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APPLICATION OF ADOMIAN DECOMPOSITION METHOD FOR SOLVING LINEAR AND NONLINEAR KLEIN - GORDON EQUATIONS

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Abstract

In this paper,Adomian Decomposition Method is applied to solve various forms of linear and nonlinear Klein-Gordon Equations.Even this method is a non-numeric method , it can be adapted for solving nonlinear partial differential equations.The nonlinear parameters can be obtained by using Adomian polynomials.It follows that non-linearities in the equation can be handled easily and accurate solution may be obtained for any physical problem.We illustrate this technique with the help of example and represent solutions graphically by Mathematica software.

Keywords: Klein Gordon Equation,Adomian Decomposition Method , Adomian Polynomials, Mathematica.

1 Introduction

Nonlinear Partial Differential Equations have remarkable developments in different areas like gravitation ,chemical reaction ,fluid dynamics,dispersion,nonlinear optics,plasma physics,acoustics etc. Nonlinear wave propagation problems have provided solutions of different physical structures than solutions of linear wave equations . Nonlinear Partial Differential equations have been widely studied throughout recent years. The importance of obtaining the exact solutions of nonlinear equations in mathematics is still a significant problem that needs new research work. The Klein-Gordon equation [1]is considered as one of the most important mathematical models in quantum field theory . The equation appears in relativistic physics and is used to describe dispersive wave phenonmenon in general. The Klein-Gordon equation arise in physics in linear and nonlinear forms.

Recently considerable attention has been given to Adomian Decomposition Method for solving Klein-Gordon Equations. The ADM was introduced by Adomian[2,3] in the early 1980 to solve nonlinear ordinary and partial differential equations. This method also discuss the appearance of Noise Terms in inhomogeneous equations. It is used for obtaining solution in a closed form with only two successive iterations. It avoids artifical boundary conditions , linearization and yields an efficient numerical solution with high accuracy.

The organization of this paper as follows . In Section 2 basic idea of Klein - Gordon equation is presented, In section 3 , we describe the ADM to solve Klein - Gordon equation. In section 4 , we present examples to show the efficiency of using ADM to solve Klein - Gordon equation. Finally , relevant conclusions are drawn in section 5.

2 Klein - Gordon Equation

In this section, we study about Klein-Gordon equations.

Definition 2.1 *The linear Klein-Gordon equation in its standard form is given by*

$$\begin{aligned} u_{tt}(x,t) - u_{xx}(x,t) + au(x,t) = h(x,t) \\ \textit{initial condition} : u(x,0) = f(x), u_t(x,0) = g(x) \end{aligned}$$

Where a is constant and h(x,t) is the source term.

Definition 2.2 *The non linear Klein-Gordon equation in its standard form is given by*

$$\begin{aligned} u_{tt}(x,t) - u_{xx}(x,t) + au(x,t) + Nu(x,t) = h(x,t) \\ \textit{initial condition} : u(x,0) = f(x), u_t(x,0) = g(x) \end{aligned}$$

Where a is constant,h(x,t) is the source term and Nu(x,t) is nonlinear function of u(x,t).

Definition 2.3 *Noise Term Phenomenon-*

This phenomenon is applicable to inhomogeneous Klein-Gordon equations. The noise terms are defined as the identical terms with opposite signs that arise in the components u_0 *and* u_1. *By canceling the noise terms between* u_0 *and* u_1 *, the remaining non canceled terms of* u_0 *may give the exact solution of the Klein-Gordon inhomogeneous equations. The noise term phenomenon is not applicable for homogeneous equations.*

In the next section, we develop the Adomain decomposition method for Klein - Gordon nonlinear partial differential equation.

3 The Adomian Decomposition Method (ADM)

To illustrate the basic idea of this method , we consider a general nonlinear inhomogeneous Klein-Gordon with the initial conditions as follows

$$u_{tt}(x,t) - u_{xx}(x,t) + au(x,t) + Nu(x,t) = h(x,t) \tag{3.1}$$

$$\text{initial condition} : u(x,0) = f(x), u_t(x,0) = g(x)$$

In an operator form equation (3.1) can be rewritten as

$$L_t u(x,t) = u_{xx}(x,t) - au(x,t) - Nu(x,t) + h(x,t) \tag{3.2}$$

$$\text{initial condition} : u(x,0) = f(x), u_t(x,0) = g(x)$$

Where L_t is a second order differential operator and the inverse operator L_t^{-1} is a two - fold integral operator defined by

$$L_t^{-1} = \int_0^t \int_0^t (.)dtdt \tag{3.3}$$

Applying $L_t^{-1}$to both sides of equation (3.2),we have

$$u(x,t) = f(x) + tg(x) + L_t^{-1}[u_{xx}(x,t) - au(x,t)] - L_t^{-1}[Nu(x,t)] + L_t^{-1}[h(x,t)] \tag{3.4}$$

Now, we decompose the unknown function $u(x,t)$ into sum of an infinite number of components given by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{3.5}$$

The nonlinear terms $Nu(x,t)$ are decomposed in the following form:

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n \tag{3.6}$$

where the Adomian polynomial can be determined as follows:

$$A_n = \frac{1}{n!}\left[\frac{d^n N}{d\lambda^n}(\sum_{k=0}^{n} \lambda^k u_k)\right]_{\lambda=0} \tag{3.7}$$

where A_n is called Adomian polynomial and that can be easily calculated by Mathematica software.
Substituting the decomposition series (3.5) and(3.6) into both sides of equation (3.4) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x)+tg(x)+L_t^{-1}[\sum_{n=0}^{\infty} u_n(x,t)_{xx} - a\sum_{n=0}^{\infty} u_n(x,t)] - L_t^{-1}\sum_{n=0}^{\infty} A_n + L_t^{-1}[h(x,t)] \tag{3.8}$$

The components $u_n(x,t)$, $n \geq 0$ of the solution $u(x,t)$ can be recursively determined by using the relation as follows:

$$u_0(x,t) = f(x) + tg(x) + L_t^{-1}[h(x,t),$$

$$u_{k+1}(x,t) = L_t^{-1}([u_k(x,t)]_{xx} - au_k(x,t)) - L_t^{-1}A_k$$

That leads to

$$\begin{aligned}
u_0(x,t) &= f(x) + tg(x) + L_t^{-1}[h(x,t) \\
u_1(x,t) &= L_t^{-1}([u_0(x,t)]_{xx} - au_0(x,t)) - L_t^{-1}A_0 \\
u_2(x,t) &= L_t^{-1}([u_1(x,t)]_{xx} - au_1(x,t)) - L_t^{-1}A_1 \\
u_3(x,t) &= L_t^{-1}([u_2(x,t)]_{xx} - au_2(x,t)) - L_t^{-1}A_2 \\
&\vdots
\end{aligned}$$

This completes the determination of the components of $u(x,t)$ .Based on this determination ,the solution in a series form is obtained.In many cases a closed form solution can be obtained.

In the next section, we illustrate some examples.

4 Applications

ADM for linear Klein-Gordon equation:
In order to elucidate the solution procedure of the ADM,we consider the linear and nonlinear Klein Gordon equations.
Test Problem (i): Consider the following linear homogeneous Klein-Gordon equation

$$\begin{aligned}
&u_{tt}(x,t) - u_{xx}(x,t) - u(x,t) = 0 \\
&\quad \text{initial condition : } u(x,0) = 0, u_t(x,0) = sinx
\end{aligned}$$

By using ADM, we have following recursive relation

$$\begin{aligned}
u_0(x,t) &= tsinx \\
u_{k+1}(x,t) &= L_t^{-1}([u_k(x,t)]_{xx} + u_k(x,t)), k \geq 0 \\
u_0(x,t) &= tsinx
\end{aligned}$$

Calculating the values of u_1, as follow-

$$\begin{aligned}
u_1(x,t) &= L_t^{-1}([u_0(x,t)]_{xx} + u_0(x,t)), \\
u_1(x,t) &= L_t^{-1}(-tsinx + tsinx), \\
u_1(x,t) &= 0
\end{aligned}$$

Similarly calculating values of u_2,we get-

$$\begin{aligned}
u_2(x,t) &= L_t^{-1}([u_1(x,t)]_{xx} + u_1(x,t)), \\
u_2(x,t) &= L_t^{-1}(-tsinx + tsinx), \\
u_2(x,t) &= 0
\end{aligned}$$

$$\vdots$$

Therefore, the series solution for the IBVP is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots.$$

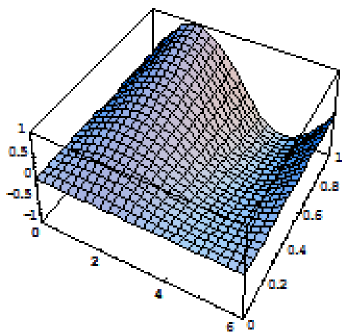
Substituting values of components in above equation, we get

$$u(x,t) = tsinx + 0 + 0 + \cdots$$

The exact solution of the original IVP is given by

$$u(x,t) = tsinx.$$

The graphical representation of the solution is as follow:



$Fig.4.2$: The exact solution of linear homogeneous Klein-Gordon equation.

Test Problem (ii): Consider the following linear inhomogeneous Klein-Gordon equation

$$\begin{gathered} u_{tt}(x,t) - u_{xx}(x,t) + u(x,t) = 2cosx \\ \text{initial condition} : u(x,0) = cosx, u_t(x,0) = 1 \end{gathered}$$

By using ADM, we have following recursive relation

$$\begin{aligned} u_0(x,t) &= cosx + t + L_t^{-1}h(x,t) \\ u_0(x,t) &= cosx + t + t^2cosx \\ u_{k+1}(x,t) &= L_t^{-1}([u_k(x,t)]_{xx} - u_k(x,t)), k \geq 0 \end{aligned}$$

$$\begin{aligned} u_1(x,t) &= L_t^{-1}([u_0(x,t)]_{xx} - u_0(x,t)), \\ u_1(x,t) &= -t^2cosx - \frac{t^3}{3!} - \frac{t^4cosx}{6}, \end{aligned}$$

$$u_2(x,t) = L_t^{-1}(\big[u_1(x,t)\big]_{xx} - u_1(x,t)),$$
$$u_2(x,t) = \frac{t^5}{5!} + \frac{t^6 cosx}{90!} + \frac{t^4 cosx}{6},$$
$$\vdots$$

Therefore, the series solution for the IBVP is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...$$

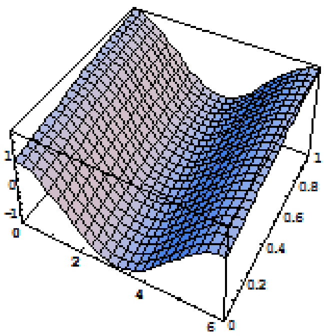
Substituting values of components in above equation, we get

$$u(x,t) = cosx + (t - \frac{t^3}{3!} + \frac{t^5}{5!}) + 0 + \cdots$$

Where noise term vanish in the limit.The solution in a closed term. The exact solution of the original IVP is given by

$$u(x,t) = cosx + sint.$$

The graphical representation of the solution is as follow:



$Fig.4.1$: The exact solution of linear inhomogeneous Klein-Gordon equation.

Test Problem (iii): Consider the following linear inhomogeneous Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u^2 = 1 + 2xt + x^2t^2$$
$$\text{initial condition} : u(x,0) = 1, u_t(x,0) = x$$

By using ADM, we have following recursive relation

$$u_0(x,t) = 1 + 2xt + x^2t^2 + L_t^{-1}h(x,t)$$
$$u_0(x,t) = 1 + xt + \frac{t^2}{2} + \frac{xt^3}{3} + \frac{x^2t^4}{12!}$$
$$u_{k+1}(x,t) = L_t^{-1}\big[u_k(x,t)\big]_{xx} - L_t^{-1}A_k, k \geq 0$$

$$u_1(x,t) = L_t^{-1}\left[u_0(x,t)\right]_{xx} - L_t^{-1}A_0,$$
$$u_1(x,t) = L_t^{-1}(\frac{t^4}{6}) - L_t^{-1}A_0,$$

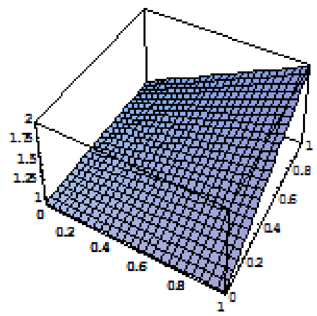
$$A_0 = u_0^2,$$
$$A_0 = (1 + xt + \frac{t^2}{2} + \frac{xt^3}{3} + \frac{x^2t^4}{12!})^2,$$
$$A_0 = 1 + x^2t^2 + \frac{t^4}{4} + \frac{x^2t^9}{9} + 2xt + \cdots$$

$$u_1(x,t) = \frac{t^6}{180} - \frac{t^2}{2} - \frac{x^2t^4}{12} - \frac{xt^3}{3} + \cdots,$$

Canceling the noise terms $\frac{t^2}{2} + \frac{x^2t^4}{12} + \frac{xt^3}{3}$ from the component u_0 and verifying that the remaining non canceled term satisfies the equation, the exact solution of the original IBP is given by -

$$u(x,t) = 1 + xt.$$

The graphical representation of the solution is as follow:



$Fig.4.3$: The exact solution of linear inhomogeneous Klein-Gordon equation.

Conclusions

The main objective of this work is to obtain a solution for linear and non linear Klein-Gordon partial differential equations.We observe that ADM is a powerful method to solve linear and nonlinear Klein-Gordon partial differential equations.Noise Term phenomenon is applicable to inhomogeneous partial differential equations. To, show the applicability and efficiency of the proposed method ,the method is applied to obtain the solutions of several examples. The obtained results demonstrate the reliability of the algorithm. It is worth mentioning that the proposed technique is capable of reducing the volume of the computational work as compared to the classical methods. Finally, we come to the conclusion that the ADM is very powerful and efficient in finding solutions for wide class of linear and nonlinear Klein - Gordon partial differential equations.

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