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SINGULAR STOCHASTIC LEONTIEFF TYPE EQUATIONS WITH DEPENDING ON TIME DIFFUSION COEFFICIENTS

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Abstract. By a singular stochastic Leontief type equation we mean a special class of stochastic differential equations in the Ito form, in which the left and right hand sides have rectangular numerical matrices that form a singular pencil. Also in the right hand side there is a deterministic term, which depends only on time. It is assumed that the diffusion coefficient of the given system is given by a matrix that depends only on time. To study the equations under consideration, it is required to consider derivatives of sufficiently high orders from the free terms, including the Wiener process. In connection with this, to differentiate the Wiener process, we apply the machinery of the Nelson mean derivatives of random processes, which allows us to avoid using the theory of generalized functions to the study of the equation. As a result, physically meaningful formulas are obtained for solving the equation in terms of mean derivatives of random processes.

Introduction

By a singular stochastic Leontief type equation, we mean a system of stochastic differential equations in R^n of the form

$$\tilde{L}\xi(t) = \tilde{M}\int_0^t \xi(s)ds + \int_0^t f(s)ds + \int_0^t P(s)dw(s),\ \ 0 \le t \le T,$$

where $\lambda\tilde{L}+\tilde{M}$ is a singular pencil of constant $n\times m$ matrices such that in the case of square matrices $\tilde{L}$ is degenerate, the diffusion coefficient $P(t)$ is a smooth enough $n \times m$ matrix, $f(t)$ is a smooth enough deterministic vector-function depending on time, $w(t)$ is a Wiener process, $\xi(t)$ is a process that we are looking for. These systems arise in applications for the mathematical modeling of technical systems [1]. Here, the white noise process $\frac{dw(t)}{dt}$ describes the interference in the system. The systems under consideration with constant diffusion coefficient were studied in [2, 3, 4].

To study this class of equations, it is required to consider higher-order derivatives of free terms – in this case, the deterministic term and the Wiener process or white noise. It is known that the derivatives of the Wiener process exist only in the sense of generalized functions, which are extremely difficult to use in specific

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equations. This circumstance makes a direct investigation of our system complicated.

Following [2, 3, 4], we use the machinery of Nelson's mean derivatives of random processes to study the solutions of the equations under consideration, for the description of which generalized functions are not involved. Namely, we apply the symmetric mean derivatives (the current velocities) of the Wiener process. Current velocities, according to the general ideology of mean derivatives, are natural analogues of the physical velocity of deterministic processes. As a result, for the system under consideration, we obtain physically sensible analytical formulas for solutions in terms of symmetric mean derivatives in the mean of random processes.

1. Mean derivatives

Consider the stochastic process $\xi(t)$ in R^n, $t \in [0,T]$, defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is L_1 a random variable for all t. It is known that such a process generates a family of σ -subalgebras of σ -algebra $\mathcal{F}$, "present" $\mathcal{N}_t^\xi$ (it is the minimal σ-subalgebra that includes the preimages of all Borel sets in R^n under the mapping $\xi(t)$), which we assume to be complete, that is, it is completed by all sets of probability zero.

For convenience, we denote the conditional mathematical expectation $E(\cdot|\mathcal{N}_t^\xi)$ relative to the "present" $\mathcal{N}_t^\xi$ for $\xi(t)$ by the symbol E_t^ξ. The usual ("unconditional") mathematical expectation is denoted by the symbol E.

Generally speaking, almost all sample trajectories of the process $\xi(t)$ are not differentiable, so that its derivatives exist only in the sense of generalized functions. To avoid using generalized functions, according to Nelson [5, 6, 7] we give the following definition:

Definition 1.1. (i) The forward mean derivative $D\xi(t)$ of the process $\xi(t)$ at the time instant t is a L_1-random variable of the form

$$D\xi(t) = \lim_{\triangle t \to +0} E_t^\xi\Big(\frac{\xi(t+\triangle t) - \xi(t)}{\triangle t}\Big),$$

where the limit is assumed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\triangle t \to +0$ means that $\triangle t$ tends to 0 and $\triangle t > 0$.

(ii) The backward mean derivative $D_*\ xi(t)$ of the process $\xi(t)$ at time instant t is L_1 - random variable

$$D_*\xi(t) = \lim_{\triangle t \to +0} E_t^\xi\Big(\frac{\xi(t) - \xi(t-\triangle t)}{\triangle t}\Big),$$

where (as in (i)) the limit is assumed to exist in $L_1(\Omega, \mathcal{F},\ sfP)$ and $\triangle t\ to + 0$ means that $\triangle t$ tends to 0 and $\triangle t > 0$.

It should be noted that, in general, $D\xi(t) \neq D\ _{ast}\xi(t)$, but if, for example, $\xi(t)$ almost surely has smooth sample paths, these derivatives obviously coincide.

From the properties of conditional expectation (see [9]), it follows that $D\xi(t)$ and $D_*\xi(t)$ can be represented as superpositions of $\xi(t)$ and Borel vector fields

(regressions)

$$Y^0(t,x) = \lim_{\triangle t \to +0} E_t^\xi \Big(\frac{\xi(t+\triangle t) - \xi(t)}{\triangle t} \big| \xi(t) = x\Big)$$
$$Y_*^0(t,x) = \lim_{\triangle t \to +0} E_t^\xi \Big(\frac{\xi(t) - \xi(t-\triangle t)}{\triangle t} \big| \xi(t) = x\Big)$$

on R^n, that is, $D\xi(t) = Y^0(t,\xi(t))$ and $D_*\xi(t) = Y_*^0(t,\xi(t))$.

Definition 1.2 ([8])**.** The derivative $D_S = \frac{1}{2}(D + D_*)$ is called symmetric mean derivative. The derivative $D_A = \frac{1}{2}(D - D_*)$ is called antisymmetric mean derivative.

Consider vector fields

$$v^\xi(t,x) = \frac{1}{2}(Y^0(t,x) + Y_*^0(t,x))$$

and

$$u^\xi(t,x) = \frac{1}{2}(Y^0(t,x) - Y_*^0(t,x))$$

.

Definition 1.3 ([8])**.** $v^\xi(t) = v^\xi(t,\xi(t)) = D_S\xi(t)$ is called current velocity of $\xi(t)$; $u^\xi(t) = u^\xi(t,\xi(t)) = D_A\xi(t)$ is called osmatic velocity of $\xi(t)$.

The current velocity is a direct analog of the usual physical velocity of deterministic processes for random processes (see [8]). Osmotic velocity measures how fast accrues "randomness" of the process.

The Wiener process plays a decisive role in our constructions. The Wiener process is denotet by the symbol $w(t)$. The following proposition takes place:

Lemma 1.4 ([10])**.** *Let $w(t)$ be a n-dimensional Wiener process, $P(t)$ be a sufficiently smooth $k \times n$-matrix, $t \in (0,l)$. Then for any t we have the formula*

$$D_S^w \int_0^t P(s)dw(s) = P(t)\frac{w(t)}{2t}.$$

Lemma 1.5 ([2, 11])**.** *For $t \in (0,l)$, the following equalities hold*

$$Dw(t) = 0, \ \ D_*w(t) = \frac{w(t)}{t}, \ \ D_Sw(t) = \frac{w(t)}{2t}.$$

For integer $k \geq 2$

$$D_S^k w(t) = (-1)^{k-1} \frac{\prod_{i=1}^{k-1}(2i-1)}{2^k} \frac{w(t)}{t^k}.$$

2. The main result

As already mentioned above, we study the system of stochastic differential equations in R^n of the form

$$\tilde{L}\xi(t) = \int_0^t \tilde{M}\xi(s)ds + \int_0^t f(s)ds + \int_0^t P(s)dw(s), \ \ 0 \leq t \leq T, \tag{2.1}$$

where $\tilde{M}+\lambda\tilde{L}$ is a singular pencil of $n\times m$ matrices, $\xi(t)$ is the process that we are looking for, $\tilde{w}(t)$ is a Wiener process in R^m, $P(t)$ is a smooth enough $n\times m$ matrix, $f(t)$ is a smooth enough n-dimensional vector-function. For simplicity we suppose that the columns and rows of the pencil $\tilde{M}+\lambda\tilde{L}$ are not connected by linear dependences with constant coefficients.

For the singular pencil of matrices $\tilde{M}+\lambda\tilde{L}$ there is the Kronecker-Weierstrass transformation (described by a pair of non-degenerate matrices P_L and P_R of sizes $n\times n$ and $m\times m$, respectively), for which the matrix $P_L\tilde{M}P_R+\lambda P_L\tilde{L}P_R$ is quasi-diagonal (see, e.g., [12]). Then equation (2.1) is transformed as follows

$$P_L\tilde{L}P_RP_R^{-1}\xi(t)=\int_0^t P_L\tilde{M}P_RP_R^{-1}\xi(s)ds+\int_0^t P_Lf(s)ds+\int_0^t P_LP(s)dw(s)$$

With the corresponding numbering of the basis vectors, the following types of blocks in $L=P_L\tilde{L}P_R$ along the main diagonal stand in the following order: N – Jordan cells with zeros along the main diagonal, E is the identity matrix, A and G are rectangular matrices of the form indicated below (singular cells). In $M=P_L\tilde{M}P_R$ in the rows, corresponding to the blocks in L, the following blocks stand in the indicated order: E is the identity matrix, K is some square matrix, B and H are rectangular matrices of the form indicated below (singular cells). We introduce the natation $\eta(t)=P_R^{-l}\xi(t)$, $C(t)=P_LP(t)$. In the new notation, (2.1) takes the form

$$L\eta(t)=\int_0^t M\eta(s)ds+\int_0^t P_Lf(s)ds+\int_0^t C(s)dw(s), \tag{2.2}$$

from which it is clear that (for simplicity) the initial condition for the solution of (2.2) is assumed to be $\eta(0)=0$. We say at once that this condition is not satisfied for the solutions constructed below. Therefore, we approximate solutions by processes that satisfy this initial condition, but they become solutions only from a certain (arbitrarily small) time moment $t_0>0$ (see below).

Remark 2.1. As it is already mentioned in the introduction, to study equation (2.1), and, hence, equation (2.2), it is required to consider derivatives of higher orders from the Wiener process. In this paper, to calculate higher-order symmetric derivatives, we use the σ-algebra of the "real" Wiener process. Note that in order to calculate the mean derivatives, one can also use some other σ-algebras, but then the formulas for calculating symmetric derivatives of higher orders of the Wiener process will be changed.

Translate the matrices A and B, G and H to the general explicit form:

$$A=\begin{pmatrix}0&1&0&\dots&0&0\\0&0&1&\dots&0&0\\\vdots&\vdots&\vdots&\dots&\vdots&\vdots\\0&0&0&\dots&1&0\\0&0&0&\dots&0&1\end{pmatrix},B=\begin{pmatrix}1&0&\dots&0&0&0\\0&1&\dots&0&0&0\\\vdots&\vdots&\dots&\vdots&\vdots&\vdots\\0&0&\dots&1&0&0\\0&0&\dots&0&1&0\end{pmatrix},$$

$$G = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

It is not difficult to see, taking into account the quasi-diagonal structure of the matrices L and M, that the system (2.2) splits into several independent systems of equations of four types (a certain type of equation corresponds to each pair of corresponding blocks in L and M). We denote by $\varsigma(t)$, $\vartheta(t)$, $\eta(t)$, $\theta(t)$ the components of the vector $\eta(t)$ corresponding to the pairs of blocks N and E, E and K, A and B, G, and H, respectively. We also denote by $u(t), v(t), g(t), z(t)$, the corresponding components of the vector $P_L f(t)$, and by $C_{p+1}(t)$, we denote the corresponding blocks of the matrix $C(t)$ by $C_{q+1}(t)$, $C_l(t)$, $C_{k+1}(t)$. Let us investigate each type of equations.

The pair of matrices N and E of size $(p+1)\times(p+1)$ corresponds to the system of type

$$N\varsigma(t) = \int_0^t \varsigma(s)ds + \int_0^t u(s)ds + \int_0^t C_{p+1}(s)dw(s).$$

In coordinates this equation takes the form

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \varsigma^1(t) \\ \varsigma^2(t) \\ \vdots \\ \varsigma^p(t) \\ \varsigma^{p+1}(t) \end{pmatrix} = \begin{pmatrix} \int_0^t(\varsigma^1(s) + u^1(s))ds \\ \int_0^t(\varsigma^2(s) + u^2(s))ds \\ \vdots \\ \int_0^t(\varsigma^p(s) + u^p(s))ds \\ \int_0^t(\varsigma^{p+1}(s) + u^{p+1}(s))ds \end{pmatrix} +$$

$$+\int_0^t \begin{pmatrix} c_1^1(s) & c_2^1(s) & \dots & c_{m-1}^1(s) & c_m^1(s) \\ c_1^2(s) & c_2^2(t) & \dots & c_{m-1}^2(s) & c_m^2(s) \\ \vdots & \vdots & \dots & \vdots & \vdots \\ c_1^p(s) & c_2^p(s) & \dots & c_{m-1}^p(s) & c_m^p(s) \\ c_1^{p+1}(s) & c_2^{p+1}(s) & \dots & c_{m-1}^{p+1}(s) & c_m^{p+1}(s) \end{pmatrix} d \begin{pmatrix} w^1(s) \\ w^2(s) \\ \vdots \\ w^{m-1}(s) \\ w^m(s) \end{pmatrix}. \quad (2.3)$$

From the last equation of system (2.3) we obtain that

$$\int_0^t (\varsigma^{p+1}(s) + u^{p+1}(s))ds = -\sum_{j=1}^m \int_0^t c_j^{p+1}(s)dw^j(s).$$

Since the current velocity (symmetric mean derivative) corresponds to the physical velocity, for $0 < t < T$ we derive from this equation that

$$\varsigma^{p+1}(t) = -u^{p+1}(t) - \sum_{j=1}^m c_j^{p+1}(t)\frac{w^j(t)}{2t}. \quad (2.4)$$

From the last equation of system (2.3) we obtain that

$$\varsigma^{p+1}(t)=\int_0^t(\varsigma^p(s)+u^p(s))ds+\sum_{j=1}^m\int_0^t c_j^p(s)dw^j(s)$$

from which, after the arguments analogous to given above, for $0<t<T$ we derive

$$\varsigma^p(t)=-u^p(t)+D_S^w\varsigma^{p+1}(t)-D_S^w\sum_{j=1}^m\int_0^t c_j^p(s)dw^j(s).$$

On substituting the expression for $\varsigma^{p+1}(t)$ from (2.4) and applying Lemma 1.4, for $0<t<T$ we obtain

$$\varsigma^p(t)=-\frac{du^{p+1}}{dt}-u^p(t)-\sum_{l=1}^m\frac{dc_l^{p+1}}{dt}\frac{w^l}{2t}+\sum_{l=1}^m c_l^{p+1}\frac{w^l}{4t^2}-\sum_{l=1}^m c_l^p\frac{w^l}{2t} \tag{2.5}$$

We also obtain

$$\begin{aligned}\varsigma^{p-1}&=-\frac{d^2u^{p+1}}{dt^2}-\frac{du^p}{dt}-u^{p-1}-\\ -\sum_{l=1}^n\frac{d^2c_l^{p+1}}{dt^2}\frac{w^l}{2t}&+2\sum_{l=1}^n\frac{dc_l^{p+1}}{dt}\frac{w^l}{4t^2}-\sum_{l=1}^n c_l^{p+1}\frac{3w^l}{8t^3}-\\ -\sum_{l=1}^n\frac{dc_l^p}{dt}\frac{w^l}{2t}&+\sum_{l=1}^n c_l^p\frac{w^l}{4t^2}-\sum_{l=1}^n c_l^{p-1}\frac{w^l}{2t}\end{aligned} \tag{2.6}$$

By complete analogy, for $1\le i\le p$ and $0<t<T$ we obtain the recursion formula

$$\varsigma^i(t)=D_S^w\varsigma^{i+1}(t)-D_S^w\sum_{j=1}^m\int_0^t c_j^i(s)dw^j(s)-u^i. \tag{2.7}$$

Using Lemma 1.4 and 1.5 and using formula (2.7) for $0<t<T$, we obtain the explicit expression for any $\varsigma^i(t)$

$$\begin{aligned}\varsigma^i=&-\sum_{k=i}^p\frac{d^{k-i+1}u^{k+1}}{dt^{k-i+1}}-u^i-\sum_{l=1}^m\frac{dc_l^{i+1}}{dt}\frac{w^l}{2t}+\\ &+\sum_{l=1}^m c_l^{i+1}\frac{w^l}{4t^2}-\sum_{s=i+1}^p\sum_{l=1}^m\{\frac{d^{s-i+1}c_l^{s+1}}{dt^{s-i+1}}\frac{w^l}{2t}+\\ &+c_l^{s+1}(-1)^{s-i+1}\frac{\prod_{j=1}^{s-i+1}(2j-1)}{2^{s-i+2}}\frac{w^l(t)}{t^{s-i+2}}+\\ &+\sum_{k=1}^{s-i}C_{s-i+1}^k\frac{d^{s-i+1-k}c_l^{s+1}}{dt^{s-i+1-k}}(-1)^k\frac{\prod_{j=1}^k(2j-1)}{2^{k+1}}\frac{w^l(t)}{t^{k+1}}\}-\sum_{l=1}^m c_l^i\frac{w^l}{2t},\end{aligned} \tag{2.8}$$

$$1\le i\le p-1,\ \ C_{n_1}^{k_1}=\frac{n_1!}{k_1!(n_1-k_1)!}$$

For the pair of matrices E and K of sizes $(q+1)\times(q+1)$, we obtain the system in R^{q+1} of the type

$$\vartheta(t)=K\cdot\int_0^t\vartheta(s)ds+\int_0^t v(s)ds+\int_0^t C_{q+1}(s)dw(s). \tag{2.9}$$

For equation (2.9), the analytical formula for solutions is known (see [13])

$$\vartheta(t)=\int_0^t e^{K(t-\tau)}v(\tau)d\tau+\int_0^t e^{K(t-\tau)}C_{q+1}(\tau)dw(\tau).$$

Considering the pair of matrices A and B of size $l\times(l+1)$, we obtain the system of the form

$$A\eta(t)=\int_0^t B\eta(s)ds+\int_0^t g(s)ds+\int_0^t C_l(s)dw(s).$$

In coordinates this equation takes the form

$$\begin{pmatrix}0&1&0&\dots&0&0\\0&0&1&\dots&0&0\\\vdots&\vdots&\vdots&\dots&\vdots&\vdots\\0&0&0&\dots&1&0\\0&0&0&\dots&0&1\end{pmatrix}\begin{pmatrix}\eta^1(t)\\\eta^2(t)\\\vdots\\\eta^l(t)\\\eta^{l+1}(t)\end{pmatrix}=\int_0^t\begin{pmatrix}1&0&\dots&0&0&0\\0&1&\dots&0&0&0\\\vdots&\vdots&\dots&\vdots&\vdots&\\0&0&\dots&1&0&0\\0&0&\dots&0&1&0\end{pmatrix}\begin{pmatrix}\eta^1(s)\\\eta^2(s)\\\vdots\\\eta^l(s)\\\eta^{l+1}(s)\end{pmatrix}ds$$
$$+\int_0^t\begin{pmatrix}g^1(s)\\g^2(s)\\\vdots\\g^{l-1}(s)\\g^l(s)\end{pmatrix}ds+\int_0^t\begin{pmatrix}\check{c}_1^1(s)&\check{c}_2^1(s)&\dots&\check{c}_m^1(s)\\\check{c}_1^2(s)&\check{c}_2^2(t)&\dots&\check{c}_m^2(s)\\\vdots&\vdots&\dots&\vdots\\\check{c}_1^{l-1}(s)&\check{c}_2^{l-1}(s)&\dots&\check{c}_m^{l-1}(s)\\\check{c}_1^l(s)&\check{c}_2^l(s)&\dots&\check{c}_m^l(s)\end{pmatrix}d\begin{pmatrix}w^1(s)\\w^2(s)\\\vdots\\w^{m-1}(s)\\w^m(s)\end{pmatrix} \tag{2.10}$$

i.e.,

$$\eta^2(t)=\int_0^t(\eta^1(s)+g^1(s))ds+\sum_{j=1}^m\int_0^t\check{c}_j^1(s)dw^j(s),$$

$$\eta^3(t)=\int_0^t(\eta^2(s)+g^2(s))ds+\sum_{j=1}^m\int_0^t\check{c}_j^2(s)dw^j(s),$$
$$\eta^{l+1}(t)=\int_0^t(\eta^l(s)+g^l(s))ds+\sum_{j=1}^m\int_0^t\check{c}_j^l(s)dw^j(s).$$

This means that as η^{l+1} we can take an arbitrary random process on $[0,T]$ that takes a zero value for $t=0$ and for which on $0<t<T$ one can calculate the symmetric derivative of order l, and then recursively obtain all other components of the process η. This is possible because in the system the number of unknowns is one more than the number of equations, that is, the system is underdetermined.

Analogously to the case of the first independent system, the following formulas hold:

$$\eta^l(t) = D_S^w \eta^{l+1} - \sum_{j=1}^m \check{c}_j^l \frac{w^j}{2t} - g^l(t), \ \ 0 < t < T; \tag{2.11}$$

$$\eta^l(t) = \int_0^t (\eta^{l-1}(s) + g^{l-1}(s))ds + \sum_{j=1}^m \int_0^t \check{c}_j^{l-1}(s)dw^j(s);$$

$$\eta^{l-1}(t) = D_S^2 \eta^{l+1} - \sum_{j=1}^m \frac{d\check{c}_j^l}{dt}\frac{w^j}{2t} + \sum_{j=1}^m \check{c}_j^l \frac{w^j}{4t^2} - \sum_{j=1}^m \check{c}_j^{l-1}\frac{w^j}{2t} - \frac{dg^l}{dt} - g^{l-1}, \quad 0 < t < T. \tag{2.12}$$

By complete analogy, for $1 \leq i \leq l$ and $0 < t < T$ we obtain

$$\eta^i(t) = D_S^w \eta^{i+1} - D_S^w \sum_{j=1}^m \int_0^t \check{c}_j^i(s)dw^j(s) - g^i(t). \tag{2.13}$$

Using Lemma 1.4 and 1.5 and using formula (2.13) for $0 < t < T$, we obtain the explicit expression for any $\eta^i(t)$

$$\eta^i = -\sum_{k=i}^{l-1} \frac{d^{k-i+1}g^{k+1}}{dt^{k-i+1}} - g^i - \sum_{j=1}^m \frac{d\check{c}_j^{i+1}}{dt}\frac{w^j}{2t} + \\ + \sum_{j=1}^m \check{c}_j^{i+1}\frac{w^j}{4t^2} - \sum_{s=i+1}^{l-1}\sum_{j=1}^m \{ \frac{d^{s-i+1}\check{c}_j^{s+1}}{dt^{s-i+1}}\frac{w^j}{2t} + \\ + \check{c}_j^{s+1}(-1)^{s-i+1}\frac{\prod_{r=1}^{s-i+1}(2r-1)}{2^{s-i+2}}\frac{w^j(t)}{t^{s-i+2}} + \\ + \sum_{k=1}^{s-i} C_{s-i+1}^k \frac{d^{s-i+1-k}\check{c}_j^{s+1}}{dt^{s-i+1-k}}(-1)^k \frac{\prod_{r=1}^k (2r-1)}{2^{k+1}}\frac{w^j(t)}{t^{k+1}} \} - \sum_{j=1}^m \check{c}_j^i \frac{w^j}{2t} + D_S^{l+1-i}\eta^{l+1}, \tag{2.14}$$

$$1 \leq i \leq l-2$$

Finally, for matrices G and H of size $(k+1)\times k$ we have the system of the type

$$G\theta(t) = \int_0^t H\theta(s)ds + \int_0^t z(s)ds + \int_0^t C_{k+1}(s)dw(s).$$

In coordinates we obtain the system of the form

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \theta^1(t) \\ \theta^2(t) \\ \vdots \\ \theta^{k-1}(t) \\ \theta^k(t) \end{pmatrix} = \int_0^t \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta^1(s) \\ \theta^2(s) \\ \vdots \\ \theta^{k-1}(s) \\ \theta^k(s) \end{pmatrix} ds +$$

$$\int_0^t \begin{pmatrix} z^1(s) \\ z^2(s) \\ \vdots \\ z^k(s) \\ z^{k+1}(s) \end{pmatrix} ds + \int_0^t \begin{pmatrix} \hat{c}_1^1(s) & \hat{c}_2^1(s) & \dots & \hat{c}_m^1(s) \\ \hat{c}_1^2(s) & \hat{c}_2^2(t) & \dots & \hat{c}_m^2(s) \\ \vdots & \vdots & \dots & \vdots \\ \hat{c}_1^k(s) & \hat{c}_2^k(s) & \dots & \hat{c}_m^k(s) \\ \hat{c}_1^{k+1}(s) & \hat{c}_2^{k+1}(s) & \dots & \hat{c}_m^{k+1}(s) \end{pmatrix} d \begin{pmatrix} w^1(s) \\ w^2(s) \\ \vdots \\ w^{m-1}(s) \\ w^m(s) \end{pmatrix}. \tag{2.15}$$

or

$$0 = \int_0^t (\theta^1(s) + z^1(s))ds + \sum_{j=1}^m \int_0^t \hat{c}_j^1(s)dw^j(s),$$

$$\theta^1(t) = \int_0^t (\theta^2(s) + z^2(s))ds + \sum_{j=1}^m \int_0^t \hat{c}_j^2(s)dw^j(s),$$

$$\theta^{k-1}(t) = \int_0^t (\theta^k(s) + z^k(s))ds + \sum_{j=1}^m \int_0^t \hat{c}_j^k(s)dw^j(s),$$

$$\theta^k(t) = \int_0^t z^{k+1}(s)ds + \sum_{j=1}^m \int_0^t \hat{c}_j^{k+1}(s)dw^j(s).$$

Starting from the first equation, for $0 < t < T$ we consequently obtain

$$\theta^1(t) = -z^1(t) - D_S^w \sum_{j=1}^m \int_0^t \hat{c}_j^1(s)dw^j(s) = -z^1 - \sum_{j=1}^m \hat{c}_j^1 \frac{w^j(t)}{2t}, \tag{2.16}$$

$$\theta^2(t) = -z^2(t) + D_S^w \theta^1 - D_S^w \sum_{j=1}^m \int_0^t \hat{c}_j^2(s)dw^j(s) =$$

$$= -z^2(t) - \frac{dz^1(t)}{dt} - \sum_{j=1}^m \frac{d\hat{c}_j^1}{dt}\frac{w^j}{2t} + \sum_{j=1}^m \hat{c}_j^1 \frac{w^j(t)}{4t^2} - \sum_j \hat{c}_j^2 \frac{w^j(t)}{2t}, \tag{2.17}$$

$$\theta^k(t) = -z^k(t) - \frac{dz^{k-1}(t)}{dt} - ... - \frac{d^{k-1}z^1(t)}{dt^{k-1}} -$$

$$-D_S \sum_{j=1}^m \int_0^t \hat{c}_j^k(s)dw^j(s) - D_S^2 \sum_{j=1}^m \int_0^t \hat{c}_j^{k-1}(s)dw^j(s) - ... - D_S^k \sum_{j=1}^m \int_0^t \hat{c}_j^1(s)dw^j(s) =$$

$$= -\sum_{j=1}^m \hat{c}_j^k \frac{w^j}{2t} - \sum_{j=1}^m \sum_{p=1}^{k-1} D_S^p (\hat{c}_j^{k-p} \frac{w^j}{2t}).$$

Hence, for $k \geq 3$ we obtain

$$\theta^k(t) = -\sum_{p=1}^{k-1} \frac{d^{k-p} z^p}{dt^{k-p}} - z^k - \sum_{j=1}^{m} \hat{c}_j^k \frac{w^j}{2t} - \sum_{j=1}^{m} \frac{d\hat{c}_j^{k-1}}{dt} \frac{w^j}{2t} +$$
$$+\sum_{j=1}^{m} \hat{c}_j^{k-1} \frac{w^j}{4t^2} - \sum_{j=1}^{m} \sum_{p=2}^{k-1} \{ \frac{d^p \hat{c}_j^{k-p}}{dt^p} \frac{w^j}{2t} +$$
$$+\sum_{l=1}^{p-1} C_p^l \frac{d^{p-l} \hat{c}_j^{k-p}}{dt^{p-l}} (-1)^l \frac{\prod_{r=1}^{l}(2r-1)}{2^{l+1}} \frac{w^j}{t^{l+1}} + \hat{c}_j^{k-p} (-1)^p \frac{\prod_{r=1}^{p}(2r-1)}{2^{p+1}} \frac{w^j}{t^{p+1}} \}. \tag{2.18}$$

For $0 < t < T$ we also obtain the compatibility condition

$$\int_0^t z^{k+1}(s)ds + w^{k+1}(t) = \theta^k(t).$$

If the components z^i and w^i do not satisfy this condition, the system has no solutions. Here the number of equations is greater than the number of unknowns, i.e., this subsystem is overdetermined. As above, for $2 \leq i \leq k$ and $0 < t < T$ the following recurrence formula holds

$$\theta^i(t) = -z^i(t) + D_S^w \theta^{i-1} - D_S^w \sum_{j=1}^{m} \int_o^t \hat{c}_j^i(s) dw^j(s).$$

Now turn to the question of zero initial conditions for solutions of systems (2.3), (2.10) and (2.15). Taking into account the definition of symmetric mean derivatives, it is easy to see that they are well-posed only on the open intervals of time since in their construction both time increments to the right and to the left are used. Then from the formulas (2.4), (2.5), (2.6), (2.8), (2.11), (2.12), (2.14), (2.16) , (2.17) and (2.18) we see that the solutions $\eta^l(t)$ are described as sums, in which each summand contains a factor of the form $\frac{w^j(t)}{t^k}$, $k \geq 1$. Hence, the solutions tend to infinity as $t \to 0$, i.e., the values of solutions for $t = 0$ do not exist. One of the possibilities for resolving this situation (as in [2]) is as follows. Specify an arbitrarily small time instant $t_0 \in (0, l)$ and define the function $t_0(t)$ by the formula

$$t_0(t) = \begin{cases} t_0, & 0 \leq t \leq t_0; \\ t, & t_0 \leq t. \end{cases} \tag{2.19}$$

We replace the elements $\frac{w^j(t)}{t^k}$ in formulas (2.4), (2.5), (2.6), (2.8), (2.11), (2.12), (2.14), (2.16), (2.17) and (2.18) by $\frac{w^j(t)}{(t_0(t))^k}$. The resulting processes at $t = 0$ take zero values but they become the solutions only at $t_0 \leq t < T$. Note that for two different time instants $t_0^{(1)}$ and $t_0^{(2)}$ for $t \geq \max(t_0^{(1)}, t_0^{(2)})$ the values of corresponding processes a.s. coincide.

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