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A STUDY OF EUROPEAN FUZZY PUT OPTION BUYER'S MODEL ON FUTURE CONTRACTS INVOLVING GENERAL TRAPEZOIDAL FUZZY NUMBERS

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Abstract. In real life situations, when we come across a small increase or decrease in the stocks prices, we need fuzzy numbers that would either be overlapping or non- overlapping partially or completely (called general trapezoidal fuzzy numbers) to describe the primary fuzzy put option pricing parameters that will be affecting fuzzy put option prices of the fuzzy underlying asset in the next time period. A new fuzzy risk-neutral probability measure involving general trapezoidal fuzzy numbers was defined by us [5] to study *American Fuzzy Put Option Buyer's Model (AFPOBM)* in early 2019.
A discrete-time European call or put option model with uncertainty was proposed by Yoshida [7] in 2003 wherein he used triangular fuzzy numbers which were non-overlapping to represent the stock price process involved in his financial model. American Fuzzy Put Option Model using fuzzy risk-neutral probability measure studied by S.Muzzioli and H.Reynaerts [6] who also employed triangular and trapezoidal fuzzy numbers that were non-overlapping.
In this paper, we validate *European Fuzzy Put Option Buyer's Model (EFPOBM) on future contracts* using the fuzzy risk-neutral probability measure defined by us involving general trapezoidal fuzzy numbers. We provide a computational procedure to obtain the profit and loss values of EFPOBM / AFPOBM with the fuzzy future price as the fuzzy underlying security. We elucidate the same using the data obtained from a website that includes Microsoft Corporation shares [12]. It is used to estimate the profit and loss values of EFPOBM and AFPOBM on future contracts and compare them.

1. Introduction

The concept of the binomial tree model for option pricing theory was first proposed in the year 1979 by Cox et.al [2] in a crisp setup and its fuzzy analogue was introduced by Yoshida [8] in 2003. He realized that two kinds of uncertainties namely randomness and fuzziness would arise in option pricing theory. The two jump factors namely, up and down u, d and the interest rate r involved in his model were considered crisp and he evaluated the optimal expected price, the permissible range of the writer's (seller's) expected price and an optimal exercise time in both the American put option and the European call or put option model [7, 8].
In this study, EFPOBM is dealt with in detail by introducing the fuzzy risk-neutral

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probability measure for future contracts involving general trapezoidal fuzzy numbers. The paper is organized as follows: In Section 2, we provide the prerequisites required for this paper. In Section 3, we define the fuzzy intrinsic values of EFPOBM on future contracts involving general trapezoidal fuzzy numbers. Also we investigate that the discounted fuzzy intrinsic values of EFPOBM turns out to be a $Q-$ fuzzy martingale. Further, we propose an algorithm to compute the profit and loss (PL) values of EFPOBM/AFPOBM on future contracts with the fuzzy future price as the fuzzy underlying security and illustrate the same through a real time application using the data [12] and compare them. In Section 4, we record our conclusion.

2. Preliminaries

For the sake of completeness, we recall the required definitions [1, 4, 7, 8] and the problem involving general trapezoidal fuzzy numbers in pricing AFPOBM [5]. The concepts of a fuzzy set, fuzzy numbers, trapezoidal fuzzy number were first introduced by Zadeh [9, 10, 11]. In this paper, we consider the cases where the trapezoidal fuzzy numbers could be either non-overlapping or overlapping partially or completely. Hence we call these numbers as general trapezoidal fuzzy numbers which are used in the study of EFPOBM and AFPOBM.

Definition 2.1. [4] A fuzzy number $\widetilde{A}$ was called a trapezoidal fuzzy number denoted by $\widetilde{A} = (a_1, a_2, a_3, a_4)$ where $a_1 \leq a_2 \leq a_3 \leq a_4$ were real numbers with membership function $\mu_{\widetilde{A}}(x)$ was given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{(x-a_1)}{(a_2-a_1)}, & \text{for} \quad a_1 \leq x \leq a_2 \\ 1, & \text{for} \quad a_2 \leq x \leq a_3 \\ \frac{(a_4-x)}{(a_4-a_3)}, & \text{for} \quad a_3 \leq x \leq a_4 \\ 0, & \text{otherwise.} \end{cases}$$

$\widetilde{A}$ was said to be non-negative if $a_1 \geq 0$.

Definition 2.2. [4] If $* \in \{+, -, \times, /\}$ was a binary operation between two non-negative trapezoidal fuzzy numbers $\widetilde{A} = (a_1, a_2, a_3, a_4)$ and $\widetilde{B} = (b_1, b_2, b_3, b_4)$, then $\widetilde{A} * \widetilde{B}$ were also non-negative trapezoidal fuzzy numbers defined by

$$\begin{aligned} \widetilde{A} + \widetilde{B} =& (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3,\ a_4 + b_4) \\ \widetilde{A} - \widetilde{B} =& (a_1 - b_4,\ a_2 - b_3,\ a_3 - b_2,\ a_4 - b_1) \\ \widetilde{A} \times \widetilde{B} =& (a_1 \times b_1,\ a_2 \times b_2,\ a_3 \times b_3,\ a_4 \times b_4) \\ \widetilde{c}\widetilde{A} =& (ca_1, ca_2, ca_3, ca_4),\ \text{where } c \in \mathbb{R}^*, \mathbb{R}^* \text{ set of all non-negative real numbers} \\ \widetilde{A}/\widetilde{B} =& (a_1/b_4,\ a_2/b_3,\ a_3/b_2,\ a_4/b_1), \text{where } b_1 > 0 \end{aligned}$$

Definition 2.3. [3] If $\widetilde{A}$ and $\widetilde{B}$ were two non-negative trapezoidal fuzzy numbers and $c \in \mathbb{R}^*$, then

$$\max(\widetilde{A}, \widetilde{B}) = (\max(a_1, b_1), \max(a_2, b_2), \max(a_3, b_3), \max(a_4, b_4))$$

Also, if $\widetilde{c} = (c, c, c, c)$ was a constant trapezoidal fuzzy number where $c \in \mathbb{R}^*$, then

$$\max(\widetilde{A}, \widetilde{c}) = (\max(a_1, c), \max(a_2, c), \max(a_3, c), \max(a_4, c))$$

Definition 2.4. [1] The measure of a trapezoidal fuzzy number $\widetilde{A}$, denoted $M(\widetilde{A})$ was defined by

$$M(\widetilde{A}) = \frac{1}{4}\left[a_1 + a_2 + a_3 + a_4\right].$$

Using the measure given in Definition 2.4, we can compare any two trapezoidal fuzzy numbers.

Definition 2.5. [1] If $\widetilde{A}$ and $\widetilde{B}$ were any two trapezoidal fuzzy numbers, then (i)$\widetilde{A} \succcurlyeq \widetilde{B}$ if $M(\widetilde{A}) \geq M(\widetilde{B})$ (ii)$\widetilde{A} \preccurlyeq \widetilde{B}$ if $M(\widetilde{A}) \leq M(\widetilde{B})$ (iii)$\widetilde{A} \approx \widetilde{B}$ if $M(\widetilde{A}) = M(\widetilde{B})$.

Definition 2.6. [5] An *AFPOBM for future contracts* gave the holder the right but not the obligation, to sell a particular fuzzy stock on or before a specified date at a predetermined constant fuzzy strike price $\widetilde{K}$ to the fuzzy put option seller in the future. The buyer of the fuzzy put option paid premium to the fuzzy put option seller for his right to sell the fuzzy stock at $\widetilde{K}$.

Remark 2.7. In the U.S. generally the option expires the 3rd Saturday of the expiration month; however, the last trading day was third friday before expiration. If this falls on a holiday, then the expiration day was thursday prior to the third friday. The risk-free interest rate was the interest rate on a three-month U.S. Treasury bill which was often used as the risk-free rate for U.S.-based investors.

Definition 2.8. [5] The fuzzy intrinsic values of AFPOBM on future contracts was defined by

$$\widetilde{v}_N^{\mathcal{A}}(\widetilde{F}_{N,i}) = \max\left\{\widetilde{K} - \widetilde{F}_{N,i}, \widetilde{0}\right\}, i = 0, 1, \ldots, N$$

and

$$\widetilde{v}_n^{\mathcal{A}}(\widetilde{F}_{n,i}) = \max\left\{\widetilde{K} - \widetilde{F}_{n,i}, \frac{\widetilde{1}}{\widetilde{1} + \widetilde{r}}\widetilde{E}_n^Q(\widetilde{v}_{n+1}^{\mathcal{A}}(\widetilde{F}_{n+1,i}))\right\}$$

where $\widetilde{E}_n^Q(\widetilde{v}_{n+1}^{\mathcal{A}}(\widetilde{F}_{n+1,i})) = (\widetilde{p}_u\widetilde{v}_{n+1}^{\mathcal{A}}(\widetilde{F}_{n+1,i}) + \widetilde{p}_d\widetilde{v}_{n+1}^{\mathcal{A}}(\widetilde{F}_{n+1,i}))$ for $i = 0, 1, \ldots, n$, and $n = N-1, N-2, \ldots, 0$ and $\widetilde{E}_n^Q$ denotes the expectation with respect to the fuzzy risk-neutral probability measure Q.

Definition 2.9. [5] A discrete time fuzzy stochastic process $\widetilde{X} \approx \{\widetilde{X}_n\}_{n=0}^N$ was called a $Q-$ fuzzy martingale with respect to a filtration $\{\mathcal{M}_n\}_{n=0}^N$ if

(i). $\widetilde{E}_n^Q(\widetilde{X}_{n+1}) \approx \widetilde{X}_n$, where $\widetilde{E}_n^Q$ was the expectation with respect to the fuzzy risk-neutral probability measure Q. Further if $\approx$ in (i) was replaced by ($\preccurlyeq$ *or* $\succcurlyeq$), we have

(ii). $\widetilde{E}_n^Q(\widetilde{X}_{n+1}) \preccurlyeq \widetilde{X}_n$, then $\widetilde{X} = \{\widetilde{X}_n\}_{n=0}^N$ was called a $Q-$ fuzzy supermartingale with respect to a filtration $\{\mathcal{M}_n\}_{n=0}^N$ and

(iii). $\widetilde{E}_n^Q(\widetilde{X}_{n+1}) \succcurlyeq \widetilde{X}_n$, then $\widetilde{X} = \{\widetilde{X}_n\}_{n=0}^N$ was called a $Q-$ fuzzy submartingale with respect to a filtration $\{\mathcal{M}_n\}_{n=0}^N$.

Remark 2.10. Let $\widetilde{F}_n, \widetilde{S}_n$ denoted the fuzzy future price and the price of the fuzzy stock at time n respectively and if the contract expired after a total period of N, the fuzzy future price at time n can be computed to be

$$\widetilde{F}_n = (\widetilde{1} + \widetilde{r})^{N-n} \widetilde{S}_n, n = 0, 1, \dots, N \tag{2.1}$$

which on expiration date $n = N$ yielding $\widetilde{F}_n = \widetilde{S}_n$.

3. Valuation of EFPOBM

In this section, we execute EFPOBM using general trapezoidal fuzzy numbers for future contracts.

Definition 3.1. An *EFPOBM on future contract* is an option contract between the buyer and seller which gives the buyer the right, but it is not an obligation to sell a particular underlying fuzzy stock on a predetermined expiration date at a predecided price, called the constant fuzzy strike price $\widetilde{K}$ to the fuzzy put option seller in the future. Selling European fuzzy put option requires the buyers to pay premium to the fuzzy put option sellers.

Remark 3.2. For European style option, the last trading day is the last thursday of the expiry month.

Definition 3.3. The two up and down estimated jump factors and the risk-free interest rate of the fuzzy stock are represented using general trapezoidal fuzzy numbers: $\widetilde{u} = (u_1, u_2, u_3, u_4), \widetilde{d} = (d_1, d_2, d_3, d_4)$ and $\widetilde{r} = (r_1, r_2, r_3, r_4)$. The positions of the two jump factors and the risk-free interest rates of the fuzzy stock satisfying the following no arbitrage condition given by
$d_1 < u_1 < d_2 \leq (1 + r_1) \leq (1 + r_2) \leq (1 + r_3) \leq (1 + r_4) < u_2 < d_3 < u_3 < d_4 < u_4$
and we define the up $\widetilde{p}_u$ and down $\widetilde{p}_d$ fuzzy risk-neutral probability measures as

$$\widetilde{p}_u = \left[\frac{1 + r_1 - d_2}{u_4 - d_2}, \frac{1 + r_2 - d_2}{u_3 - d_2}, \frac{1 + r_3 - d_2}{u_2 - d_2}, \frac{1 + r_4 - d_1}{u_2 - d_1} \right] \tag{3.1}$$

$$\widetilde{p}_d = \left[\frac{u_2 - 1 - r_4}{u_2 - d_1}, \frac{u_2 - 1 - r_3}{u_2 - d_2}, \frac{u_3 - 1 - r_2}{u_3 - d_2}, \frac{u_4 - 1 - r_1}{u_4 - d_2} \right] \tag{3.2}$$

which are general trapezoidal fuzzy numbers $Q(\widetilde{p}_u, \widetilde{p}_d)$.

In a view of remark 2.10, we have

Definition 3.4. The fuzzy intrinsic values of the EFPOBM on future contracts is defined by

$$\widetilde{v}_N^{\mathcal{E}}(\widetilde{F}_{N,i}) = \max \left\{ \widetilde{K} - \widetilde{F}_{N,i}, \widetilde{0} \right\}, i = 0, 1, \dots, N$$

and

$$\widetilde{v}_n^{\mathcal{E}}(\widetilde{F}_{n,i}) = \left\{ \frac{\widetilde{1}}{\widetilde{1} + \widetilde{r}} \widetilde{E}_n^Q (\widetilde{v}_{n+1}^{\mathcal{E}}(\widetilde{F}_{n+1,i})) \right\}$$

where $\widetilde{E}_n^Q (\widetilde{v}_{n+1}^{\mathcal{E}}(\widetilde{F}_{n+1,i})) = (\widetilde{p}_u \widetilde{v}_{n+1}^{\mathcal{E}}(\widetilde{F}_{n+1,i}) + \widetilde{p}_d \widetilde{v}_{n+1}^{\mathcal{E}}(\widetilde{F}_{n+1,i}))$ for $i = 0, 1, \dots, n$, and $n = N - 1, N - 2, \dots, 0$ and $\widetilde{E}_n^Q$ denotes the expectation with respect to the fuzzy risk-neutral probability measure Q.

The proof of the following proposition can be deduced from Definition 2.5.

Proposition 3.5. *The fuzzy intrinsic values of AFPOBM are greater than or equal to the the fuzzy intrinsic values of EFPOBM with respect to the same fuzzy put option pricing parameters. i.e.,* $\widetilde{v}_n^{\mathcal{A}}(\widetilde{F}_{n,i}) \succcurlyeq \widetilde{v}_n^{\mathcal{E}}(\widetilde{F}_{n,i})$.

Proof. $\widetilde{v}_n^{\mathcal{A}}(\widetilde{F}_{n,i}) = \max\left\{\widetilde{K} - \widetilde{F}_{n,i}, \frac{\widetilde{1}}{\widetilde{1}+\widetilde{r}}\widetilde{E}_n^Q(\widetilde{v}_{n+1}^{\mathcal{A}}(\widetilde{F}_{n+1,i}))\right\}$
$\Longrightarrow \widetilde{v}_n^{\mathcal{A}}(\widetilde{F}_{n,i}) \succcurlyeq \frac{\widetilde{1}}{\widetilde{1}+\widetilde{r}}\left(\widetilde{E}_n^Q(\widetilde{v}_{n+1}^{\mathcal{E}}(\widetilde{F}_{n+1,i})\right)$
$\Longrightarrow \widetilde{v}_n^{\mathcal{A}}(\widetilde{F}_{n,i}) \succcurlyeq \widetilde{v}_n^{\mathcal{E}}(\widetilde{F}_{n,i})$. □

Definition 3.6. The expected European fuzzy put option price at time $(n+1)$ is defined as $\widetilde{E}_n^Q(\widetilde{v}_{n+1}^{\mathcal{E}}(\widetilde{S}_{n+1,i})) = (\widetilde{p}_u\widetilde{v}_{n+1}^{\mathcal{E}}(\widetilde{u}\widetilde{S}_{n,i}) + \widetilde{p}_d\widetilde{v}_{n+1}^{\mathcal{E}}(\widetilde{d}\widetilde{S}_{n,i}))$ for $n = 0,1,2\ldots,N-1$ and $i = 0,1,\ldots,n$. Similarly, we can define for other states.

Remark 3.7. The European fuzzy put option price is defined by

$$\widetilde{v}_N^{\mathcal{E}}(\widetilde{S}_{N,i}) = \max\left\{\widetilde{K} - \widetilde{S}_{N,i}, \widetilde{0}\right\}, i = 0,1,\ldots,N$$

and

$$\widetilde{v}_n^{\mathcal{E}}(\widetilde{S}_{n,i}) = \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})}\left(\widetilde{p}_u\widetilde{v}_{n+1}^{\mathcal{E}}(\widetilde{u}\widetilde{S}_{n,i}) + \widetilde{p}_d\widetilde{v}_{n+1}^{\mathcal{E}}(\widetilde{d}\widetilde{S}_{n,i})\right)$$

for $i = 0,1,\ldots,n$, and $n = N-1, N-2,\ldots,0$

Proposition 3.8. *The discounted European fuzzy intrinsic values of EFPOBM*

$$\left\{\frac{\widetilde{v}_n^{\mathcal{A}}}{(\widetilde{1}+\widetilde{r})^n}\right\}_{n=0}^{N}$$

is a $Q-$ *fuzzy martingale with respect to the fuzzy risk-neutral probability measure* Q *whenever (i).* $\widetilde{u}\widetilde{p}_u + \widetilde{d}\widetilde{p}_d \approx \widetilde{1}+\widetilde{r}$ *(ii).* $\widetilde{u}\widetilde{p}_u + \widetilde{d}\widetilde{p}_d \succcurlyeq \widetilde{1}+\widetilde{r}$ *(iii).* $\widetilde{u}\widetilde{p}_u + \widetilde{d}\widetilde{p}_d \preccurlyeq \widetilde{1}+\widetilde{r}$.

Proof. We prove (i) by induction on n.
When $n=0$: Using Definition 3.6 and Remark 3.7 we have,
if $\widetilde{u}\widetilde{p}_u + \widetilde{d}\widetilde{p}_d \approx \widetilde{1}+\widetilde{r}$, then $\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^1}\widetilde{E}_0^Q(\widetilde{v}_1^{\mathcal{E}}(\widetilde{F}_1)) \approx \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^1}\widetilde{E}_0^Q(\widetilde{v}_1^{\mathcal{E}}(\widetilde{1}+\widetilde{r})^{N-1}\widetilde{S}_1)$
$\Longrightarrow \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^1}\widetilde{E}_0^Q(\widetilde{v}_1^{\mathcal{E}}(\widetilde{F}_1)) \approx \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^1}\left(\widetilde{p}_u\widetilde{v}_1^{\mathcal{E}}((\widetilde{1}+\widetilde{r})^{N-1}\widetilde{u}\widetilde{S}_0) + \widetilde{p}_d\widetilde{v}_1^{\mathcal{E}}((\widetilde{1}+\widetilde{r})^{N-1}\widetilde{d}\widetilde{S}_0)\right)$
$\Longrightarrow \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^1}\widetilde{E}_0^Q(\widetilde{v}_1^{\mathcal{E}}(\widetilde{F}_1)) \approx \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^1}\left(\widetilde{p}_u\widetilde{v}_1^{\mathcal{E}}((\widetilde{1}+\widetilde{r})^{N-1}\widetilde{S}_1^u) + \widetilde{p}_d\widetilde{v}_1^{\mathcal{E}}((\widetilde{1}+\widetilde{r})^{N-1}\widetilde{S}_1^d)\right)$
$\Longrightarrow \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^1}\widetilde{E}_0^Q(\widetilde{v}_1^{\mathcal{E}}(\widetilde{F}_1)) \approx \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^0}\widetilde{v}_0((\widetilde{1}+\widetilde{r})^{N-0}\widetilde{S}_0)$
$\Longrightarrow \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^1}\widetilde{E}_0^Q(\widetilde{v}_1^{\mathcal{E}}(\widetilde{F}_1)) \approx \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^0}\widetilde{v}_0^{\mathcal{E}}(\widetilde{F}_0)$

Assume (i) holds for $n \le p$.
i.e., $\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^p}\widetilde{v}_p^{\mathcal{E}}(\widetilde{F}_p^{uuu\ldots u}) \approx \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{E}_p^Q(\widetilde{v}_{p+1}^{\mathcal{E}}(\widetilde{F}_{p+1}^{uuu\ldots u}))$
Now to prove (i) for $n = p+1$.
$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{v}_{p+1}^{\mathcal{E}}(\widetilde{F}_{p+1}^{uuu\ldots u}) \approx \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{v}_{p+1}^{\mathcal{E}}((\widetilde{1}+\widetilde{r})^{N-(p+1)}\widetilde{S}_{p+1}^{uuu\ldots u})$
$\Longrightarrow \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{v}_{p+1}^{\mathcal{E}}(\widetilde{F}_{p+1}^{uuu\ldots u}) \approx$
$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^{p+2}}\left(\widetilde{p}_u\widetilde{v}_{p+2}^{\mathcal{E}}((\widetilde{1}+\widetilde{r})^{N-(p+2)}\widetilde{u}\widetilde{S}_{p+1}^{uuu\ldots u}) + \widetilde{p}_d\widetilde{v}_{p+2}^{\mathcal{E}}((\widetilde{1}+\widetilde{r})^{N-(p+2)}\widetilde{d}\widetilde{S}_{p+1}^{uuu\ldots u})\right)$

$\Longrightarrow \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{v}^{\mathcal{E}}_{p+1}(\widetilde{F}^{uuu...u}_{p+1}) \approx$
$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^{p+2}}\left(\widetilde{p}_u\widetilde{v}^{\mathcal{E}}_{p+2}((\widetilde{1}+\widetilde{r})^{N-(p+2)}\widetilde{S}^{uuu...u}_{p+2}) + \widetilde{p}_d\widetilde{v}^{\mathcal{E}}_{p+2}((\widetilde{1}+\widetilde{r})^{N-(p+2)}\widetilde{S}^{uuu...d}_{p+2})\right)$
$\Longrightarrow \frac{1}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{v}^{\mathcal{E}}_{p+1}(\widetilde{F}^{uuu...u}_{p+1}) \approx \frac{1}{(\widetilde{1}+\widetilde{r})^{p+2}}\widetilde{E}^{Q}_{p+1}(\widetilde{v}^{\mathcal{E}}_{p+2}((\widetilde{1}+\widetilde{r})^{N-(p+2)}\widetilde{S}^{uuu...u}_{p+2}))$
$\Longrightarrow \frac{1}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{v}^{\mathcal{E}}_{p+1}(\widetilde{F}^{uuu...u}_{p+1}) \approx \frac{1}{(\widetilde{1}+\widetilde{r})^{p+2}}\widetilde{E}^{Q}_{p+1}(\widetilde{v}^{\mathcal{E}}_{p+2}(\widetilde{F}^{uuu...u}_{p+2}))$

Similarly we can prove for other states.
We prove (ii) by induction on n.
When $n = 0$: Using Definition 3.6 and Remark 3.7 we have,
if $\widetilde{u}\widetilde{p}_u + \widetilde{d}\widetilde{p}_d \succcurlyeq 1+\widetilde{r}$, then $(\widetilde{1}+\widetilde{r})\widetilde{v}^{\mathcal{E}}_0(\widetilde{F}_0) \approx (\widetilde{1}+\widetilde{r})\widetilde{v}^{\mathcal{E}}_0((\widetilde{1}+\widetilde{r})^{N-0}\widetilde{S}_0)$
$\Longrightarrow (\widetilde{1}+\widetilde{r})\widetilde{v}^{\mathcal{E}}_0(\widetilde{F}_0) \succcurlyeq (\widetilde{p}_u\widetilde{v}^{\mathcal{E}}_1((\widetilde{1}+\widetilde{r})^{N-0}\widetilde{u}\widetilde{S}_0) + \widetilde{p}_d\widetilde{v}^{\mathcal{E}}_1((\widetilde{1}+\widetilde{r})^{N-0}\widetilde{d}\widetilde{S}_0))$
$\Longrightarrow (\widetilde{1}+\widetilde{r})\widetilde{v}^{\mathcal{E}}_0(\widetilde{F}_0) \succcurlyeq (\widetilde{p}_u\widetilde{v}^{\mathcal{E}}_1((\widetilde{1}+\widetilde{r})^{N-0}\widetilde{S}^{u}_1) + \widetilde{p}_d\widetilde{v}^{\mathcal{E}}_1((\widetilde{1}+\widetilde{r})^{N-0}\widetilde{S}^{d}_1))$
$\Longrightarrow (\widetilde{1}+\widetilde{r})\widetilde{v}^{\mathcal{E}}_0(\widetilde{F}_0) \succcurlyeq \widetilde{p}_u\widetilde{v}^{\mathcal{E}}_1(\widetilde{F}^{u}_1) + \widetilde{p}_d\widetilde{v}^{\mathcal{E}}_1(\widetilde{F}^{d}_1)$
$\Longrightarrow \frac{1}{(\widetilde{1}+\widetilde{r})^{0}}\widetilde{v}^{\mathcal{E}}_0(\widetilde{F}_0) \approx \frac{1}{(\widetilde{1}+\widetilde{r})^{1}}\widetilde{E}^{Q}_0(\widetilde{v}_1(\widetilde{F}_1))$
Similarly, we can prove for down state.
Assume (ii) holds for $n \leq p$. i.e., $\frac{1}{(\widetilde{1}+\widetilde{r})^{p}}\widetilde{v}^{\mathcal{E}}_p(\widetilde{F}^{uuu...u}_p) \approx \frac{1}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{E}^{Q}_p(\widetilde{v}^{\mathcal{E}}_{p+1}(\widetilde{F}^{uuu...u}_{p+1}))$
Now to prove (ii) for $n = p+1$.
$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{v}^{\mathcal{E}}_{p+1}(\widetilde{F}^{uuu...u}_{p+1}) \approx \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{v}^{\mathcal{E}}_{p+1}((\widetilde{1}+\widetilde{r})^{N-(p+1)}\widetilde{S}^{uuu...u}_{p+1})$
$\Longrightarrow \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{v}^{\mathcal{E}}_{p+1}(\widetilde{F}^{uuu...u}_{p+1}) \approx$
$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^{p+2}}\left(\widetilde{p}_u\widetilde{v}^{\mathcal{E}}_{p+2}((\widetilde{1}+\widetilde{r})^{N-(p+2)}\widetilde{u}\widetilde{S}^{uuu...u}_{p+1}) + \widetilde{p}_d\widetilde{v}^{\mathcal{E}}_{p+2}((\widetilde{1}+\widetilde{r})^{N-(p+2)}\widetilde{d}\widetilde{S}^{uuu...u}_{p+1})\right)$
$\Longrightarrow \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{v}^{\mathcal{E}}_{p+1}(\widetilde{F}^{uuu...u}_{p+1}) \approx$
$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{r})^{p+2}}\left(\widetilde{p}_u\widetilde{v}^{\mathcal{E}}_{p+2}((\widetilde{1}+\widetilde{r})^{N-(p+2)}\widetilde{S}^{uuu...u}_{p+2}) + \widetilde{p}_d\widetilde{v}^{\mathcal{E}}_{p+2}((\widetilde{1}+\widetilde{r})^{N-(p+2)}\widetilde{S}^{uuu...d}_{p+2})\right)$
$\Longrightarrow \frac{1}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{v}^{\mathcal{E}}_{p+1}(\widetilde{F}^{uuu...u}_{p+1}) \approx \frac{1}{(\widetilde{1}+\widetilde{r})^{p+2}}\widetilde{E}^{Q}_{p+1}(\widetilde{v}^{\mathcal{E}}_{p+2}((\widetilde{1}+\widetilde{r})^{N-(p+2)}\widetilde{S}^{uuu...u}_{p+2}))$
$\Longrightarrow \frac{1}{(\widetilde{1}+\widetilde{r})^{p+1}}\widetilde{v}^{\mathcal{E}}_{p+1}(\widetilde{F}^{uuu...u}_{p+1}) \approx \frac{1}{(\widetilde{1}+\widetilde{r})^{p+2}}\widetilde{E}^{Q}_{p+1}(\widetilde{v}^{\mathcal{E}}_{p+2}(\widetilde{F}^{uuu...u}_{p+2}))$
Similarly we can prove for other states.

We can prove (iii) by using induction on n similar to (ii). □

We record a computational procedure to estimate the PL values of EFPOBM / AFPOBM for future contracts using general trapezoidal fuzzy numbers.

Step 1: Compute the fuzzy stock prices $\widetilde{S}_{n,i}$
Step 2: Estimate the fuzzy future prices $\widetilde{F}_{n,i}$ of the given fuzzy stock using Remark 2.10.
Step 3: Calculate the up and down fuzzy risk-neutral probability measures $\widetilde{p}_u$ and $\widetilde{p}_d$ using equations 3.1 and 3.2 .
Step 4: Obtain the fuzzy intrinsic values of EFPOBM/AFPOBM with the fuzzy future price as the fuzzy underlying security using Definitions 2.8 and 3.4.
Step 5: Defuzzify the fuzzy future prices of EFPOBM/AFPOBM using the measure M given in Definition 2.4 and compare the same with the crisp strike price at time $n = 2$ / at time $n = 1$ and at time $n = 2$ during all the nodes of the fuzzy binomial tree.

Step 6: Estimate the fuzzy PL values by computing fuzzy intrinsic values of EFPOBM / AFPOBM + constant fuzzy premium paid and obtain the crisp PL values using the measure M given in step 4 and compare them.

3.1. Numerical Illustration. Here we deploy the real data of 22-days Microsoft Corporation (MSFT) put option future contract with the following specifications to study the PL values of EFPOBM and AFPOBM (Table 1).

Symbol	**Initial stock price S_0**	**Strike price K**	**Premium**	**Expiry Date**	**Option style**	**Risk-free interest rate**
MSFT	\$102.075	\$103	\$2.73	13/07/2018	PE/AE	0.88

Table 1. Quotes of MSFT put option future contract

Here, we consider positive general trapezoidal fuzzy numbers for the fuzzy stock and fuzzy future prices and non-negative general trapezoidal fuzzy numbers for the fuzzy intrinsic value price processes.

Description of Data in Table 1:
Consider the European/American MSFT103 put option future contract was currently trading at \$102.075 per share. It's underlying security was European/ American Microsoft corporation. The buyer of the put option would expect that the price of European/American MSFT103 put option will drop by the time of expiry so that he could purchase more quantity at lower price. Based on this belief, put option buyers enter into a put option agreement to make profits. As per the contract, put option buyers bought the right, but not the obligation to sell the European/American MSFT103 put option future contract at a strike price of \$103 per share on July 13 of 2018 to the put option sellers. To buy this right, put option buyers have to pay a premium of \$2.73 per share, to the put option sellers through the stock exchange. Once the buyer of the put option exercised their right, the seller of the put option was obligated (as he received premium from the buyer) to purchase the European/American MSFT103 put option contract at the above quoted strike price at which it was originally agreed from the put buyer.

Problem Description of EFPOBM and AFPOBM:
Consider the case when the European/American fuzzy put option buyers would choose to exercise the fuzzy put option if the fuzzy future price drops below the constant fuzzy strike price $\widetilde{K}$ i.e, if $\widetilde{K} \succcurlyeq \widetilde{F}_{n,i}$, the maximum profit for the buyers of the fuzzy put option is the difference between the fuzzy intrinsic values of EFPOBM/AFPOBM and the premium paid for the option. However if the fuzzy future price stays at $\widetilde{K}$ or headed above $\widetilde{K}$, then there will be a possibility of an entire investment be eroded and the maximum loss incurred by the buyer is the premium paid in this transaction and this will be the profit to the fuzzy put option seller.

Estimating the PL values of EFPOBM and AFPOBM:
Let the maturity date of the option be T= 22/360. The one-period risk-free interest rate and the estimated values of up and down jump factors are fuzzified into

the following general trapezoidal fuzzy numbers:
$\widetilde{r} = [0.84, 0.88, 0.91, 0.92]$,
$\widetilde{u} = [0.9538, 1.004, 1.0191, 1.0542]$,
$\widetilde{d} = [0.9405, 0.99, 1.0248, 1.0395]$.
The number of time steps involved in the fuzzy binomial tree is $n = 2$.

Step 1: We obtain the Fuzzy Binomial tree of fuzzy stock prices $\widetilde{S}_{n,i}$ and the values are tabulated in Table 2.

Nodes	**Fuzzy Binomial tree of fuzzy stock prices**
$\widetilde{S}_2^{uu}$	(92.8611, 102.8932, 106.0115, 113.4398)
$\widetilde{S}_2^{ud}$	(91.5663, 101.4585, 106.6044, 111.8580)
$\widetilde{S}_2^{dd}$	(90.2894, 100.0437, 107.2007, 110.2982)
$\widetilde{S}_1^{u}$	(97.3591, 102.4833, 104.0246, 107.6075)
$\widetilde{S}_1^{d}$	(96.0015, 101.0542, 104.6065, 106.1070)
$\widetilde{S}_0$	(102.075, 102.075, 102.075, 102.075)

Table 2. Fuzzy Binomial tree: fuzzy stock prices

Step 2: We compute the Fuzzy Binomial tree of fuzzy future prices $\widetilde{F}_{n,i}$ and the values are tabulated in Table 3.

Nodes	**Fuzzy Binomial tree of fuzzy future prices**
$\widetilde{F}_2^{uu}$	((92.8611, 102.8932, 106.0115, 113.4398)
$\widetilde{F}_2^{ud}$	((91.5663, 101.4585, 106.6044, 111.8580)
$\widetilde{F}_2^{dd}$	(90.2894, 100.0437, 107.2007, 110.2982)
$\widetilde{F}_1^{u}$	(97.3841, 102.5109, 104.0536, 107.6377)
$\widetilde{F}_1^{d}$	(96.0262, 101.0814, 104.6355, 106.1368)
$\widetilde{F}_0$	(102.1275, 102.1299, 102.1318, 102.1324)

Table 3. Fuzzy Binomial tree: fuzzy future prices

Step 3: We obtain the following up and down fuzzy risk-neutral probability measures

$$\widetilde{p}_u = [0.1598, 0.3529, 0.7341, 0.9410], \widetilde{p}_d = [0.0586, 0.2659, 0.6471, 0.8402]$$

.
Step 4: We compute the fuzzy intrinsic values of EFPOBM and AFPOBM and the same are recorded in the form of a tree diagram in Figure 1/ Table 4.

$$\begin{tikzcd}
 & & \widetilde{v}_0(\widetilde{F}_0)\text{[redacted]} \arrow[dl] \arrow[dr] & & \\
 & \widetilde{v}_1(\widetilde{F}_1)^d\text{[redacted]} \arrow[dl] \arrow[dr] & & \widetilde{v}_1(\widetilde{F}_1)^u\text{[redacted]} \arrow[dl] \arrow[dr] & \\
\widetilde{v}_2(\widetilde{F}_2)^{dd}\text{[redacted]} & & \widetilde{v}_2(\widetilde{F}_2)^{ud}\text{[redacted]} & & \widetilde{v}_2(\widetilde{F}_2)^{uu}\text{[redacted]}
\end{tikzcd}$$

Figure 1. Fuzzy Intrinsic values of EFPOBM / AFPOBM

Nodes	**Fuzzy intrinsic values: EFPOBM**	**Fuzzy intrinsic values: AFPOBM**
$\widetilde{v}_2(\widetilde{F}_2)^{uu}$	(0, 0, 0.1068, 10.1389)	(0, 0, 0.1068, 10.1389)
$\widetilde{v}_2(\widetilde{F}_2)^{ud}$	(0, 0, 1.5415, 11.4337)	(0, 0 ,1.5415, 11.4337)
$\widetilde{v}_2(\widetilde{F}_2)^{dd}$	(0, 0, 2.9563, 12.7106)	(0, 0, 2.9563, 12.7106)
$\widetilde{v}_1(\widetilde{F}_1)^{u}$	(0, 0, 1.0756, 19.1472)	(0, 0, 1.0756, 19.1472)
$\widetilde{v}_1(\widetilde{F}_1)^{d}$	(0, 0, 3.0440, 21.4385)	(0, 0, 3.0440, 21.4385)
$\widetilde{v}_0(\widetilde{F}_0)$	(0, 0, 2.7587, 36.0301)	(0.8676, 0.8682, 2.7587, 36.0301)

Table 4. Fuzzy intrinsic values of EFPOBM / AFPOBM

Step 5: We obtain the following crisp future prices by using the measure M.
$M(\widetilde{F}_2^{uu}) = 103.8014; M(\widetilde{F}_2^{ud}) = 102.8718; M(\widetilde{F}_2^{dd}) = 101.958;$
$M(\widetilde{F}_1^{u}) = 102.8966; M(\widetilde{F}_1^{d}) = 101.9700.$
Step 6: We compute the PL values of EFPOBM / AFPOBM and compare them and the values are recorded in Tables 5 and 6.

Nodes	**Fuzzy intrinsic values of EFPOBM + Premium paid**	**PL values**
uu	(-2.73, -2.73, -2.6232, 7.4089)	-0.1686
ud	(-2.73, -2.73, -1.1885, 8.7037)	+0.5138
dd	(-2.73, -2.73, 0.2263, 9.9806)	+1.1867
u	—	—
d	—	—

Table 5. PL values of EFPOBM and AFPOBM

Remark 3.9. In EFPOBM, the fuzzy intrinsic value at the initial node $M(\widetilde{v}_0^{\mathcal{E}}(\widetilde{F}_0))$ is (0, 0, 2.7587, 36.0301) (see Table 4) which is the theoretical fuzzy put option price. The defuzzified value of theoretical fuzzy put option price at that node is \$9.6972 whereas in AFPOBM the fuzzy intrinsic value at the initial node is

Nodes	**Fuzzy intrinsic values of AFPOBM + Premium paid**	**PL values**
uu	(-2.73, -2.73, -2.6232, 7.4089)	-0.1686
ud	(-2.73, -2.73, -1.1885, 8.7037)	+0.5138
dd	(-2.73, -2.73, 0.2263, 9.9806)	+1.1867
u	(-2.73, -2.73, -1.6544, 16.4172)	+2.3257
d	(-2.73, -2.73, 0.314, 18.7085)	+3.3906

A negative sign indicates that a cash outflow from the trading account.

Table 6. PL values of AFPOBM

(0.8676, 0.8682, 2.7587, 36.0301) and its defuzzified value is \$10.1311. This higher price is due to the fact that early exercise allowed in AFPOBM. As the observed market price \$2.73 is below the theoretical put option price in both the models, the put option contract is underpriced. It helps the option buyers to make optimum decision in option trading. Hence buyers would make profit when the put option price increases. Therefore in order to make profit, buyers will buy a security for a lower price and sell it for high.

The optimal exercise time and the optimal exercise price of AFPOBM and EFPOBM is shown in Table 7.

Nodes	EFPOBM		Optimal Price	AFPOBM		Optimal Price
ud	—	sell	+0.5138	—	sell	+0.5138
dd	—	sell	+1.1867	—	sell	+1.1867
u	—	—	—	sell	—	+2.3257
d	—	—	—	sell	—	+3.3906
Optimal Exercise Time	n=1	n=2		n=1	n=2	

Table 7. Optimal Exercise Time of AFPOBM / EFPOBM

Comparision of PL values between AFPOBM and EFPOBM:
The profit earned by EFPOBM and AFPOBM (see Tables 5 and 6) are substanially increasing at maturity using general trapezoidal fuzzy numbers. Also the buyer obtained loss on expiration day when $\widetilde{K}$ declines below the fuzzy future price at the node uu. In this case, the premium amount \$2.73 is the reward for the option seller in both AFPOBM and EFPOBM. Since in AFPOBM, buyers have the facility to exercise the option any time before or on expiry, they can execute the option early at the node d and collect the maximum profit \$+3.3906. However in EFPOBM as early exercise is not permitted, AFPOBM price would be atleast equal to or higher than the EFPOBM price at every earlier node(see Proposition 3.5). As the discounted fuzzy intrinsic values of EFPOBM process is Q - fuzzy martingale with respect to the fuzzy risk-neutral probability measure Q, the intrinsic values of EFPOBM would not be rise or fall (see Proposition 3.8). Hence AFPOBM price is optimal. However the risk involved in AFPOBM is higher than that of EFPOBM. This is due to the fact that early exercise is allowed

in AFPOBM. Though, on expiry both the models yield the same prices.

Remark 3.10. In particular, if $a_2 = a_3$, the trapezoidal fuzzy number reduces to a triangular fuzzy number given by $\widetilde{A} = (a_1, a_2, a_4)$.

Remark 3.11. The above described problem is also handled for EFPOBM and AFPOBM involving general triangular fuzzy numbers and the PL values are tabulated in Tables 8 and 9. Note that the profit and loss values obtained by

Nodes	**Fuzzy intrinsic values of EFPOBM + Premium paid**	**PL values**
uu	(-2.73, -2.6232, 7.4089)	-0.1419
ud	(-2.73, -1.1885, 8.7037)	+0.8992
dd	(-2.73, 0.2263, 9.9806)	+1.9258
u	—	—
d	—	—

Table 8. EFPOBM- PL values using general triangular fuzzy numbers

Nodes	**Fuzzy intrinsic values of AFPOBM + Premium paid**	**PL values**
uu	(-2.73,-2.6232,7.4089)	-0.1419
ud	(-2.73, -1.1885, 8.7037)	+0.8992
dd	(-2.73, 0.2263, 9.9806)	+1.9258
u	(-2.73, -2.2409, 16.4172)	+2.3013
d	(-2.73, -0.8114, 18.7085)	+3.5889

Table 9. AFPOBM- PL values using general triangular fuzzy numbers

EFPOBM and AFPOBM for future contracts using general trapezoidal gives optimum result than using general triangular fuzzy numbers at the earlier nodes whereas the PL values obtained by EFPOBM and AFPOBM involving general triangular fuzzy numbers gives better result than using general trapezoidal fuzzy numbers at the expiry nodes. Also the crisp theoretical put option prices of EFPOBM and AFPOBM involving general triangular fuzzy numbers are \$9.4425 and \$9.6595 respectively which is lower than the crisp theoretical put option prices of EFPOBM and AFPOBM using general trapezoidal fuzzy numbers (see Remark 3.9).

4. Conclusion and Future Work:

We have studied EFPOBM/AFPOBM for future contracts involving general trapezoidal fuzzy numbers with respect to a fuzzy risk-neutral probability measure defined by us. Also we obtained the PL values in both the models and compared them. Our future work is directed to seller's of European Fuzzy Put Option Model for future contracts using general triangular, trapezoidal and octagonal fuzzy numbers.

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