{k-1}}{k!} + \sum_{i=0}^{k-2} \frac{\rho^{k-i-1}(i+1)!}{k!} \left[\zeta_1\eta_1^{i+1} + \zeta_2\eta_2^{i+1}\right]\right) P_{1,0} & \text{if } k \le C \\ \left(\left(\frac{\rho}{C}\right)^{k-C} A(C,0) + \sum_{i=1}^{k-C} \left(\frac{\rho}{C}\right)^{i} \left[\zeta_1\eta_1^{k-i} + \zeta_2\eta_2^{k-i}\right]\right) P_{1,0} & \text{if } C+1 \le k \le k_2 \\ \left(\left(\frac{\rho}{C}\right)^{k-k_2} A(k_2,0) + \frac{\rho\zeta_1\eta_1^{k_2}}{C\eta_1-\rho} \left[\eta_1^{k-k_2} - \left(\frac{\rho}{C}\right)^{k-k_2}\right]\right) P_{1,0} & \text{if } k_2 \le k \le k_1 \\ \left(\left(\frac{\rho}{C}\right)^{k-k_1} A(k_1,0)\right) P_{1,0} & \text{if } k \ge k_1 \end{cases}$$

(ii) $P_{k,1} = \zeta_1\eta_1^k P_{1,0},\ k = 0, 1, ...k_1 - 1$

(iii) $P_{k,2} = \zeta_2\eta_2^k P_{1,0}, k = 0, 1, ...k_2 - 1$

where C is the number of servers in the system, $\rho = \frac{\lambda}{\mu}$ is the offered load, $\zeta_1 = \frac{\mu}{(\lambda+\omega_1)}, \zeta_2 = \frac{\mu\omega_1}{\lambda(\lambda+\omega_1)}, \eta_1 = \frac{\lambda}{\lambda+\omega_1} < 1,\ \eta_2 = \frac{\lambda}{\lambda+\omega_2} < 1,$

$$A(C,0) = \frac{\rho^{C-1}}{C!} + \sum_{i=0}^{C-2} \frac{\rho^{C-i-1}(i+1)!}{C!} \left[\zeta_1\eta_1^{i+1} + \zeta_2\eta_2^{i+1}\right],$$

$$A(k_2,0) = \left(\frac{\rho}{C}\right)^{k_2-C} A(C,0) + \sum_{i=1}^{k_2-C} \left(\frac{\rho}{C}\right)^{i} \left[\zeta_1\eta_1^{k_2-i} + \zeta_2\eta_2^{k_2-i}\right],$$

$$A(k_1,0)=\left(\frac{\rho}{C}\right)^{k_1-k_2}A(k_2,0)+\frac{\rho\zeta_1\eta_1^{k_2}}{C\eta_1-\rho}\left[\eta_1^{k_1-k_2}-\left(\frac{\rho}{C}\right)^{k_1-k_2}\right].$$

The steady-state probability $P_{1,0}=\frac{1}{S_1+S_2+S_3+S_4+S_5+S_6}$*, where*

$$S_1=\sum_{k=1}^{C}\left(\frac{\rho^{k-1}}{k!}+\sum_{i=0}^{k-2}\frac{\rho^{k-i-1}(i+1)!}{k!}\left[\zeta_1\eta_1^{i+1}+\zeta_2\eta_2^{i+1}\right]\right)$$

$$S_2=\frac{\rho}{C-\rho}\left(1-\left(\frac{\rho}{C}\right)^{k_2-1-C}\right)A(C,0)+\sum_{k=C+1}^{k_2-1}\left\{\frac{\rho\zeta_1\eta_1^{C}}{C\eta_1-\rho}\left[\eta_1^{k-C}-\left(\frac{\rho}{C}\right)^{k-C}\right]+\frac{\rho\zeta_2\eta_2^{C}}{C\eta_2-\rho}\left[\eta_2^{k-C}-\left(\frac{\rho}{C}\right)^{k-C}\right]\right\}$$

$$S_3=A(K_2,0)\frac{C}{C-\rho}\left(1-\left(\frac{\rho}{C}\right)^{k_1-k_2}\right)+\frac{\rho\zeta_1\eta_1^{k_2}}{C\eta_1-\rho}\left\{\frac{1-\eta_1^{k_1-k_2}}{1-\eta_1}-\frac{C}{C-\rho}\left(1-\left(\frac{\rho}{C}\right)^{k_1-k_2}\right)\right\}$$

$S_4=\frac{C}{C-\rho}A(K_1,0)$, $S_5=\frac{1-\eta_1^{k_1}}{1-\eta_1}$ *and* $S_6=\frac{1-\eta_1^{k_2}}{1-\eta_2}$.

The average queue length of the system is given by $E(m)=Q_1+Q_2+Q_3+Q_4+Q_5$*, where*

$$Q_1=\sum_{k=C}^{k_2-1}(k-C)P_{k,0}=\left\{\frac{C\rho}{(C-\rho)^2}A(C,0)\left((k_2-1-C)\left(\frac{\rho}{C}\right)^{k_2-C}-(k_2-C)\left(\frac{\rho}{C}\right)^{k_2-1-C}+1\right)+\sum_{k=C+1}^{k_2-1}(k-C)\left\{\frac{\rho\zeta_1\eta_1^{C}}{C\eta_1-\rho}\left[\eta_1^{k-C}-\left(\frac{\rho}{C}\right)^{k-C}\right]+\frac{\rho\zeta_2\eta_2^{C}}{C\eta_2-\rho}\left[\eta_2^{k-C}-\left(\frac{\rho}{C}\right)^{k-C}\right]\right\}\right\}P_{1,0}$$

$$Q_2=\sum_{k=k_2}^{k_1-1}(k-C)P_{k,0}=\left\{A(k_2,0)\left\{\frac{C\rho}{(C-\rho)^2}\left[(k_1-k_2-1)\left(\frac{\rho}{C}\right)^{k_1-k_2}-(k_1-k_2)\left(\frac{\rho}{C}\right)^{k_1-k_2-1}+1\right]+\frac{C(k_2-C)\left(1-(\frac{\rho}{C})^{k_1-k_2}\right)}{C-\rho}\right\}+\frac{\rho\zeta_1\eta_1^{k_2}}{C\eta_1-\rho}\left\{\frac{(k_2-C)\left(1-\eta_1^{k_1-k_2}\right)}{1-\eta_1}+\frac{\eta_1\left((k_1-k_2-1)\eta_1^{k_1-k_2}-(k_1-k_2)\eta_1^{k_1-k_2-1}+1\right)}{(1-\eta_1)^2}-\frac{C(k_2-C)\left(1-(\frac{\rho}{C})^{k_1-k_2}\right)}{C-\rho}-\frac{C\rho}{(C-\rho)^2}\left[(k_1-k_2-1)\left(\frac{\rho}{C}\right)^{k_1-k_2}-(k_1-k_2)\left(\frac{\rho}{C}\right)^{k_1-k_2-1}+1\right]\right\}\right\}P_{1,0}$$

$$Q_3=\sum_{k=k_1}^{\infty}(k-C)P_{k,0}=\left\{\left(\frac{C(C-\rho)(k_1-C)+C\rho}{(C-\rho)^2}\right)A(k_1,0)\right\}P_{1,0}$$

$$Q_4=\sum_{k=C}^{k_1-1}(k-C)P_{k,1}=\left\{\zeta_1\eta_1^{C+1}\left(\frac{(k_1-C-1)\eta_1^{k_1-C}-(k_1-C)\eta_1^{k_1-C-1}+1}{(1-\eta_1)^2}\right)\right\}P_{1,0}$$

$$Q_5 = \sum_{k=C}^{k_2-1}(k-C)P_{k,2} = \left\{\zeta_2\eta_2^{C+1}\left(\frac{(k_2-C-1)\eta_2^{k_2-C}-(k_2-C)\eta_2^{k_2-C-1}+1}{(1-\eta_2)^2}\right)\right\}P_{1,0}$$

With the aid of $E(m)$ the average waiting time that a customer spends in the system(or mean delay) is given by

$$E(v) = \frac{E(m)}{\lambda} + \frac{1}{\mu}$$

Proof. : By applying the global balances in Fig. 1 we obtain

$$\mu P_{1,0} = (\lambda+\omega_1)P_{0,1} \tag{3.1}$$

$$\lambda P_{0,2} = \omega_1 P_{0,1}. \tag{3.2}$$

$$\lambda P_{k-1,1} = (\lambda+\omega_1)P_{k,1},\ \ k=1,2,\ldots,k_1-1 \tag{3.3}$$

$$\lambda P_{k-1,2} = (\lambda+\omega_2)P_{k,2},\ \ k=1,2,\ldots,k_2-1 \tag{3.4}$$

Hence

$$P_{0,1} = \frac{\mu}{(\lambda+\omega_1)}P_{1,0} = \zeta_1 P_{1,0}, \tag{3.5}$$

$$P_{0,2} = \frac{\omega_1}{\lambda}P_{0,1} = \zeta_2 P_{1,0}, \tag{3.6}$$

$$P_{k,1} = \left(\frac{\lambda}{\lambda+\omega_1}\right)^k P_{0,1} = \zeta_1\eta_1^k P_{1,0},\ \ k=0,1,\ldots k_1-1 \tag{3.7}$$

$$P_{k,2} = \left(\frac{\lambda}{\lambda+\omega_2}\right)^k P_{0,2} = \zeta_2\eta_2^k P_{1,0},\ \ k=0,1,\ldots k_2-1 \tag{3.8}$$

where $\zeta_1 = \frac{\mu}{(\lambda+\omega_1)}$, $\zeta_2 = \frac{\mu\omega_1}{\lambda(\lambda+\omega_1)}$. $\eta_1 = \frac{\lambda}{\lambda+\omega_1} < 1$ and $\eta_2 = \frac{\lambda}{\lambda+\omega_2} < 1$.

By considering the local balances for the steady-state probabilities of busy periods we get

$$\lambda P_{k-1,0} + \lambda P_{k-1,1} + \lambda P_{k-1,2} = k\mu P_{k,0},\ \ k \le C \tag{3.9}$$

and

$$\lambda P_{k-1,0} + \lambda P_{k-1,1} + \lambda P_{k-1,2} = C\mu P_{k,0},\ \ C+1 \le k \le k_2 \tag{3.10}$$

By letting $\rho = \frac{\lambda}{\mu}$ and by using equations (3.7) and (3.8) we can write the equations (3.9) and (3.10) as

$$P_{k,0} = \frac{\rho}{k}\left(P_{k-1,0} + \zeta_1\eta_1^{k-1}P_{1,0} + \zeta_2\eta_2^{k-1}P_{1,0}\right),\ \ k \le C \tag{3.11}$$

and

$$P_{k,0} = \frac{\rho}{C}\left(P_{k-1,0} + \zeta_1\eta_1^{k-1}P_{1,0} + \zeta_2\eta_2^{k-1}P_{1,0}\right),\ \ C+1 \le k \le k_2. \tag{3.12}$$

respectively. By solving equations (3.11) and (3.12) recursively we obtain,

$$P_{k,0} = \left(\frac{\rho^{k-1}}{k!} + \sum_{i=0}^{k-2}\frac{\rho^{k-i-1}(i+1)!}{k!}[\zeta_1\eta_1^{i+1} + \zeta_2\eta_2^{i+1}]\right)P_{1,0},\ \ k \le C \tag{3.13}$$

and

$$P_{k,0} = \left(\left(\frac{\rho}{C}\right)^{k-C} A(C,0) + \sum_{i=1}^{k-C} \left(\frac{\rho}{C}\right)^{i} [\zeta_1 \eta_1^{k-i} + \zeta_2 \eta_2^{k-i}] \right) P_{1,0}, \quad C+1 \le k \le k_2. \tag{3.14}$$

where $A(C,0)$ is defined by

$$A(C,0) = \frac{\rho^{C-1}}{C!} + \sum_{i=0}^{C-2} \frac{\rho^{C-i-1}(i+1)!}{C!} \left[\zeta_1 \eta_1^{i+1} + \zeta_2 \eta_2^{i+1}\right]. \tag{3.15}$$

Simillarly by considering the local balance for the busy state when $k_2+1 \le k \le k_1$, we obtain

$$\lambda P_{k-1,0} + \lambda P_{k-1,1} = C\mu P_{k,0}. \tag{3.16}$$

By solving equation (3.16) recursively we get,

$$P_{k,0} = \left(\left(\frac{\rho}{C}\right)^{k-k_2} A(k_2,0) + \frac{\rho \zeta_1 \eta_1^{k_2}}{C\eta_1 - \rho} \left[\eta_1^{k-k_2} - \left(\frac{\rho}{C}\right)^{k-k_2} \right] \right) P_{1,0} \tag{3.17}$$

where

$$A(k_2,0) = \left(\frac{\rho}{C}\right)^{k_2-C} A(C,0) + \sum_{i=1}^{k_2-C} \left(\frac{\rho}{C}\right)^{i} \left[\zeta_1 \eta_1^{k_2-i} + \zeta_2 \eta_2^{k_2-i}\right] \tag{3.18}$$

From equation (3.17) we have

$$\begin{aligned} P_{k_1,0} &= \left(\left(\tfrac{\rho}{C}\right)^{k_1-k_2} A(k_2,0) + \tfrac{\rho \zeta_1 \eta_1^{k_2}}{C\eta_1 - \rho} \left[\eta_1^{k_1-k_2} - \left(\tfrac{\rho}{C}\right)^{k_1-k_2} \right] \right) P_{1,0} \\ &= A(k_1,0) P_{1,0} \ (say) \end{aligned} \tag{3.19}$$

By observing the local balance for $k > k_1$ we can write

$$\lambda P_{k-1,0} = C\mu P_{k,0} \tag{3.20}$$

That is

$$P_{k,0} = \frac{\rho}{C} P_{k-1,0} \tag{3.21}$$

By using equation (3.19) solve the above equation recursively. Then we get

$$P_{k,0} = \left(\left(\frac{\rho}{C}\right)^{k-k_1} A(k_1,0) \right) P_{1,0}, \quad k \ge k_1 \tag{3.22}$$

To find the value of $P_{1,0}$ applying the law of total probability,
i.e.,

$$\sum_{k=1}^{\infty} P_{k,0} + \sum_{k=0}^{k_1-1} P_{k,1} + \sum_{k=0}^{k_2-1} P_{k,2} = 1. \tag{3.23}$$

i.e.,

$$\sum_{k=1}^{C} P_{k,0} + \sum_{k=C+1}^{k_2-1} P_{k,0} + \sum_{k=k_2}^{k_1-1} P_{k,0} + \sum_{k=k_1}^{\infty} P_{k,0} + \sum_{k=0}^{k_1-1} P_{k,1} + \sum_{k=0}^{k_2-1} P_{k,2} = 1. \tag{3.24}$$

By substituting the respective probabilities in the above summations and by following some algebraic manupulations in each summation we can derive that the steady-state probability

$$P_{1,0} = \frac{1}{S_1 + S_2 + S_3 + S_4 + S_5 + S_6} \tag{3.25}$$

where

$$S_1 = \sum_{k=1}^{C} \left(\frac{\rho^{k-1}}{k!} + \sum_{i=0}^{k-2} \frac{\rho^{k-i-1}(i+1)!}{k!} \left[\zeta_1 \eta_1^{i+1} + \zeta_2 \eta_2^{i+1} \right] \right) \tag{3.26}$$

$$\begin{aligned} S_2 &= \frac{\rho}{C-\rho} \left(1 - \left(\frac{\rho}{C} \right)^{k_2-1-C} \right) A(C,0) \\ &+ \sum_{k=C+1}^{k_2-1} \left\{ \frac{\rho \zeta_1 \eta_1^C}{C\eta_1 - \rho} \left[\eta_1^{k-C} - \left(\frac{\rho}{C} \right)^{k-C} \right] + \right. && (3.27) \\ &\left. \frac{\rho \zeta_2 \eta_2^C}{C\eta_2 - \rho} \left[\eta_2^{k-C} - \left(\frac{\rho}{C} \right)^{k-C} \right] \right\} && (3.28) \\ S_3 &= A(K_2, 0) \frac{C}{C-\rho} \left(1 - \left(\frac{\rho}{C} \right)^{k_1-k_2} \right) + \\ & \frac{\rho \zeta_1 \eta_1^{k_2}}{C\eta_1 - \rho} \left\{ \frac{1 - \eta_1^{k_1-k_2}}{1-\eta_1} - \frac{C}{C-\rho} \left(1 - \left(\frac{\rho}{C} \right)^{k_1-k_2} \right) \right\} && (3.29) \\ S_4 &= \frac{C}{C-\rho} A(K_1, 0) && (3.30) \\ S_5 &= \frac{1-\eta_1^{k_1}}{1-\eta_1} && (3.31) \end{aligned}$$

and

$$S_6 = \frac{1-\eta_1^{k_2}}{1-\eta_2} \tag{3.32}$$

The average queue length of the queueing system with C- service channels can be calculated by

$$E(m) = \sum_{K=C}^{\infty} (k-C) P_k \tag{3.33}$$

But for the proposed model $E(m)$ is given by

$$\begin{aligned} E(m) &= \sum_{k=C}^{k_2-1} (k-C) P_{k,0} + \sum_{k=k_2}^{k_1-1} (k-C) P_{k,0} + \sum_{k=k_1}^{\infty} (k-C) P_{k,0} \\ + & \sum_{k=C}^{k_1-1} (k-C) P_{k,1} + \sum_{k=C}^{k_2-1} (k-C) P_{k,2} && (3.34) \end{aligned}$$

Consider each summation in the above equation seperately with their corresponding probabilities and by evaluating them with some algebraic calculations we find,

$$\begin{aligned}\sum_{k=C}^{k_2-1}(k-C)P_{k,0} \quad = \quad & \left\{\frac{C\rho}{(C-\rho)^2}A(C,0)\left((k_2-1-C)\left(\frac{\rho}{C}\right)^{k_2-C}\right.\right. \\ & \left. -(k_2-C)\left(\frac{\rho}{C}\right)^{k_2-1-C}+1\right) \\ & +\sum_{k=C+1}^{k_2-1}(k-C)\left\{\frac{\rho\zeta_1\eta_1^C}{C\eta_1-\rho}\left[\eta_1^{k-C}-\left(\frac{\rho}{C}\right)^{k-C}\right]+\right. \\ & \left.\left.\frac{\rho\zeta_2\eta_2^C}{C\eta_2-\rho}\left[\eta_2^{k-C}-\left(\frac{\rho}{C}\right)^{k-C}\right]\right\}\right\}P_{1,0} \qquad (3.35)\end{aligned}$$

$$\begin{aligned}\sum_{k=k_2}^{k_1-1}(k-C)P_{k,0} = & \left\{A(k_2,0)\left\{\frac{C\rho}{(C-\rho)^2}\left[(k_1-k_2-1)\left(\frac{\rho}{C}\right)^{k_1-k_2}-(k_1-k_2)\left(\frac{\rho}{C}\right)^{k_1-k_2-1}+1\right]\right.\right. \\ & \left.+\frac{C(k_2-C)\left(1-(\frac{\rho}{C})^{k_1-k_2}\right)}{C-\rho}\right\}+\frac{\rho\zeta_1\eta_1^{k_2}}{C\eta_1-\rho}\left\{\frac{(k_2-C)\left(1-\eta_1^{k_1-k_2}\right)}{1-\eta_1}\right. \\ & +\frac{\eta_1\left((k_1-k_2-1)\eta_1^{k_1-k_2}-(k_1-k_2)\eta_1^{k_1-k_2-1}+1\right)}{(1-\eta_1)^2}-\frac{C(k_2-C)\left(1-(\frac{\rho}{C})^{k_1-k_2}\right)}{C-\rho} \\ & \left.\left.-\frac{C\rho}{(C-\rho)^2}\left[(k_1-k_2-1)\left(\frac{\rho}{C}\right)^{k_1-k_2}-(k_1-k_2)\left(\frac{\rho}{C}\right)^{k_1-k_2-1}+1\right]\right\}\right\}P_{1,0}\end{aligned} \tag{3.36}$$

$$\sum_{k=k_1}^{\infty}(k-C)P_{k,0} = \left\{\left(\frac{C(C-\rho)(k_1-C)+C\rho}{(C-\rho)^2}\right)A(k_1,0)\right\}P_{1,0} \tag{3.37}$$

$$\sum_{k=C}^{k_1-1}(k-C)P_{k,1} = \left\{\zeta_1\eta_1^{C+1}\left(\frac{(k_1-C-1)\eta_1^{k_1-C}-(k_1-C)\eta_1^{k_1-C-1}+1}{(1-\eta_1)^2}\right)\right\}P_{1,0} \tag{3.38}$$

$$\sum_{k=C}^{k_2-1}(k-C)P_{k,2} = \left\{\zeta_2\eta_2^{C+1}\left(\frac{(k_2-C-1)\eta_2^{k_2-C}-(k_2-C)\eta_2^{k_2-C-1}+1}{(1-\eta_2)^2}\right)\right\}P_{1,0} \tag{3.39}$$

By substituting the equations (3.35), (3.36), (3.37),(3.38) and (3.39) in (3.34) we obtain the formula for calculating $E(m)$ of the recommended model. Finally, from Little's formula[14] the average waiting time ($E(v)$) generated in the system is given by

$$E(v) = \frac{E(m)}{\lambda}+\frac{1}{\mu} \tag{3.40}$$

This completes the proof. □

4. Numerical Analysis

In this section we investigate the effect of parameter measures such as the mean durations of vacations, thresholds for vacation interruptions and number of severs on the average waiting time generated in the system under complete vacation interruption policies. Thoughout the discussion we fix the service rate μ as 0.25 and assume two values for the number of servers (C), $C = 2$ and $C = 3$. As mentioned earlier under complete vacation interruption policies both types of vacations can be interrupted accordingly when the number of customers in the system reaches k_2 and k_1 for type 2 and type 1 vacations respectively. First we fix the value of k_2 as 6 and varies the value of k_1 as 10,15 and 20 respectively in three different cases of vacation durations: when the duration $(1/\omega_1)$ of type 1 vacation is longer than type 2 vacation $(1/\omega_2)$ (*i.e.*, $\omega_1 < \omega_2$), when both vacation types are of the same durations (*i.e.*, $\omega_1 = \omega_2$) and when the duration of type 1 vacation is shorter than type 2 vacation (*i.e.*, $\omega_1 > \omega_2$). Figures 2, 3 and 4 depict the variations of $E(v)$ against the offered load ρ for the three respective cases.

From figure 2 it can be observed that for each value of C, $E(v)$ decreases correspondingily as the value of k_1 decreases. Thus when $\omega_1 < \omega_2$ the early interruption of type 1 vacation has a significant impact on the average waiting time generated by both 2 and 3 servers. Figure 3 reveals that when $\omega_1 = \omega_2$ there does not occur any remarkable changes in $E(v)$ by varing k_1 in the system. Thus interrupting vacations of same mean durations with larger values of k_1 will not satisfy the customers by avoiding the excess mean delay originated in the system. Figure 4 picturized that when $\omega_1 > \omega_2$ also the variations of k_1 does not affect $E(v)$ significantly. But there is a considerable reduction in the value of $E(v)$ for each value of ρ when $\omega_1 \geq \omega_2$. This is due to the fact that we are interrupting first a vacation whose duration is equal or longer than the other vacation type. Thus it can be concluded that among the three cases of vacation durations $E(v)$ is more responsive to the variations of k_1 only when $\omega_1 < \omega_2$. But in all the three cases the average waiting time generated by 2 servers is greater than that of 3 servers.

Next we fix the value of k_1 as 20 and varies the value of k_2 as 6,8 and 10 respectively for the same 3 cases of vacation durations as above. By analysing the graphs given in figure 5 which interpret the behaviour of $E(v)$ for $\omega_1 > \omega_2$ it can be observed that the average waiting time of the system reduces systematically when k_2 decreases in its value. So the early interruption of a vacation with longer duration can reduce mean delay of the system remarkably.

Figures 6 and 7 show the performance of the system for $\omega_1 < \omega_2$ and $\omega_1 = \omega_2$ respectively. By examine these figures it can be concluded that interrupting first a vacation of shorter or equal duration does not make that much significant effect on the average waiting time as $\omega_1 > \omega_2$. Hence we conclude that k_2 has the most notable impact on the system delay when $\omega_1 > \omega_2$. The entire analysis enable us to realize that the complete vacation interruption become more effective in the recommended model when $\omega_1 > \omega_2$ with early interruption of type 2 vacation. Moreover by increasing the number of servers in the system, it is practicable to enlarge the thresholds of k_1 and k_2 which results in not extending the mean delay of the system noticeably.

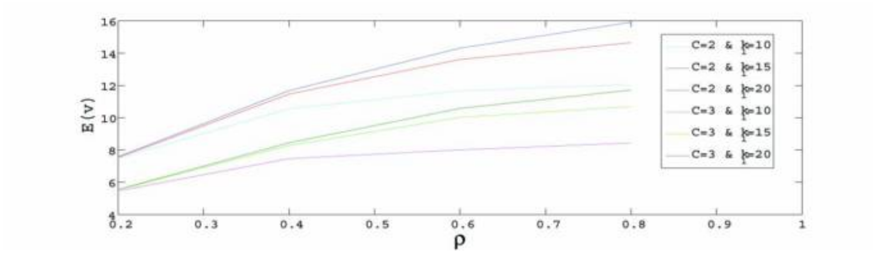

FIGURE 2. Mean time in the system by varing ρ for $\omega_1 = 0.05$, $\omega_2 = 0.1$ and $k_2 = 6$

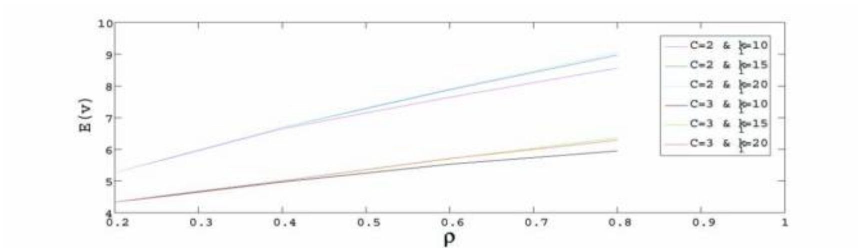

FIGURE 3. Mean time in the system by varing ρ for $\omega_1 = 0.1 = \omega_2$ and $k_2 = 6$

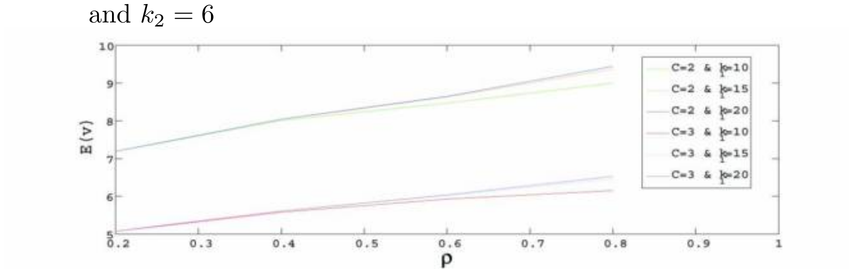

FIGURE 4. Mean time in the system by varing ρ for $\omega_1 = 0.1$, $\omega_2 = 0.05$ and $k_2 = 6$

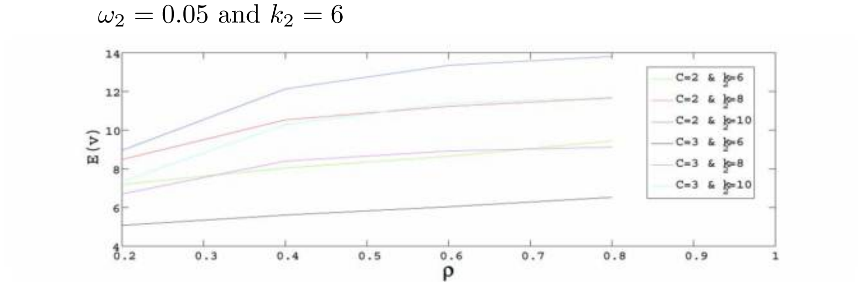

FIGURE 5. Mean time in the system by varing ρ for $\omega_1 = 0.1$, $\omega_2 = 0.05$ and $k_1 = 20$

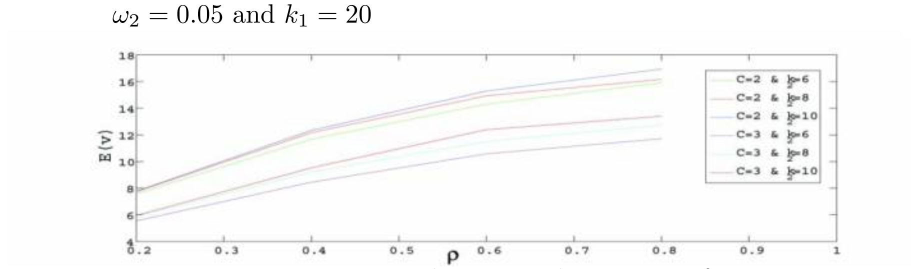

FIGURE 6. Mean time in the system by varing ρ for $\omega_1 = 0.05$, $\omega_2 = 0.1$ and $k_1 = 20$

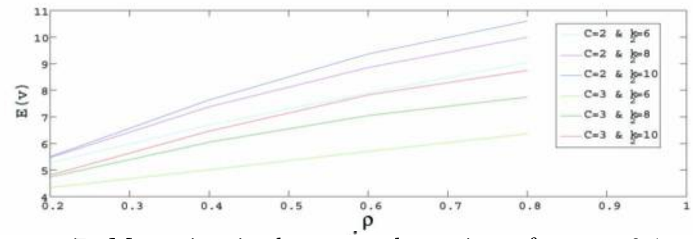

FIGURE 7. Mean time in the system by varing ρ for $\omega_1 = 0.1 = \omega_2$ and $k_1 = 20$

5. Conclusion

In this paper we proposed a multiple servers differentiated vacation queueing system with complete vacation interruption strategies. Hence the servers of the system can be interrupted during any types of vacations under consideration. We derived the formulas for calculating the steady-state probabilities and the average waiting time generated in the recommended model. The results indicate that under complete vacation interruption policies the average waiting time is greately affected by the vacation termination when $\omega_1 > \omega_2$ and modestly affected by the the vacation termination when $\omega_1 < \omega_2$. The graphical interpretations show that when both types of vacations have unequal durations the average waiting time will be increased gradually as the thresholds k_1 and k_2 increase. So the early interruption of both types of vacation have significant impact on the mean delay of the system. In other words the vacation termination has no appreciable advantage over there was no termination with higher values of k_1 and k_2. The most significant conclusion from the entire analysis is by increasing the number of servers in the system, it is possible to make delay for the vacation interruption without affecting the average waiting time generated in the system noticeably.

The recommended model can be widely applied in many real life situations for example it can be introduced in medical centers to handle effectively the pandemic situation like COVID-19. As we all know that the COVID-19 pandemic has frightened the entire people of the world. The intense contagious of COVID-19 had tremendously shaken and impasse the whole countries of the world. Current situation indicates that there is going to be an imbalance between demand and availability of hospital beds, ICU beds, ventilators, PPE and trained medical personals throughout the country. Therefore, in such circumstances, authorities will find it difficult to provide appropriate health care facilities to all sections of society and the COVID effected people. So, the hospitals need indispensable medical facilities and well preparedness to face the pandemic outbreak. For proper care and preventions, transmission of infection, it is important to train doctors, nurses, technicians, support staff and sanitation workers in each hospital quickly. Through these toilsome acts we could restrain the spreading of virus at least to an extent. Hence as soon possible we have to arrange more hospitals with all needs to fight against the next wave of corona virus. Let us model the intensive care units with the concept of differentiated vacation and vacation interruption of complete type. Since the pandemic like COVID-19 is an easily spreading disease, organize the emergency rooms for the patients with more hygienic and high facilities like ventilators, PPE, well trained doctors, nurses, lab technicians and all. Once a patient was occupied in an emergency room the type 1 vacation can be used to sanitize, sterilize and to arrange the equipment with some necessary clean ups for the next rush of patients. Where as type 2 vacation can be made useful for the actual rest of staffs in the emergency room provided the mandatory setups and clean ups have been done already. Moreover, the interruption of both type1 and type 2 vacations in accordance with the predefined thresholds will help to handle the next wave of patients systematically without any inconvenience for both patients and medical staffs. The concept of differentiated vacations and vacation interruption can also be applied in between doctors, nurses, pharmacists, ambulance drivers and so on.

Acknowledgment. The authors would like to express their sincere thanks to associate editors and the anonymous reviewers for their constructive comments and valuable suggestions which have improved the quality of the manuscript.

References

1. Ayyappan, G., Sekar, G. and Ganapathi, A. M.: M/M/1 retrial queueing system with vacation interruptions under ERLANG-K service, *Int. J. Comput. Appl.* **2** (2010) 52–57.
2. Baba, Y.: Analysis of a $G1/M/1$ queue with multiple working vacations, *Oper. Res. Lett.* **33** (2005) 201–209.
3. Baba, Y.: The $M^X/M/1$ queue with multiple working vacation, *Am. J. Oper. Res.* **2** (2012) 217–224 .
4. Doshi, B. T.: Queueing systems with vacations- a survey, *QUESTA* **1** (1986) 29–66.
5. Doshi, B. T.: Single server queues with vacations, *Stochastic Analysis of Computer and Communication Systems*, (1990) 217–265.
6. Ibe, O. C. and Isijola, O. A.: M/M/1 multiple Vacation Queueing systems with Differentiated vacations, *Model. Simul. Eng.* **1** (2014) 1–6.
7. Ibe, O. C. and Isijola, O. A.: M/M/1 multiple vacation queueing systems with differentiated vacations and vacation interruptions, *IEEE Access.* (2014).
8. Jain, M. and Agrawal, P. K.: *N*-policy for state-dependent batch arrival queueing system with l-large service and modified Bernoulli schedule vacation, *Qual. Technol Quant. Manag.* **7** (2010) 215–230.
9. Jain, M. and Upadhyaya, S.: Threshold N-policy for degraded machining system with mulitiple type spares and multiple vacations, *Qual. Technol Quant. Manag.* **6** (2009) 185–203.
10. Krishnamoorthy, A. and Sreenivasan, C.: An M/M/2 queueing system with heterogeneous servers including one with working vacation, *Int.J. Stochastic Anal.* (2012).
11. Levy, Y. and Yechiali, U.: Utilization of idle time in an M/G/1 queueing system, *Manage. Sci.* **22** (1975) 202–211.
12. Li, J. and Tian, N.: The M/M/1 queue with working vacations and vacation interruption, *J. Syst. Sci .Syst. Eng.* **16** (2007) 121–127.
13. Li, J. and Tian, N.: The discrete-time GI/Geo/1 queue with working vacations and vacations interruption, *Appl. Math. Comput.* **185** (2007) 1-10 .
14. Little, J. D. C.: A proof for the queueing formula: $L - \lambda W$, *Oper. Res.* **9** (1961) 383–387.
15. Servi, L. D. and Finn, S. G.: M/M/1 queue with working vacations (M/M/1/WV), *Perform. Eval.* **50** (2002) 41–52.
16. Sreenivasan, C., Chakravarthy, S. R. and Krishnamoorthy, A.: MAP/PH/1 queue with working vacations, vacation interruptions and N policy, *Appl. Math. Model.* **37** (2013) 3879–3893.
17. Takagi, H.: Queueing Analysis: A Foundation of Performance Analysis, *vol. 1 of vacation and priority Systems* part 1(1991).

18. Tian, N. and Zhang, Z. G.: Vacation Queueing Models: Theory and Applications, *Springer*, (2006).
19. Tian, N., Ma, Z. and Liu, M.: The discrete time Geom/Geom/1 queue with multiple working vacations, *Appl. Math. Model* **32** (2008) 2941–2953.
20. Vyshna Unni and Julia Rose Mary, K.: Queueing systems with multiple servers under differentiated working vacations, *Int. j. res. advent technol.* **7** (2019) 392–396.
21. Wu, D. and Takagi, H.: M/G/1 queue with multiple working vacations, *Perform. Eval.* **63** (2006) 654–681.
22. Xu, X., Zhang, Z. and Tian, N. : Analysis for the $M^X/M/1$ working vacation queue, *Int. J. Inf. Manag. Sci.* **20** (2009) 379—394.
23. Zhang, M. and Hou, Z.: Performance analysis of M/G/1 queue with working vacations and vacations interruption, *J. Comput. Appl. Math.*

Vyshna Unni: Department of Mathematics,Nirmala College for Women, Coimbatore - 641 018, Tamilnadu, India
Email address: `vyshnaunni@gmail.com`

K. Julia Rose Mary:Department of Mathematics,Nirmala College for Women, Coimbatore - 641 018, Tamilnadu, India
Email address: `juliakulandaisamy@gmail.com`