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STOCHASTIC DIFFERENTIAL EQUATIONS WITH NON-INSTANTANEOUS IMPULSES, NON-LOCAL CONDITIONS AND INFINITE DELAY

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ABSTRACT. For a system of differential equations modeling a problem, impulses, delays, non-local conditions and noises are intrinsic phenomena that under certain conditions do not change certain properties of the system, such as stability, synchronization, controllability. In other words, if these elements, not taken into account in many mathematical models, are added as disturbances of the system, the existences of solution and controllability hold through. In this regard, we study the existence and uniqueness of solutions for retarded stochastic differential equations with non-instantaneous impulses, non-local conditions and infinite delay. To this end, first of all, we select the phase space adequately in such a way that it satisfies the axiomatic theory formulated by Hale and Kato to study differential equations with infinite delay. Then the problem of the existence of solutions is reduced to the problem of finding the fixed points of an operator equation; to do so, we apply Karakosta's Fixed Point Theorem, which is an Extension of Krasnosel'skii's Fixed Point Theorem. After that, under certain conditions, we prove that the solutions of our problem are unique. Next, we study the prolongation of solutions, and we prove that, under certain conditions, these solutions are globally defined. Finally, we present an example to illustrate our results.

♣ Note to author: Use 2000 Mathematics Subject Classification.

1. Introduction

For a system of differential equations modeling a real life problem, impulses, delays, non-local conditions and noises are intrinsic phenomena that under certain conditions do not change certain properties of the system, such as stability, synchronization and controllability. That is, if we consider these elements as disturbances of the system, it turns out that the existences of solution and controllability hold through under these influences not taken into account in many mathematical models that represent extremely important problems in the area of applied mathematics. On the other hand, stochastic differential equations result from the generalization of ordinary differential equations and partial differential equations. This is due to the need to include noises into the equation in order to have more realistic models. Example of stochastic models, come from financial markets or

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financial derivatives in which the price of the underlying assets evolve randomly, for which normal distributions are used in their modeling. Other stochastic models also emerge in population dynamics where the model of the number of births and deaths can be seen as a counting process, say $(N_t)_{t\in\mathbb{R}}$ is the population size subject to a birth rate λ and a death rate μ. Other examples can be found in epidemiological contagion. But, it turns out that to have a more accuracy model, which considers these intrinsic phenomena that appears naturally in the real life problem, we must include abrupt changes or impulses, nonlocal conditions and delays. That is to say, the stochastic differential equation along is not enough to have a mathematical model caring out all these phenomena. That is why stochastic differential equations has intensified study in recent years in which each model studied includes instantaneous or non-instantaneous impulses or non-local condition and delay, or even all of the previous ones. For more information about this one can see [7] and nearby references.
Without further ado, in this work we will study the existence and uniqueness of solutions for the following semi linear stochastic differential equations with multiplicative noise, non-instantaneous impulses, non-local conditions and infinite delay

$$\begin{cases} dz(t) = \{A(t)z(t) + f(t, z_t)\}dt + \sigma(t, z_t)dW(t), & t \in I_k, \quad k = 0, 1, 2, \cdots, \\ z(s) + g(z)(s) = \phi(s), & s \in (-\infty, 0], \\ z(t) = \mathcal{G}_k(t, z(t_k^-)), & t \in J_k, \quad k = 1, 2, \cdots, \end{cases} \tag{1.1}$$

where $A(t) \in \mathbb{R}^{n\times n}$ is a continuous matrix, $I_0 = (0, t_1]$, $I_k = (s_k, t_{k+1}]$, $J_k = (t_k, s_k]$, $0 = t_0 = s_0 < t_1 < s_1 < t_2 < s_2 < \cdots < s_k < t_{k+1} \longrightarrow \infty$, as $k \longrightarrow \infty$. $g : \mathfrak{Q} \to \mathfrak{Q}$, $\phi : (-\infty, 0] \times \Omega \longrightarrow \mathbb{R}^n$, $f : (0, +\infty) \times \mathfrak{Q} \longrightarrow \mathbb{R}^n$, $\sigma : (0, +\infty) \times \mathfrak{Q} \longrightarrow \mathbb{R}^{n\times n}$, are given functions satisfying some assumptions, $\mathcal{G}_k : [t_k, s_k] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for $k = 1, 2, 3, \cdots$, represents the impulsive effect of the system (1.1), $\{W(s) : s \geq 0\}$ is an n-dimensional Wiener process defined on the probability space $(\Omega, \mathcal{F}, P)$ with an increasing family $\mathcal{F}_t$ of sub-σ-algebra of $\mathcal{F}$.

The advantage of having nonlocal conditions is that measurements at more places can be incorporated to get better accuracy models. For more details and physical interpretations see [3, 4, 5, 6, 21] and references therein. $\phi \in \mathfrak{Q}$, $\mathfrak{Q}$ is the phase space to be specified later. $\mathcal{G}_k : J_k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $k = 1, 2, 3, \cdots$, are continuous and represents the impulsive effect in the system (1.1), i.e., we are considering that the system can have abrupt changes that stay there for an interval of time. These alterations in state might be due to certain external factors, which cannot be well described by pure ordinary stochastic differential equations, (see, for instance, [11] and reference therein). For this type of problems the phase space for initial functions plays an important role in the study of both qualitative and quantitative theory, for more details, in the non stochastic case without impulses and non local conditions, we refer to Hale and Kato [8], Hino et al [9] and Shin [18, 19]. Here, $z_t(\theta) = z(t + \theta)$ for $\theta \in (-\infty, 0]$ illustrate the history of the state up to the time t, and also remembers much of the historical past of ϕ, carrying part of the present to the past.

2. Preliminaries

This section is dedicated mainly to select the appropriate phase space $\mathfrak{Q}$ where our problem will be set, which satisfies the axiomatic theory proposed by Hale and Kato to study differential equations with infinite delay; this is on the one hand, while on the other hand, throughout this paper we will use the following standard notations. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with measure P on Ω and filtration $\{\mathcal{F}_s, s \geq 0\}$ generated by an n-dimensional Wiener process $\{W(s), s \geq 0\}$ defined on $(\Omega, \mathcal{F}, P)$. The filtration satisfies $\mathcal{F}_s \subset \mathcal{F}$ for $s \geq 0$.

If $z(\cdot,\cdot) : (0, \tau] \times \Omega \to \mathbb{R}^n$, then for any $t \geq 0$, $z_t(\cdot,\cdot) : (-\infty, 0] \times \Omega \to \mathbb{R}^n$ is given by,

$$z_t(s, \omega) = z(t+s, \omega), \quad s \in (-\infty, 0], \quad \omega \in \Omega.$$

We will define the linear space of measurable random functions $\phi(t, \omega)$, denoted by $\mathcal{CP} = \mathcal{CP}((-\infty, 0] \times \Omega; \mathbb{R}^n)$, as the linear space of almost surely (a.s.) normalized piecewise continuous paths. i.e., the restriction of $\phi(\cdot, \omega)$ to the interval $[a, 0]$ is measurable with respect to $\mathcal{F}_0$ on and piecewise continuous on any $[a, 0]$, $a < 0$, which can be written as follows:

$$\begin{aligned}\mathcal{CP} = \{\varphi :\ & (-\infty, 0] \times \Omega \longrightarrow \mathbb{R}^n : \mathcal{F}_0 - \text{measurable and} \\ & \varphi|_{[a,0]} \quad \text{is a.s. piecewise continuous function}, \quad \forall a < 0 \Big\}.\end{aligned}$$

Now, we will define the phase space C_h for our system (1.1). By using some ideas from [15], we consider a function $h : \mathbb{R} \to \mathbb{R}_+$ such that

a) $h(0) = 1$,
b) $h(-\infty) = +\infty$,
c) h is decreasing.

Remark 2.1. A particular function h is $h(s) = \exp(-as)$, with $a > 0$.

Using a function h of the foregoing type, we define the following linear space of functions:

$$C_h = \left\{ \phi : \phi \in \mathcal{CP} \text{ and } \sup_{s \leq 0} \frac{\mathbf{E}(\|\phi(s)\|^2_{\mathbb{R}^n})}{h(s)} < \infty \right\},$$

the space C_h endowed with the norm

$$\|\phi\|_{C_h} = \left(\sup_{s \leq 0} \frac{\mathbf{E}\|\phi(s)\|^2_{\mathbb{R}^n}}{h(s)} \right)^{1/2}, \quad \phi \in C_h,$$

is a Banach space(see [22]).

Proof. It is enough to show that C_h is a complete normed space. Let $\{\phi_n\}$ be a Cauchy sequence in C_h with probability 1. Now, from $\{\phi_n\}$ one can extract a subsequence that converges for almost every (t, ω). To do so, we take $n_1 = 1$ and define n_k inductively as the smallest $n > n_{k-1}$ such that

$$\sup_{s \leq 0} \frac{\mathbf{E}\|\phi_m(s, \omega) - \phi_{m'}(s, \omega)\|^2}{h(s)} < 2^{-3r} \text{ for all } m > n, \quad m' > n.$$

Then, applying Chebyshev inequality, we get that

$$\mathbf{P}\left\{\omega : \frac{\|\phi_{n_{r+1}}(s,\omega) - \phi_{n_r}(s,\omega)\|}{h(s)} > 2^{-r}\right\} \leq \frac{\mathbf{E}\|\phi_{n_{r+1}}(s,\omega) - \phi_{n_r}(s,\omega)\|^2}{h(s)2^{-2r}} < 2^{-r}.$$

Then, the series

$$\sum_{r=0}^{\infty} \mathbf{P}\left\{\omega : \frac{\|\phi_{n_{r+1}}(s,\omega) - \phi_{n_r}(s,\omega)\|}{h(s)} > 2^{-r}\right\} \leq \sum_{r=0}^{\infty} 2^{-r} < \infty,$$

and by Borel-Cantelli theorem, the series

$$\sum_{r=1}^{\infty} \frac{\|\phi_{n_{r+1}}(s,\omega) - \phi_{n_r}(s,\omega)\|}{h(s)} < \infty,$$

almost surely converges uniformly in $s \in (-\infty, 0]$.
Let $\psi_{n_r}(s,\omega) = \frac{\phi_{n_r}(s,\omega)}{h(s)}$ and $\mathcal{N} = \{\omega : \|\psi_{n_{r+1}}(s,\omega) - \psi_{n_r}(s,\omega)\| = \infty\}$,
then, if we put

$$\psi(s,\omega) = \begin{cases} \psi_{n_1}(s,\omega) + \sum_{k=0}^{\infty}(\psi_{n_{k+1}}(s,\omega) - \psi_{n_k}(s,\omega)), & \omega \in \Omega \setminus \mathcal{N}, \\ 0, & \omega \in \mathcal{N}, \end{cases}$$

we obtain ψ_{n_k} converge to ψ almost surely. A standard argument yields to the conclusion that the whole sequence $\{\psi_n\}$ converges to ψ. To finish the proof it is enough to see that $h\psi \in C_h$. □

Now, we shall consider the space $\mathcal{CP}_\tau$ of $\mathcal{F}_\tau$-adapted processes defined on $(-\infty, \tau]$, for a fixed $\tau > 0$:

$$\mathcal{CP}_\tau = \Big\{ z : (-\infty, \tau] \times \Omega \to \mathbb{R}^n : z\Big|_{(-\infty,0]} \in C_h, z\Big|_{(t_k,t_{k+1}]} \text{ is continuous for all } \omega \in \Omega,\ k = 0, 1, \cdots, p; s_{p-1} < \tau,\ z_0 \in C_h, \text{ and there exist } z(t_k^+), z(t_k^-) \text{ with } z(t_k^-) = z(t_k), k = 1, 2, \cdots, p \text{ and } \sup_{t\in[0,\tau]} \mathbf{E}\|z(t)\|^2_{\mathbb{R}^n} < \infty \Big\}.$$

Lemma 2.2. *$\mathcal{CP}_\tau$ is a Banach space endowed with the norm*

$$\|z\|^2_{\mathcal{CP}_\tau} = \|z(s)\|^2_{C_h} + \sup_{0\leq s\leq\tau}\Big(\mathbf{E}\,\|z(s)\|^2_{\mathbb{R}^n}\Big) \tag{2.1}$$

The set

$$\mathfrak{Q} := C_h,$$

equipped with the norm

$$\|z\|_{\mathfrak{Q}} := \|z\|_{C_h},$$

will be our phase space.
It is not hard to verify that $\mathfrak{Q}$ satisfies the Hale and Kato axiomatic theory [8] for the phase space of retarded differential equations with infinite delay, but defined as in the book [9]:

(A1) If $z \in \mathcal{CP}_\tau$ and $z_0 \in \mathfrak{Q}$, then for every $t \in [0, \tau]$ the following conditions hold:
  (i) z_t is in $\mathfrak{Q}$;
  (ii) $\|z(t)\|_{\mathcal{CP}_\tau} \leq H\|z_t\|_{\mathfrak{Q}}$;

(iii) $\|z_t\|_{\mathfrak{Q}} \leq K(t)\sup\{\|z(s)\|_{\mathcal{CP}_\tau} : 0 < s \leq t\} + M(t)\|z_0\|_{\mathfrak{Q}}$, where $H \geq 0$ is a constant, $K,\ M : [0,\infty) \to [0,\infty)$, K is continuous and M is locally bounded, and $H,\ K,\ M$ are independent of z.

(A2) For the function z in (A1), z_t is a $\mathfrak{Q}-$valued function on $[0,\tau]$.

(A3) The space $\mathfrak{Q}$ is complete.

The following result is stronger than axiom (A1)-iii), and is due to the fact that the function h is defined in the all real line:

Lemma 2.3. *For all function $z \in \mathcal{CP}_\tau$ the following estimate holds for all $t \in [0,\tau]$:*

$$\|z_t\|_{\mathfrak{Q}} \leq \|z\|_{\mathcal{CP}_\tau} = \|z\|.$$

Proof.

$$\begin{aligned}
\|z_t\|^2_{\mathfrak{Q}} &= \sup_{\theta\leq 0}\frac{\mathbf{E}\|z_t(\theta)\|^2}{h(\theta)} = \sup_{\theta\leq 0}\frac{\mathbf{E}\|z(t+\theta)\|^2}{h(\theta)} = \sup_{\theta\leq 0}\frac{\mathbf{E}\|z(t+\theta)\|^2}{h(t+\theta)}\frac{h(t+\theta)}{h(\theta)} \\
&\leq \sup_{\theta\leq 0}\frac{\mathbf{E}\|z(t+\theta)\|^2}{h(t+\theta)} = \sup_{l\leq t}\frac{\mathbf{E}\|z(l)\|^2}{h(l)} \leq \sup_{l\leq 0}\frac{\mathbf{E}\|z(l)\|^2}{h(l)} + \sup_{0\leq l\leq t}\mathbf{E}\|z(l)\|^2 \\
&\leq \|z\|^2_{\mathfrak{Q}} + \sup_{0\leq s\leq \tau}\mathbf{E}\|z(s)\|^2 = \|z\|^2_{\mathcal{CP}_\tau}.
\end{aligned}$$

□

3. Main Results

In this section, we shall prove that under suitable conditions on f, $\mathcal{G}_k$, g and σ, the Problem (1.1) admits a mild solution on $(-\infty,\tau]$, for some $\tau > 0$, given, for $k = 1,2,\cdots$, by

$$z(t) = \begin{cases} \mathcal{S}(t,0)[\phi(0) - g(z)(0)] + \displaystyle\int_0^t \mathcal{S}(t,s)f(s,z_s)ds \\ + \displaystyle\int_0^t \mathcal{S}(t,s)\sigma(s,z_s)dW(s),\ t \in I_0 = (0,t_1] \\ \mathcal{S}(t,s_k)z(s_k) + \displaystyle\int_{s_k}^t \mathcal{S}(t,s)f(s,z_s)ds + \int_0^t \mathcal{S}(t,s)\sigma(s,z_s)dW(s),\ t \in I_k \\ \mathcal{G}_k(t, z(t_k^-)),\ t \in J_k, \\ \phi(t) - g(z)(t),\ t \in (-\infty,0] \end{cases} \tag{3.1}$$

where $\mathcal{S}(t,s_)$ is the evolution operator o transition matrix associated with the linear system defined below.

Remark 3.1. It is necessary to consider the condition $z(t) = \mathcal{G}_k(t,z(t_k^-))$ for $t \in (t_k,s_k] = J_k$ and $k = 1,2,\cdots$, since in this case $z(t_k^+)) = \mathcal{G}_k(t_k,z(t_k^-))$.

Additionally, the solution z shall be defined on each s_k by:

$$z(s_k) = z(s_k^+) := z(s_k^-) = \mathcal{G}_k(s_k, z(t_k^-)).$$

We denote by Φ the fundamental matrix of the linear system

$$z'(t) = A(t)z(t), \quad t \in \mathbb{R} \tag{3.2}$$

i.e.,

$$\begin{cases} \dfrac{d\Phi(t)}{dt} = A(t)\Phi(t), \\ \Phi(0) = I. \end{cases}$$

Then the evolution operator $\mathcal{S}(t,s)$ is defined by $\mathcal{S}(t,s) = \Phi(t)\Phi^{-1}(s), t, s \in \mathbb{R}$.

For $\tau > 0$, we consider the following bound for the evolution operator

$$M = \sup_{t,s\in(0,\tau]} \|\mathcal{S}(t,s)\|.$$

Definition 3.2. Given $\phi \in \mathfrak{Q}$, an $\mathbb{R}^n$-valued process $\{z(t), t \in (-\infty,\tau]\}$ is called a mild solution of Eq.(1.1) if

(1) $z(\cdot) \in \mathcal{CP}_\tau$.
(2) For arbitrary $t \in (\infty,\tau]$, we have the Eq. (3.1). It is to be understood that (3.1) hold P-a.s.,that is, for any fixed t there is an $\Omega_t \in \mathcal{F}_t$ of full P-measure such that (3.1) hold for all $\omega \in \Omega$.

3.1. Existence and uniqueness Theorems. In this subsection, we shall assume the hypotheses that will allow us to prove the existence and uniqueness of solutions for problem (1.1).

H1. The functions $f : \mathbb{R}_+ \times \mathfrak{Q} \longrightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}_+ \times \mathfrak{Q} \to \mathbb{R}^{n\times n}$ satisfies the following conditions for $\forall\varphi_1, \varphi_2 \in \mathfrak{Q}, \quad \forall t \in I_k$:
i) $\|f(t,\varphi_1)-f(t,\varphi_2)\|^2_{\mathbb{R}^n}+\|\sigma(t,\varphi_1)-\sigma(t,\varphi_2)\|^2_{\mathbb{R}^{n\times n}} \le \mathcal{K}(\|\varphi_1\|_{\mathfrak{Q}}, \|\varphi_2\|_{\mathfrak{Q}})\,\|\varphi_1-\varphi_2\|^2_{\mathfrak{Q}}$
where

$$\|\sigma(t,\varphi_1)\|_{\mathbb{R}^{n\times n}} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |\sigma_{ij}(t,\varphi_1)|^2\right)^{1/2},$$

ii) $\|f(t,\varphi)\|^2_{\mathbb{R}^n} + \|\sigma(t,\varphi)\|^2_{\mathbb{R}^{n\times n}} \le \tilde{\psi}(\|\varphi\|^2_{\mathfrak{Q}}), \forall\varphi \in \mathfrak{Q}$, where $\mathcal{K} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\tilde{\psi} : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous and increasing functions respectively.

H2. The impulse $\mathcal{G}_k : [t_k, s_k] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions: There exist constants $L > 0$ such that for all $k = 1, 2, ..., y, z \in \mathbb{R}^n, \ell, t \in J_k$ we have that $\mathcal{G}_k(t, 0) = 0$ and
i) $\mathbf{E}\,\|\mathcal{G}_k(t,y)) - \mathcal{G}_k(\ell,z)\|^2_{\mathbb{R}^n} \le L\left\{|t-\ell|^2 + \|y-z\|^2_{\mathbb{R}^n}\right\}$.
ii) The function $g : \mathfrak{Q} \to \mathbb{R}^n$ satisfies the following condition: There exist constant $d_q > 0$ such that

$$\mathbf{E}\,\|g(x) - g(y)\|^2_{\mathbb{R}^n} \quad \le d_q\,\|x-y\|^2_{\mathfrak{Q}}, \quad \forall x, y \in \mathfrak{Q},$$

with $g(0) = 0$ and

$$M^2L + d_q < \frac{1}{2}$$

H3. There exist $\tau, \rho > 0$ such that

$$3M^2\left\{ (d_q + L)\,(\|\tilde{\phi}\| + \rho)^2 + \tau(\tau+1)\tilde{\psi}((\|\tilde{\phi}\| + \rho)^2)\right\} < \frac{\rho}{2},$$

where the function $\tilde{\phi} \in \mathcal{CP}_\tau$ is defined for $\phi \in \mathfrak{Q}$ by

$$\tilde{\phi} = \begin{cases} \mathcal{S}(t,0)\phi(0), & t \in I_0, \\ \phi(t), & t \in \mathbb{R}_-, \\ 0, & t \in I_k, \\ 0, & t \in J_k. \end{cases} \tag{3.3}$$

To prove our main existence theorem, we shall use some ideas from [12, 13] where the following known result is applied.

Theorem 3.3. *(See* [10]*) (G.L. Karakostas Fixed Point Theorem) Let Z and Y be Banach spaces and D be a closed convex subset of Z, and let $\mathcal{C} : D \to Y$ be a continuous operator such that $\mathcal{C}(D)$ is a relatively compact subset of Y. Let*

$$\mathcal{T} : D \times \overline{\mathcal{C}(D)} \to D$$

is a continuous operator such that the family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{C}(D)}\}$ is an equicontractive family. Then, the operator equation

$$\mathcal{T}(z, \mathcal{C}(z)) = z$$

admits a solution on D.

The following closed and convex set will be considered

$$D = D(\rho, \tau, \tilde{\phi}) = \left\{ y \in \mathcal{CP}_\tau : \|y - \tilde{\phi}\|_{\mathcal{CP}_\tau} \leq \rho \right\}, \tag{3.4}$$

where the function $\tilde{\phi}$ is defined in (3.3).

Theorem 3.4. *Let's consider an initial function $\phi \in \mathfrak{Q}$ such that the hypothesis* **H1**-**H3** *hold. Then, the system* (1.1) *admits a mild solution on $(-\infty, \tau]$.*

Proof. Let's consider the following operators:

$$\begin{aligned} \mathcal{T} :& \mathcal{CP}_\tau \times \mathcal{CP}_\tau \longrightarrow \mathcal{CP}_\tau, \\ \mathcal{C} :& \mathcal{CP}_\tau \longrightarrow \mathcal{CP}_\tau, \end{aligned}$$

where

$$\mathcal{T}(z,y)(t) = \begin{cases} \phi(t) - g(z)(t), \ t \in (-\infty, 0], \\ y(t), \ t \in I_0, \\ y(t) + \mathcal{S}(t, s_k)\mathcal{G}_k(s_k, z(t_k^-)), \ t \in I_k, \\ \mathcal{G}_k(t, z(t_k^-)), \ t \in J_k, \end{cases}$$

and

$$\mathcal{C}(z)(t) = \begin{cases} \mathcal{S}(t,0)[\phi(0) - g(z)] + \int_0^t \mathcal{S}(t,s) f(s, z_s) ds + \int_0^t \mathcal{S}(t,s)\sigma(s, z_s) dW(s), t \in I_0, \\ \int_{s_k}^t \mathcal{S}(t,s) f(s, z_s) ds + \int_{s_k}^t \mathcal{S}(t,s)\sigma(s, z_s) dW(s), \ t \in I_k, \\ \phi(t), \ t \in (-\infty, 0], \\ 0, \ t \in J_k. \end{cases}$$

Now, the problem to find a solution of (1.1) is reduced to the problem of finding a solution of the operator equation

$$\mathcal{T}(z, \mathcal{C}(z)) = z.$$

We will apply Karakostas Fixed Point Theorem to find solutions of such equation. In fact, let us verify that the operators $\mathcal{C}$ and $\mathcal{T}$ satisfies the assumptions presented in Theorem 3.3. To do so, the proof will be divided in several steps:

Step 1: $\mathcal{C}$ is a continuous operator. In order to prove this, we shall use the hypotheses (**H1**-i), (**H2**-ii) and Lemma 2.3. For $z, y \in \mathcal{CP}_\tau$ and the function h considered in the definition of our phase space, in this regard we have:

- Let $s \in (-\infty, 0]$, then

$$\mathbf{E}\|\mathcal{C}(z)(s) - \mathcal{C}(y)(s)\|^2_{\mathbb{R}^n} = \mathbf{E}\|\phi(s) - \phi(s)\|^2_{\mathbb{R}^n} = 0. \tag{3.5}$$

Now,

$$\|\mathcal{C}(z) - \mathcal{C}(y)\|^2_{\mathfrak{Q}} = \sup_{s\leq 0} \frac{\mathbf{E}\|\mathcal{C}(z)(s) - \mathcal{C}(y)(s)\|^2_{\mathbb{R}^n}}{h(s)} = 0.$$

Therefore, for $s \in (-\infty, 0]$ we get that

$$\|\mathcal{C}(z) - \mathcal{C}(y)\|^2_{\mathfrak{Q}} = \|\mathcal{C}(z) - \mathcal{C}(y)\|^2_{\mathfrak{Q}} = 0. \tag{3.6}$$

- Next, let's consider $t \in I_0$:

$$\begin{aligned}
\mathbf{E}\|\mathcal{C}(y)(t) - \mathcal{C}(z)(t)\|^2_{\mathbb{R}^n} &= \mathbf{E}\Big\|\mathcal{S}(t,0)g(z)(0) - \mathcal{S}(t,0)g(y)(0) + \int_0^t \mathcal{S}(t,s)f(s,z_s)ds \\
&- \int_0^t \mathcal{S}(t,s)f(s,y_s)ds + \int_0^t \mathcal{S}(t,s)\sigma(s,z_s)dW(s) - \int_0^t \mathcal{S}(t,s)\sigma(s,y_s)dW(s)\Big\|^2_{\mathbb{R}^n} \\
&\leq \mathbf{E}\bigg(\Big\|\mathcal{S}(t,0)g(z)(0) - \mathcal{S}(t,0)g(y)(0)\Big\| + \Big\|\int_0^t \mathcal{S}(t,s)f(s,z_s)ds - \int_0^t \mathcal{S}(t,s)f(s,y_s)ds\Big\| \\
&+ \Big\|\int_0^t \mathcal{S}(t,s)\sigma(s,z_s)dW(s) - \int_0^t \mathcal{S}(t,s)\sigma(s,y_s)dW(s)\Big\|\bigg)^2 \\
&\leq 3\mathbf{E}\Big\|\mathcal{S}(t,0)g(z)(0) - \mathcal{S}(t,0)g(y)(0)\Big\|^2 + 3\mathbf{E}\Big\|\int_0^t \mathcal{S}(t,s)\{f(s,z_s) - f(s,y_s)\}ds\Big\|^2 \\
&+ 3\mathbf{E}\Big\|\int_0^t \mathcal{S}(t,s)\{\sigma(s,z_s) - \sigma(s,y_s)\}dW(s)\Big\|^2 \\
&\leq 3M^2\mathbf{E}\Big\|g(z)(0) - g(y)(0)\Big\|^2_{\mathbb{R}^n} + 3M^2 t\int_0^t \mathbf{E}\|f(s,z_s) - f(s,y_s)\|^2_{\mathbb{R}^n} ds \\
&+ 3M^2\int_0^t \mathbf{E}\Big\|\sigma(s,z_s) - \sigma(s,y_s)\Big\|^2_{\mathbb{R}^{n\times n}} ds \\
&\leq 3M^2 d_q\|y - z\|^2_{\mathfrak{Q}} + 3M^2 t\int_0^t \mathcal{K}(\|z_s\|^2_{\mathfrak{Q}}, \|y_s\|^2_{\mathfrak{Q}})\|y_s - z_s\|^2_{\mathfrak{Q}} ds \\
&+ 3M^2\int_0^t \mathcal{K}(\|z_s\|^2_{\mathfrak{Q}}, \|y_s\|^2_{\mathfrak{Q}})\|z_s - y_s\|^2_{\mathfrak{Q}} ds
\end{aligned}$$

$$\leq 3M^2 d_q \|y-z\|^2_{\mathfrak{Q}} + 3M^2 t_1 \int_0^t \mathcal{K}(\|z\|^2_{\mathcal{CP}_\tau}, \|y\|^2_{\mathcal{CP}_\tau}) \|y-z\|^2_{\mathcal{CP}_\tau} ds$$
$$+ 3M^2 \int_0^t \mathcal{K}(\|z\|^2_{\mathcal{CP}_\tau}, \|y\|^2_{\mathcal{CP}_\tau}) \|z-y\|^2_{\mathcal{CP}_\tau} ds$$
$$\leq \left(3M^2 d_q + 3M^2(t_1^2+t_1)\mathcal{K}(\|z\|^2_{\mathcal{CP}_\tau}, \|y\|^2_{\mathcal{CP}_\tau})\right)\|y-z\|^2_{\mathcal{CP}_\tau}.$$

Hence, on $t \in I_0$ we get

$$\mathbf{E}\|\mathcal{C}(y)(t)-\mathcal{C}(z)(t)\|^2_{\mathbb{R}^n} \leq \left(3M^2 d_q + 3M^2(t_1^2+t_1)\mathcal{K}(\|z\|^2_{\mathcal{CP}_\tau}, \|y\|^2_{\mathcal{CP}_\tau})\right)\|y-z\|^2_{\mathcal{CP}_\tau}.$$

- Now, let's consider $t \in I_k$, for $k = 1, 2, ...p$.

$$\mathbf{E}\|\mathcal{C}(z)(t)-\mathcal{C}(y)(t)\|^2_{\mathbb{R}^n}$$
$$= \mathbf{E}\left\|\int_{s_k}^t \mathcal{S}(t,s)(f(s,z_s)-f(s,y_s))ds + \int_{s_k}^t \mathcal{S}(t,s)(\sigma(s,z_s)-\sigma(s,y_s))dW(s)\right\|^2_{\mathbb{R}^n}$$
$$\leq 2\int_{s_k}^t \|\mathcal{S}(t,s)\|^2 ds \int_{s_k}^t \mathbf{E}\|f(s,z_s)-f(s,y_s)\|^2_{\mathbb{R}^n} ds$$
$$+ 2\int_{s_k}^t \|\mathcal{S}(t,s)\|^2 \mathbf{E}\|\sigma(s,z_s)-\sigma(s,y_s)\|^2_{\mathbb{R}^{n\times n}} ds$$
$$\leq 2M^2(t-s_k)\int_{s_k}^t \mathcal{K}(\|z_s\|^2_{\mathfrak{Q}}, \|y_s\|^2_{\mathfrak{Q}})\|z_s-y_s\|^2_{\mathfrak{Q}} ds$$
$$+ 2M^2 \int_{s_k}^t \mathcal{K}(\|z_s\|^2_{\mathfrak{Q}}, \|y_s\|^2_{\mathfrak{Q}})\|z_s-y_s\|^2_{\mathfrak{Q}} ds$$
$$\leq 2M^2(t-s_k)^2\mathcal{K}(\|z\|^2_{\mathcal{CP}_\tau}, \|y\|^2_{\mathcal{CP}_\tau})\|z-y\|^2_{\mathcal{CP}_\tau}$$
$$+ 2M^2(t-s_k)\mathcal{K}(\|z\|^2_{\mathcal{CP}_\tau}, \|y\|^2_{\mathcal{CP}_\tau})\|z-y\|^2_{\mathcal{CP}_\tau}$$
$$\leq 2M^2(\tau^2+\tau)\mathcal{K}(\|z\|^2_{\mathcal{CP}_\tau}, \|y\|^2_{\mathcal{CP}_\tau})\|z-y\|^2_{\mathcal{CP}_\tau}.$$

Thus, on I_k we get that

$$\|\mathcal{C}(z)(t)-\mathcal{C}(y)(t)\|^2_{\mathcal{CP}_\tau} \leq 2M^2(\tau^2+\tau)\mathcal{K}(\|z\|^2_{\mathcal{CP}_\tau}, \|y\|^2_{\mathcal{CP}_\tau})\|z-y\|^2_{\mathcal{CP}_\tau}. \tag{3.7}$$

Since $\mathbf{E}\|\mathcal{C}(z)(t)-\mathcal{C}(y)(t)\|^2_{\mathbb{R}^n} = 0$ for $t \in J_k$, $k = 1, 2, ...$, from (3.5), (3.6) and (3.7), we have that $\mathcal{C}$ is locally Lipschitz, which implies the continuity of the operator $\mathcal{C}$.

Step 2: $\mathcal{C}$ maps bounded sets of $\mathcal{CP}_\tau$ into bounded sets of $\mathcal{CP}_\tau$. It is enough to prove that for any $R > 0$ there exists $r > 0$ such that for each $y \in B_R = \left\{z \in \mathcal{CP}_\tau : \|z\|^2_{\mathcal{CP}_\tau} \leq R\right\}$, we have that $\|\mathcal{C}(y)\|^2_{\mathcal{CP}_\tau} \leq r$. Indeed, let's consider $z \in B_R$. Then, by to Lemma 2.3 and hypotheses (**H1**)-ii)-(**H2**)-ii), we obtain the following:

- For $t \in (-\infty, 0]$, we obtain that

$$\mathbf{E}\|\mathcal{C}(z)(t)\|^2_{\mathbb{R}^n} = \mathbf{E}\|\phi(t)\|^2_{\mathbb{R}^n},$$

which implies

$$\|C(z)\|^2_{\mathcal{CP}_\tau} = \|C(z)\|^2_{\mathfrak{Q}} = \|\phi\|^2_{\mathfrak{Q}} := R_1. \tag{3.8}$$

• For $t \in I_0$, we have obtain that,

$$\mathbf{E}\|\mathcal{C}(z)(t)\|^2_{\mathbb{R}^n} \leq 3\mathbf{E}\,\|\mathcal{S}(t,0)\,\{\phi(0)-g(z)(0)\}\|^2 + 3\int_0^t \|\mathcal{S}(t,s)\|^2 ds \int_0^t \mathbf{E}\|f(s,z_s)\|^2_{\mathbb{R}^n} ds$$

$$+\,3\int_0^t \|\mathcal{S}(t,s)\|^2 \mathbf{E}\|\sigma(s,z_s)\|^2_{\mathbb{R}^{n\times n}}$$

$$\leq 3M^2\mathbf{E}\|\phi(0)-g(z)(0)\|^2 + 3M^2\tilde{\psi}(\|z\|^2_{\mathcal{CP}_\tau})(t^2+t)$$

$$\leq 3M^2\mathbf{E}\|\phi(0)\|^2_{\mathbb{R}^n} + 3M^2\mathbf{E}\|g(z)(0)\|^2_{\mathbb{R}^n} + 3M^2\tilde{\psi}(\|z\|^2_{\mathcal{CP}_\tau})(t^2+t)$$

$$\leq 3M^2\mathbf{E}\|\phi(0)\|^2_{\mathbb{R}^n} + 3M^2 d_q\|z\|^2_{\mathfrak{Q}} + 3M^2\tilde{\psi}(\|z\|^2_{\mathcal{CP}_\tau})(t_1^2+t_1)$$

$$\leq 3M^2\mathbf{E}\|\phi(0)\|^2_{\mathbb{R}^n} + 3M^2 d_q\|z\|^2_{\mathcal{CP}_\tau} + 3M^2\tilde{\psi}(\|z\|^2_{\mathcal{CP}_\tau})(t_1^2+t_1)$$

$$\leq 3M^2\mathbf{E}\|\phi(0)\|^2_{\mathbb{R}^n} + 3M^2 d_q R + 3M^2\tilde{\psi}(R)(t_1^2+t_1) := R_2$$

• For $t \in I_k$, we have that

$$\mathbf{E}\|\mathcal{C}(z)(t)\|^2_{\mathbb{R}^n} \leq 2M^2(t-s_k)\int_{s_k}^t \mathbf{E}\|f(s,z_s)\|^2_{\mathbb{R}^n} ds + 2M^2\int_{s_k}^t \mathbf{E}\|\sigma(s,z_s)\|^2_{\mathbb{R}^{n\times n}} ds$$

$$\leq 2M^2\tau^2\tilde{\psi}(\|z\|^2_{\mathcal{CP}_\tau}) + 2M^2\tau\tilde{\psi}(\|z\|^2_{\mathcal{CP}_\tau})$$

$$\leq 2M^2\tilde{\psi}(\|z\|^2_{\mathcal{CP}_\tau})\tau(\tau+1)$$

$$\leq 2M^2\tilde{\psi}(R)\tau(\tau+1) := R_3.$$

Letting $r = R_1 + R_2 + R_3$, we get that $\|\mathcal{C}(z)\|^2_{\mathcal{CP}_\tau} \leq r$.

Step 3: $\mathcal{C}$ maps bounded sets of $\mathcal{CP}_{h\tau}$ into equicontinuous sets of PW_τ.
Let's consider B_R as it was previously defined in the foregoing step. We shall prove that $\mathcal{C}(B_R)$ is an equicontinuous family. Since the equicontinuity on $(-\infty, 0]$ is trivial, we only need to prove the equicontinuity in the remain part. Let's take $y \in B_R$, and consider Lemma 2.3 and hypotheses (**H1**)-ii), (**H2**)-ii). Then, we get that

• For $t_1, t_2 \in I_0$ such that $0 < t_1 < t_2$, it turns out that

$$\mathbf{E}\|\mathcal{C}(y)(t_2) - \mathcal{C}(y)(t_1)\|^2_{\mathbb{R}^n}$$

$$= \mathbf{E}\left\|\mathcal{S}(t_2,0)\,\{\phi(0)-g(y)\} + \int_0^{t_2}\mathcal{S}(t_2,s)f(s,y_s)ds + \int_0^{t_2}\mathcal{S}(t_2,s)\sigma(s,y_s)dW(s)\right.$$

$$\left. -\mathcal{S}(t_1,0)\,\{\phi(0)-g(y)\} - \int_0^{t_1}\mathcal{S}(t_1,s)f(s,y_s)ds - \int_0^{t_1}\mathcal{S}(t_1,s)\sigma(s,y_s)dW(s)\right\|^2_{\mathbb{R}^n}$$

$$\leq 5\mathbf{E}\|\big(\mathcal{S}(t_2,0)-\mathcal{S}(t_1,0)\big)\,\{\phi(0)-g(y)\}\|^2_{\mathbb{R}^n}$$

$$+\,5\mathbf{E}\left\|\int_0^{t_1}[\mathcal{S}(t_2,s)-\mathcal{S}(t_1,s)]f(s,y_s)ds\right\|^2_{\mathbb{R}^n}$$

$$+\,5\mathbf{E}\left\|\int_{t_1}^{t_2}\mathcal{S}(t_2,s)f(s,y_s)ds\right\|^2_{\mathbb{R}^n} + 5\mathbf{E}\left\|\int_0^{t_1}[\mathcal{S}(t_2,s)-\mathcal{S}(t_2,s)]\sigma(s,y_s)dW(s)\right\|^2_{\mathbb{R}^n}$$

$$+\,5\mathbf{E}\left\|\int_{t_1}^{t_2}\mathcal{S}(t_2,s)\sigma(s,y_s)dW(s)\right\|^2_{\mathbb{R}^n}$$

$$\leq 5\|\mathcal{S}(t_2,0)-\mathcal{S}(t_1,0)\|^2\mathbf{E}\|\phi(0)-g(y)(0)\|^2_{\mathbb{R}^n}$$
$$+5\int_0^{t_1}\|\mathcal{S}(t_2,s)-\mathcal{S}(t_1,s)\|^2_{\mathbb{R}^n}ds\int_0^{t_1}\mathbf{E}\|f(s,y_s)\|^2_{\mathbb{R}^n}ds$$
$$+5\int_{t_1}^{t_2}\|\mathcal{S}(t_2,s)\|^2_{\mathbb{R}^n}ds\int_{t_1}^{t_2}\mathbf{E}\|f(s,y_s)\|^2_{\mathbb{R}^n}ds$$
$$+5\int_0^{t_1}\|\mathcal{S}(t_2,s)-\mathcal{S}(t_1,s)\|^2_{\mathbb{R}^n}\mathbf{E}\|\sigma(s,y_s)\|^2_{\mathbb{R}^{n\times n}}ds$$
$$+5\int_{t_1}^{t_2}\|\mathcal{S}(t_2,s)\|^2_{\mathbb{R}^n}\mathbf{E}\|\sigma(s,y_s)\|^2_{\mathbb{R}^{n\times n}}ds$$
$$\leq 10\|\mathcal{S}(t_2,0)-\mathcal{S}(t_1,0)\|^2_{\mathbb{R}^n}\{\mathbf{E}\|\phi(0)\|^2_{\mathbb{R}^n}+d_q\|y\|^2_{\mathcal{CP}_\tau}\}$$
$$+5M^2\{(t_2-t_1)^2+(t_2-t_1)\}\tilde{\psi}(\|y\|^2_{\mathcal{CP}_\tau})+5\tilde{\psi}(\|y\|^2_{\mathcal{CP}_\tau})(t_1+1)\int_0^{t_1}\|\mathcal{S}(t_2,s)-\mathcal{S}(t_1,s)\|^2ds$$
$$\leq 10\|\mathcal{S}(t_2,0)-\mathcal{S}(t_1,0)\|^2_{\mathbb{R}^n}\left(\mathbf{E}\|\phi(0)\|^2_{\mathbb{R}^n}+d_qR\right)+5M^2\tilde{\psi}(R)\{(t_2-t_1)^2+(t_2-t_1)\}$$
$$+5\tilde{\psi}(R)(t_1+1)\int_0^{t_1}\|\mathcal{S}(t_2,s)-\mathcal{S}(t_1,s)\|^2_{\mathbb{R}^n}ds,$$

by the continuity of the evolution operator, we have that

$$\mathbf{E}\|\mathcal{C}(y)(t_2)-\mathcal{C}(y)(t_1)\|^2_{\mathbb{R}^n}\to 0 \quad as \quad t_2\to t_1, \tag{3.9}$$

independently on $y\in B_R$.

• For $t_1,t_2\in I_k$ such that $0<t_1<t_2$, we have that

$$\mathbf{E}\|\mathcal{C}(y)(t_2)-\mathcal{C}(y)(t_1)\|^2_{\mathbb{R}^n}=\mathbf{E}\Bigg\|\int_{s_k}^{t_2}\mathcal{S}(t_2,s)f(s,y_s)ds-\int_{s_k}^{t_1}\mathcal{S}(t_1,s)f(s,y_s)ds$$
$$+\int_{s_k}^{t_2}\mathcal{S}(t_2,s)\sigma(s,y_s)dW(s)-\int_{s_k}^{t_1}\mathcal{S}(t_1,s)\sigma(s,y_s)dW(s)\Bigg\|^2$$
$$\leq 4\int_{s_k}^{t_1}\|\mathcal{S}(t_2,s)-\mathcal{S}(t_1,s)\|^2ds\int_{s_k}^{t_1}\mathbf{E}\|f(s,y_s)\|^2ds$$
$$+4\int_{t_1}^{t_2}\|\mathcal{S}(t_2,s)\|^2ds\int_{t_1}^{t_2}\mathbf{E}\|f(s,y_s)\|^2ds+4\int_{s_k}^{t_1}\|\mathcal{S}(t_2,s)-\mathcal{S}(t_1,s)\|^2\mathbf{E}\|\sigma(s,y_s)\|^2_{\mathbb{R}^{n\times n}}ds$$
$$+4\int_{t_1}^{t_2}\|\mathcal{S}(t_2,s)\|^2\mathbf{E}\|\sigma(s,y_s)\|^2_{\mathbb{R}^{n\times n}}ds$$
$$\leq 4\tilde{\psi}(\|y\|^2_{\mathcal{CP}_\tau})\{(t_1-s_k)+1\}\int_{s_k}^{t_1}\|\mathcal{S}(t_2,s)-\mathcal{S}(t_1,s)\|^2ds$$
$$+4M^2\tilde{\psi}(\|y\|^2_{\mathcal{CP}_\tau})(t_2-t_1)[(t_2-t_1)+1]$$
$$\leq 4\tilde{\psi}(R)\{(t_1-s_k)+1\}\int_{s_k}^{t_1}\|\mathcal{S}(t_2,s)-\mathcal{S}(t_1,s)\|ds+4M^2\tilde{\psi}(R)(t_2-t_1)[(t_2-t_1)+1].$$

The continuity of $\mathcal{S}(t,s)$ implies that

$$\mathbf{E}\|\mathcal{C}(y)(t_2)-\mathcal{C}(y)(t_1)\|^2_{\mathbb{R}^n}\to 0 \quad as \quad t_2\to t_1, \tag{3.10}$$

independently on the chosen y.

Hence, by (3.9) and (3.10), the operator $\mathcal{C}$ maps bounded sets of $\mathcal{CP}_\tau$ into equicontinuous sets of $\mathcal{CP}_\tau$, i.e., $\mathcal{C}(B_R)$ is an equicontinuous family.

Step 4: The subset $\mathcal{C}(D)$ is relatively compact in $\mathcal{CP}_\tau$. In fact, let $D \subset \mathcal{CP}_\tau$ be the bounded set defined in (3.4). Without loss of generality, we can assume that $t_p \leq \tau$. By **steps 2** and **3**, $\mathcal{C}(D)$ it is bounded uniform and equicontinuous in $\mathcal{CP}_{h\tau}$. Let us take a sequence $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathcal{C}(D)$. Observe that $y_n|_{(-\infty,0]} = \phi$ for all $n \in \mathbb{N}$, then by Arzelá-Ascoli Theorem applied to $\left\{y_n|_{(0,t_1]}\right\}_{n\in\mathbb{N}} \subset \mathcal{C}(D)$, there exist an uniformly convergent subsequence $\{y_n^1\}_{n\in\mathbb{N}}$ on $(-\infty, t_1]$. Let's consider now the sequence $\{y_n^1\}_{n\in\mathbb{N}}$ on the interval $(t_1, t_2]$, which is also bounded and equicontinuous. Then, applying Arzelá-Ascoli Theorem, it has a convergent subsequence $\{y_n^2\}_{n\in\mathbb{N}}$ on $(t_1, t_2]$. This sequence is actually an uniformly convergent subsequence of $\{y_n\}_{n\in\mathbb{N}}$ on $(-\infty, t_2]$. We continue this process iteratively over each interval $(t_2, t_3], \cdots, (t_p, \tau]$ and finally arrived to the conclusion that the subsequence $\{y_n^p\}_{n\in\mathbb{N}} \subseteq \{y_n\}_{n\in\mathbb{N}}$ is uniformly convergent on the whole interval $(-\infty, \tau]$. This implies that $\overline{\mathcal{C}(D)}$ is compact, and therefore $\mathcal{C}(D)$ is relatively compact .

Step 5:

The family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{C}(D)}\}$ is equicontractive.

Let us take $z, x \in \mathcal{CP}_\tau$ and $y \in \overline{\mathcal{C}(D)}$. Also, consider Lemma 2.3 and (H2), then

- Let us chose $t \in (-\infty, 0]$. Then

$$\begin{aligned}\frac{\mathbf{E}\,\|\mathcal{T}(z,\mathcal{C}(y))(t) - \mathcal{T}(x,\mathcal{C}(y))(t)\|^2_{\mathbb{R}^n}}{h(t)} &= \frac{\mathbf{E}\,\|g(z)(t) - g(x)(t)\|^2_{\mathbb{R}^n}}{h(t)} \\ &\leq \|g(z) - g(x)\|^2_{\mathfrak{Q}} \\ &\leq d_q\,\|z - y\|^2_{\mathfrak{Q}} \\ &\leq d_q\|z - x\|^2_{\mathcal{CP}_\tau} \\ &\leq \frac{1}{2}\|z - x\|^2_{\mathcal{CP}_\tau}.\end{aligned}$$

By taking the supremum on $t \in (-\infty, 0]$, we have that,

$$\|\mathcal{T}(z,\mathcal{C}(y)) - \mathcal{T}(x,\mathcal{C}(y))\|^2_{\mathfrak{Q}} \leq \frac{1}{2}\|z - x\|^2_{\mathcal{CP}_\tau}. \tag{3.11}$$

- Let $t \in I_0$. Then we have that

$$\|\mathcal{T}(z,\mathcal{C}(y)(t)) - \mathcal{T}(x,\mathcal{C}(y))(t)\| = \|\mathcal{C}(y)(t) - \mathcal{C}(y))(t)\| = 0.$$

- Let $t \in I_k$. Then we have that

$$\begin{aligned}\mathbf{E}\,\|\mathcal{T}(z,\mathcal{C}(y))(t) - \mathcal{T}(x,\mathcal{C}(y))(t)\|^2_{\mathbb{R}^n} &= \left\|\mathcal{S}(t,s_k)\mathcal{G}_k(s_k,z(t_k^-)) - \mathcal{S}(t,s_k)\mathcal{G}_k(s_k,x(t_k^-))\right\|^2_{\mathbb{R}^n} \\ &\leq M^2\left\|\mathcal{G}_k(s_k,z(t_k^-)) - \mathcal{G}_k(s_k,x(t_k^-))\right\|^2_{\mathbb{R}^n} \\ &\leq M^2 L\|z(t_k^-) - x(t_k^-)\|^2_{\mathbb{R}^n} \\ &\leq \frac{1}{2}\|z - x\|^2_{\mathcal{CP}_\tau}\end{aligned}$$

Thus,

$$\mathbf{E}\,\|\mathcal{T}(z,\mathcal{C}(y))(t)-\mathcal{T}(x,\mathcal{C}(y))(t)\|^2_{\mathbb{R}^n}\leq \frac{1}{2}\|z-x\|^2_{\mathcal{CP}_\tau},\quad t\in I_k. \tag{3.12}$$

- Consider $t\in J_k$. Then we get

$$\begin{aligned}\mathbf{E}\,\|\mathcal{T}(z,\mathcal{C}(y))(t)-\mathcal{T}(x,\mathcal{C}(y))(t)\|^2_{\mathbb{R}^n} &\leq \left\|\mathcal{G}_k(t,z(t_k^-))-\mathcal{G}_k(t,x(t_k^-))\right\|^2_{\mathbb{R}^n}\\ &\leq L\|z(t_k^-)-x(t_k^-)\|^2_{\mathbb{R}^n}\\ &\leq \frac{1}{2}\|z-x\|^2_{\mathcal{CP}_\tau}.\end{aligned}$$

Hence,

$$\mathbf{E}\,\|\mathcal{T}(z,\mathcal{C}(y))(t)-\mathcal{T}(x,\mathcal{C}(y))(t)\|^2_{\mathbb{R}^n}\leq \frac{1}{2}\|z-x\|^2_{\mathcal{CP}_\tau}. \tag{3.13}$$

Therefore, from the foregoing inequalities and (3.13), we get that

$$\|\mathcal{T}(z,\mathcal{C}(y))-\mathcal{T}(x,\mathcal{C}(y))\|^2_{\mathcal{CP}_\tau}<\frac{1}{2}\|z-x\|^2_{\mathcal{CP}_\tau},$$

which is a contraction independently of $y\in\overline{\mathcal{C}(D)}$. So , the family $\{\mathcal{T}(\cdot,y): y\in\overline{\mathcal{C}(D)}\}$ is equicontractive.

Step 6: Let D defined in (3.4), we will prove that $\mathcal{T}(D,\overline{\mathcal{C}(D)}\subseteq D$. In fact, let us consider $z\in D(\rho,\tau,\tilde{\phi})$ and take into account Lemma 2.3, hypotheses (**H1**)-ii), (**H2**)-ii) and (**H3**).

- Consider $t\in(-\infty,0]$. Then we have the following estimate

$$\begin{aligned}&\frac{1}{h(t)}\mathbf{E}\|\mathcal{T}(z,\mathcal{C}(z))(t)-\tilde{\phi}(t)\|^2_{\mathbb{R}^n}=\frac{1}{h(t)}\mathbf{E}\|g(z)\|^2_{\mathbb{R}^n}\\ &\leq d_q\|z\|^2_{\mathfrak{Q}}\\ &\leq d_q\|z\|^2_{\mathcal{CP}_\tau}\\ &\leq d_q(\|\tilde{\phi}\|_{\mathcal{CP}_\tau}+\rho)^2<\rho/2<\rho.\end{aligned}$$

- Next, for $t\in I_0$, we get that

$$\begin{aligned}&\mathbf{E}\|\mathcal{T}(z,\mathcal{C}(z))(t)-\tilde{\phi}(t)\|^2_{\mathbb{R}^n}\\ &=\mathbf{E}\left\|-\mathcal{S}(t,0)g(z)(0)+\int_0^t\mathcal{S}(t,s)f(s,z_s)ds+\int_0^t\mathcal{S}(t,s)\sigma(s,z_s)dW(s)\right\|^2_{\mathbb{R}^n}\\ &\leq 3\mathbf{E}\left\|\mathcal{S}(t,0)g(z)(0)\right\|^2_{\mathbb{R}^n}+3\mathbf{E}\left\|\int_0^t\mathcal{S}(t,s)f(s,z_s)ds\right\|^2_{\mathbb{R}^n}+3\mathbf{E}\left\|\int_0^t\mathcal{S}(t,s)\sigma(s,z_s)dW(s)\right\|^2_{\mathbb{R}^n}\\ &\leq 3M^2\mathbf{E}\|g(z)(0)\|^2_{\mathbb{R}^n}+3M^2t\int_0^t\mathbf{E}\|f(s,z_s)\|^2_{\mathbb{R}^n}ds+3M^2\int_0^t\mathbf{E}\|\sigma(s,z_s)\|^2_{\mathbb{R}^{n\times n}}ds\\ &\leq 3M^2\mathbf{E}\|g(z)\|^2_{\mathfrak{Q}}+3M^2t^2\tilde{\psi}(\|z\|^2_{\mathcal{CP}_\tau})+3M^2t\tilde{\psi}(\|z\|^2_{\mathcal{CP}_\tau})\\ &\leq 3M^2d_q\,\|z\|^2_{\mathfrak{Q}}+3M^2\tilde{\psi}(\|z\|^2_{\mathcal{CP}_\tau})t(t+1)\\ &\leq 3M^2d_q\,\|z\|^2_{\mathcal{CP}_\tau}+3M^2\tilde{\psi}(\|z\|^2_{\mathcal{CP}_\tau})t(t+1)\end{aligned}$$

$$\leq 3M^2 d_q \, \|z\|^2_{\mathcal{CP}_\tau} + 3M^2 \tilde{\psi}(\|z\|^2_{\mathcal{CP}_\tau})\tau(\tau+1)$$
$$\leq 3M^2 \{d_q(\|\tilde{\phi}\|_{\mathcal{CP}_\tau} + \rho)^2 + \tilde{\psi}((\|\tilde{\phi}\|_{\mathcal{CP}_\tau} + \rho)^2)\tau(\tau+1)\}$$
$$< \rho/2 < \rho.$$

- Next, if $t \in I_k$, we get that

$$\mathbf{E}\|\mathcal{T}(z, \mathcal{C}(z))(t) - \tilde{\phi}(t)\|^2_{\mathbb{R}^n} \leq 3\|\mathcal{S}(t, s_k)\|^2 \|\mathcal{G}_k(s_k, z(t_k^-))\|^2_{\mathbb{R}^n}$$
$$+ 3\int_{s_k}^t \|\mathcal{S}(t,s)\|^2 ds \int_{s_k}^t \mathbf{E}\, \|f(s, z_s)\|^2_{\mathbb{R}^n} \, ds + 3\int_{s_k}^t \|\mathcal{S}(t,s)\|^2 \mathbf{E}\, \|\sigma(s, z_s)\|^2_{\mathbb{R}^{n\times n}}$$
$$\leq 3M^2\{L(\|\tilde{\phi}\|_{\mathcal{CP}_\tau} + \rho)^2 + \tau(\tau+1)\tilde{\psi}((\|\tilde{\phi}\|_{\mathcal{CP}_\tau} + \rho)^2)\} < \rho/2 < \rho$$

- Finally, if $t \in J_k$, we get that

$$\begin{aligned}
\mathbf{E}\|\mathcal{T}(z, \mathcal{C}(z))(t) - \tilde{\phi}(t)\|^2_{\mathbb{R}^n} &= \mathbf{E}\|\mathcal{G}_k(t, z(t_k^-))\|^2_{\mathbb{R}^n} \\
&\leq L\mathbf{E}\|z(t_k^-)\|^2_{\mathbb{R}^n} \leq L\mathbf{E}\|z\|^2_{\mathcal{CP}_\tau} \\
&\leq L(\|\tilde{\phi}\|_{\mathcal{CP}_\tau} + \rho)^2 < \rho/2 < \rho
\end{aligned}$$

Hence, $\mathcal{T}(D, \overline{\mathcal{C}(D)}) \subseteq D$.

Since **Step 1**, **Step 4** and **Step** 5 hold, the conditions of Karakostas Fixed Point Theorem are satisfied for the closed and convex set given in (3.4), and the proof of Theorem 3.4 immediately follows by applying Theorem 3.3. □

Theorem 3.5. *(***Uniqueness***) In addition to the conditions of Theorem* (3.4)*, we suppose that for $\rho, \tau > 0$ the following inequality holds*

$$3M^2\left[\tau(\tau+1)\mathcal{K}((\|\tilde{\phi}\| + \rho)^2, (\|\tilde{\phi}\| + \rho)^2) + (d_q + L)\right] < \frac{1}{2},$$

then, the problem (1.1) *has only one solution on* $(-\infty, \tau]$.

Proof. Let z^1 and z^2 be two solutions of problem (1.1). Then, we have that the following estimate holds for $t \in (-\infty, 0]$:

$$\begin{aligned}
\frac{1}{h(t)}\mathbf{E}\|z^1(t) - z^2(t)\|^2_{\mathbb{R}^n} &= \frac{1}{h(t)}\mathbf{E}\|g(z_2)(t) - g(z_1)(t)\|^2_{\mathbb{R}^n} \\
&\leq d_q\|z^2 - z^1\|^2_{\mathfrak{Q}} \\
&\leq d_q\|z^2 - z^1\|^2_{\mathcal{CP}_\tau} \\
&< \frac{1}{2}\|z^2 - z^1\|^2_{\mathcal{CP}_\tau}.
\end{aligned}$$

Now, let $t \in (0, t_1]$, then we get

$$\mathbf{E}\|z^2(t) - z^1(t)\|^2_{\mathbb{R}^n}$$
$$= \Bigg\|\mathcal{S}(t,0)[\phi(0) - g(z^2)(0)] + \int_0^t \mathcal{S}(t,s)f(s, z_s^2)ds + \int_0^t \mathcal{S}(t,s)\sigma(s, z_s^2)dW(s)$$
$$- \mathcal{S}(t,0)[\phi(0) - g(z^1)(0)] - \int_0^t \mathcal{S}(t,s)f(s, z_s^1)ds - \int_0^t \mathcal{S}(t,s)\sigma(s, z_s^1)dW(s)\Bigg\|$$

$$\leq 3\mathbf{E}\|\mathcal{S}(t,0)[g(z^1)(0)-g(z^2)(0)]\|^2_{\mathbb{R}^n}+3\mathbf{E}\|\int_0^t \mathcal{S}(t,s)(f(s,z_s^2)-f(s,z_s^1))ds\|^2_{\mathbb{R}^n}$$
$$+3\mathbf{E}\|\int_0^t \mathcal{S}(t,s)(\sigma(s,z_s^2)-\sigma(s,z_s^1))dW(s)\|^2_{\mathbb{R}^n}$$
$$\leq 3M^2\mathbf{E}\|[g(z^1)(0)-g(z^2)(0)]\|^2_{\mathbb{R}^n}+3M^2\int_0^t \mathbf{E}\|(f(s,z_s^2)-f(s,z_s^1))\|^2_{\mathbb{R}^n}ds$$
$$+3M^2\int_0^t \mathbf{E}\|(\sigma(s,z_s^2)-\sigma(s,z_s^1))\|^2_{\mathbb{R}^{n\times n}}ds$$
$$\leq 3M^2 d_q\|z^1-z^2\|^2_{\mathcal{CP}_\tau}+3M^2t(t+1)\mathcal{K}(\|z^2\|^2_{\mathcal{CP}_\tau},\|z^1\|^2_{\mathcal{CP}_\tau})\|z^2-z^1\|^2_{\mathcal{CP}_\tau}$$
$$=[3M^2d_q+3M^2t(t+1)\mathcal{K}(\|z^2\|^2_{\mathcal{CP}_\tau},\|z^1\|^2_{\mathcal{CP}_\tau})]\|z^2-z^1\|^2_{\mathcal{CP}_\tau}$$
$$\leq 3M^2\left[d_q+\tau(\tau+1)\mathcal{K}((\|\tilde{\phi}\|_{\mathcal{CP}_\tau}+\rho)^2,(\|\tilde{\phi}\|_{\mathcal{CP}_\tau}+\rho)^2)\right]\|z^2-z^1\|^2_{\mathcal{CP}_\tau}$$
$$\leq \frac{1}{2}\|z^2-z^1\|^2_{\mathcal{CP}_\tau}.$$

- Now, we consider $t\in I_k$. Then

$$\mathbf{E}\|z^2(t)-z^1(t)\|^2_{\mathbb{R}^n}=\mathbf{E}\bigg\|\mathcal{S}(t,s_k)\mathcal{G}_k(s_k,z^2(t_k^-))+\int_{s_k}^t \mathcal{S}(t,s)f(s,z_s^2)ds$$
$$+\int_{s_k}^t \mathcal{S}(t,s)\sigma(s,z_s^2)dW(s)-\mathcal{S}(t,s_k)\mathcal{G}_k(s_k,z^1(t_k^-))-\int_{s_k}^t \mathcal{S}(t,s)f(s,z_s^1)ds$$
$$-\int_{s_k}^t \mathcal{S}(t,s)\sigma(s,z_s^1)dW(s)\bigg\|^2_{\mathbb{R}^n}$$
$$\leq 3M^2\mathbf{E}\|\mathcal{G}_k(s_k,z^2(t_k^-))-\mathcal{G}_k(s_k,z^1(t_k^-))\|^2_{\mathbb{R}^n}$$
$$+3M^2(t_{k+1}-s_k)\int_{s_k}^{t_{k+1}}\mathbf{E}\|f(s,z_s^2)-f(s,z_s^1)\|^2_{\mathbb{R}^n}ds$$
$$+3M^2\int_{s_k}^{t_{k+1}}\mathbf{E}\|\sigma(s,z_s^2)-\sigma(s,z_s^1)\|^2_{\mathbb{R}^{n\times n}}ds$$
$$\leq 3M^2L\|z^2-z^1\|^2_{\mathcal{CP}_\tau}$$
$$+3M^2(t_{k+1}-s_k)[(t_{k+1}-s_k)+1]\mathcal{K}(\|z^2\|^2_{\mathcal{CP}_\tau},\|z^1\|^2_{\mathcal{CP}_\tau})\|z^2-z^1\|^2_{\mathcal{CP}_\tau}$$
$$\leq 3M^2\left[L+(t_{k+1}-s_k)[(t_{k+1}-s_k)+1]\mathcal{K}((\|\tilde{\phi}\|+\rho)^2,(\|\tilde{\phi}\|+\rho)^2)\right]\|z^2-z^1\|^2_{\mathcal{CP}_\tau}$$
$$\leq 3M^2\left[L+\tau(\tau+1)\mathcal{K}((\|\tilde{\phi}\|+\rho)^2,(\|\tilde{\phi}\|+\rho)^2)\right]\|z^2-z^1\|^2_{\mathcal{CP}_\tau}$$
$$\leq \frac{1}{2}\|z^2-z^1\|^2_{\mathcal{CP}_\tau}$$

- Consider $t \in J_k$. Then we have that

$$\begin{aligned}\mathbf{E}\|z^2(t) - z^1(t)\|^2_{\mathbb{R}^n} &= \mathbf{E}\|\mathcal{G}_k(t, z^2(t_k^-)) - \mathcal{G}_k(t, z^1(t_k^-))\|^2_{\mathbb{R}^n} \\ &\leq L\|z^2 - z^1\|^2_{\mathcal{CP}_\tau} \\ &< \frac{1}{2}\|z^2 - z^1\|^2_{\mathcal{CP}_\tau}.\end{aligned}$$

Hence, from the foregoing inequalities and the last expression, we get that

$$\|z^2 - z^1\|^2_{\mathcal{CP}_\tau} < \|z^2 - z^1\|^2_{\mathcal{CP}_\tau},$$

which implies that $z^1 = z^2$. □

3.2. Prolongation of Solutions. In this subsection we shall study the prolongation of the solutions of problem (1.1). To do so, we will consider the following subset $\tilde{D}$ of $\mathcal{CP}_\tau$:

$$\tilde{D} = \{y \in \mathcal{CP}_\tau : \|y\|^2_{\mathcal{CP}_\tau} \leq \rho\}. \tag{3.14}$$

Hence,it turns out that for all $z \in D$, we have that $z(t) - \tilde{\phi}(t) \in \tilde{D}$ for $-\infty < t \leq \tau$.

Definition 3.6. We will say that $(-\infty, \tau_1)$ is a maximal interval of existence of the solution $z(\cdot)$ of problem (1.1) if there is not solution of the (1.1) on $(-\infty, \tau_2)$ with $\tau_2 > \tau_1$.

Theorem 3.7. *Suppose that the conditions of Theorem* (3.5) *hold. If z is a solution of problem* (1.1) *on $(-\infty, \tau_1)$ and τ_1 is maximal, then either $\tau_1 = +\infty$ or there exists a sequence $\tau_n \to \tau_1$ as $n \to \infty$ such that $z(\tau_n) - \tilde{\phi}(\tau_n) \to \partial\tilde{D}$.*

Proof. Suppose, for the purpose of contradiction, that $\tau_1 < \infty$ and there exist a neighborhood N of $\partial\tilde{D}$ such that $z(t) - \tilde{\phi}(t)$ does not enter in it, for $0 < s_2 \leq t < \tau_1$. We can take $N = \tilde{D}\backslash B$, where B is a closed subset of $\tilde{D}$, then $z(t) - \tilde{\phi}(t) \in B$ for $0 < s_{p-1} < t < \tau_1$. We need to prove that $\lim_{t \to \tau_1^-} \{z(t) - \tilde{\phi}(t)\} = z_1 - \tilde{\phi}(\tau_1) \in B$.

For that purpose, it enough to prove that $\lim_{t \to \tau_1^-} z(t) = z_1$. We will divide the proof in two cases:
First, suppose that $0 < s_{p-1} < t_p \leq t < \tau_1$.

- Consider $t, \ell > 0$ such that

$$0 < t_p < \ell < t < \tau_1 \leq s_p.$$

In this case $t, l \in J_p$ and

$$\begin{aligned}\mathbf{E}\|z(t) - z(\ell)\|^2_{\mathbb{R}^n} &= \mathbf{E}\|\mathcal{G}_p(t, z(t_p^-)) - \mathcal{G}_p(\ell, z(t_p^-))\|^2_{\mathbb{R}^n} \\ &\leq L\left\{|t - \ell|^2 + \|z(t_p^-) - z(t_p^-))\|^2_{\mathbb{R}^n}\right\} \\ &= L|t - \ell|^2.\end{aligned}$$

Then

$$\mathbf{E}\|z(t) - z(\ell)\|^2_{\mathbb{R}^n} \leq L|t - \ell|^2 \to 0 \quad \text{as} \quad t, \ell \to 0.$$

Therefore, $\lim_{t \to \tau_1^-} z(t) = z_1$ exists in $\mathbb{R}^n$, and since B is closed, $z_1 - \tilde{\phi}(\tau_1)$ belongs to B.

• Suppose that $0 < s_{p-1} < \tau_1 \le t_p$. Indeed, if we consider $0 < s_{p-1} < \ell < t < \tau_1 \le t_p$, then by Cauchy's inequality, for all $t, \ell \in I_{p-1}$ we obtain the following estimate

$$
\begin{aligned}
&\mathbf{E}\|z(t)-z(\ell)\|^2_{\mathbb{R}^n} = \mathbf{E}\bigg\|\Big(\mathcal{S}(t,s_{p-1})-\mathcal{S}(\ell,s_{p-1})\Big)\mathcal{G}_{p-1}(s_{p-1},z(t^-_{p-1}))\\
&+\int_{s_{p-1}}^{\ell}\Big(\mathcal{S}(t,s)-\mathcal{S}(\ell,s)\Big)f(s,z_s)ds+\int_{\ell}^{t}\mathcal{S}(t,s)f(s,z_s)ds\\
&+\int_{s_{p-1}}^{\ell}\Big(\mathcal{S}(t,s)-\mathcal{S}(\ell,s)\Big)\sigma(s,z_s)dW(s)+\int_{\ell}^{t}\mathcal{S}(t,s)\sigma(s,z_s)dW(s)\bigg\|^2_{\mathbb{R}^n}\\
&\le 5\mathbf{E}\bigg\|\Big(\mathcal{S}(t,s_{p-1})-\mathcal{S}(\ell,s_{p-1})\Big)\mathcal{G}_{p-1}(s_{p-1},z(t^-_{p-1}))\bigg\|^2_{\mathbb{R}^n}\\
&+5\mathbf{E}\bigg\|\int_{s_{p-1}}^{\ell}\Big(\mathcal{S}(t,s)-\mathcal{S}(\ell,s)\Big)f(s,z_s)ds\bigg\|^2_{\mathbb{R}^n}+5\mathbf{E}\bigg\|\int_{\ell}^{t}\mathcal{S}(t,s)f(s,z_s)ds\bigg\|^2_{\mathbb{R}^n}\\
&+5\mathbf{E}\bigg\|\int_{s_{p-1}}^{\ell}\Big(\mathcal{S}(t,s)-\mathcal{S}(\ell,s)\Big)\sigma(s,z_s)dW(s)\bigg\|^2_{\mathbb{R}^n}+5\mathbf{E}\bigg\|\int_{\ell}^{t}\mathcal{S}(t,s)\sigma(s,z_s)dW(s)\bigg\|^2_{\mathbb{R}^n}\\
&\le 5\mathbf{E}\bigg\|\mathcal{S}(t,s_{p-1})-\mathcal{S}(\ell,s_{p-1})\bigg\|^2_{\mathbb{R}^n}\bigg\|\mathcal{G}_{p-1}(s_{p-1},z(t^-_{p-1}))\bigg\|^2_{\mathbb{R}^n}\\
&+5\int_{s_{p-1}}^{\ell}\|\mathcal{S}(t,s)-\mathcal{S}(\ell,s)\|^2_{\mathbb{R}^n}ds\int_{s_{p-1}}^{\ell}\mathbf{E}\|f(s,z_s)\|^2_{\mathbb{R}^n}ds\\
&+5\int_{\ell}^{t}\|\mathcal{S}(t,s)\|^2_{\mathbb{R}^n}ds\int_{\ell}^{t}\mathbf{E}\|f(s,z_s)\|^2_{\mathbb{R}^n}ds+5\int_{s_{p-1}}^{\ell}\|\mathcal{S}(t,s)-\mathcal{S}(\ell,s)\|^2_{\mathbb{R}^n}\mathbf{E}\|\sigma(s,z_s)\|^2_{\mathbb{R}^{n\times n}}ds\\
&+5\int_{\ell}^{t}\|\mathcal{S}(t,s)\|^2_{\mathbb{R}^n}\mathbf{E}\|\sigma(s,z_s)\|^2_{\mathbb{R}^{n\times n}}ds\\
&\le 5\bigg\|\mathcal{S}(t,s_{p-1})-\mathcal{S}(\ell,s_{p-1})\bigg\|^2_{\mathbb{R}^n}L\Big[\|z\|^2_{\mathcal{CP}_\tau}\Big]+\bigg[\int_{s_{p-1}}^{\ell}\|\mathcal{S}(t,s)-\mathcal{S}(\ell,s)\|^2_{\mathbb{R}^n}ds\\
&+\int_{\ell}^{t}\|\mathcal{S}(t,s)\|^2_{\mathbb{R}^n}ds\bigg]10(\tau_1+1)\tilde{\psi}(\|z\|^2_{\mathfrak{Q}})\\
&\le 5\bigg\|\mathcal{S}(t,s_{p-1})-\mathcal{S}(\ell,s_{p-1})\bigg\|^2_{\mathbb{R}^n}LR+\bigg[\int_{s_{p-1}}^{\ell}\|\mathcal{S}(t,s)-\mathcal{S}(\ell,s)\|^2_{\mathbb{R}^n}ds\\
&+\int_{\ell}^{t}\|\mathcal{S}(t,s)\|^2_{\mathbb{R}^n}ds\bigg]5(\tau_1+1)\tilde{\psi}(R)
\end{aligned}
$$

Since $\mathcal{S}(t,s)$ is uniformly continuous for $t \ge 0$, then $\mathbf{E}\|z(t)-z(\ell)\|^2_{\mathbb{R}^n}$ goes to zero as $\ell \to \tau_1$. Therefore, $\lim\limits_{t\to\tau_1} z(t) = z_1$ exists in $\mathbb{R}^n$, and since B is closed, $z_1 - \tilde{\phi}(\tau_1)$ belongs to B, which is a contradiction with the maximality of τ_1. In fact, we have that $z_1 \in B + \tilde{\phi}(\tau_1)$ which is contained in the interior of the ball $\tilde{D} + \tilde{\phi}(\tau_1)$. Hence, $z(\cdot)$ can be extended to $(-\infty, \tau_1]$. In this regard, for ϵ small enough, the following initial value problem admit only one solutions on $(-\infty, \tau_1 + \epsilon)$ of the initial value problem

$$\begin{cases} dv(t) = \{A(t)v(t) + f(t, v_t)\}dt + \sigma(t, v_t)dW(t), \quad t \in (\tau_1, \tau_1 + \epsilon) \\ v(s) + g(v)(s) = z(s), \; s \in (-\infty, \tau_1], \end{cases} \tag{3.15}$$

This is a contradiction with the maximality of τ_1. So, the proof is completed □

Corollary 3.8. *Under the conditions of Theorem* (3.5)*, and assuming the following condition*

$$\|f(t,\phi)\|^2_{\mathbb{R}^n} + \|\sigma(t,\phi)\|^2_{\mathbb{R}^{n\times n}} \leq \mu(t)(1 + \|\phi\|^2_{\mathfrak{Q}}), \quad \phi \in \mathfrak{Q}, \quad t \in \mathbb{R}, \tag{3.16}$$

where $\mu(\cdot)$ *is a continuous function on* $(-\infty,\infty)$*, the unique solution of problem* (1.1) *exists on* $(-\infty,\infty)$*.*

Proof. We will divide the proof in two cases:

- Suppose that $0 \leq s_{p-1} < t_p < \tau_1$. Then $0 \leq s_{p-1} < t_p < t < \tau_1 \leq s_p$. Then,

$$\mathbf{E}\|z(t)\|^2_{\mathbb{R}^n} = \|\mathcal{G}_p(t, z(t_p^-))\|^2_{\mathbb{R}^n} \leq L\mathbf{E}\|z(t_p^-)\|^2_{\mathbb{R}^n} < \infty.$$

Consequently

$$\mathbf{E}\|z(t)\|^2_{\mathbb{R}^n} \leq \infty, \quad t \in J_p.$$

- Suppose that $0 \leq s_{p-1} \leq t \leq \tau_1 \leq t_p$. Then, for $t \in I_{p-1}$ we have that

$$\begin{aligned}
\mathbf{E}\|z(t)\|^2_{\mathbb{R}^n} &= \mathbf{E}\left\|\mathcal{S}(t,s_{p-1})\mathcal{G}_{p-1}(s_{p-1}, z(t_{p-1}^-)) + \int_{s_{p-1}}^t \mathcal{S}(t,s)f(s,z_s)ds \right. \\
&\left. + \int_{s_{p-1}}^t \mathcal{S}(t,s)\sigma(s,z_s)dW(s)\right\|^2_{\mathbb{R}^n} \\
&\leq 3\|\mathcal{S}(t,s_{p-1})\mathcal{G}_{p-1}(s_{p-1}, z(t_{p-1}^-))\|^2_{\mathbb{R}^n} + 3\mathbf{E}\|\int_{s_{p-1}}^t \mathcal{S}(t,s)f(s,z_s)ds\|^2_{\mathbb{R}^n} \\
&+ 3\mathbf{E}\|\int_{s_{p-1}}^t \mathcal{S}(t,s)\sigma(s,z_s)dW(s)\|^2_{\mathbb{R}^n} \\
&\leq 3\|\mathcal{S}(t,s_{p-1})\|^2\|\mathcal{G}_{p-1}(s_{p-1}, z(t_{p-1}^-))\|^2_{\mathbb{R}^n} + 3\int_{s_{p-1}}^t \|\mathcal{S}(t,s)\|^2 ds \int_{s_{p-1}}^t \mathbf{E}\|f(s,z_s)\|^2_{\mathbb{R}^n} ds \\
&+ 3\mathbf{E}\int_{s_{p-1}}^t \|\mathcal{S}(t,s)\|^2\|\sigma(s,z_s)\|^2_{\mathbb{R}^{n\times n}} ds \\
&\leq 3M^2L\|z(t_{p-1}^-)\|^2_{\mathbb{R}^n} + 3M^2(t - s_{p-1})\int_{s_{p-1}}^t \mu(s)(1 + \mathbf{E}\|z(s)\|^2_{\mathbb{R}^n})ds \\
&+ 3M^2\int_{s_{p-1}}^t \mu(s)(1 + \mathbf{E}\|z(s)\|^2_{\mathbb{R}^n})ds
\end{aligned}$$

$$\leq \left(3M^2L\|z(t_{p-1}^-)\|_{\mathbb{R}^n}^2 + 3M^2(\tau_p - s_{p-1})\int_{s_{p-1}}^{\tau_p} \mu(s)ds\right)$$
$$+ \int_{s_{p-1}}^{t} 3M^2(\tau_p - s_{p-1})\mu(s)\mathbf{E}\|z(s)\|_{\mathbb{R}^n}^2 ds$$
$$+ 3M^2 \int_{s_{p-1}}^{\tau_p} \mu(s)ds + 3M^2 \int_{s_{p-1}}^{t} \mathbf{E}\|z(s)\|_{\mathbb{R}^n}^2 ds$$
$$= 3M^2\left[L\|z(t_{p-1}^-)\|_{\mathbb{R}^n}^2 + \left((\tau_p - s_{p-1}) + 1\right)\int_{s_{p-1}}^{\tau_p} \mu(s)ds\right]$$
$$+ \int_{s_{p-1}}^{t} 3M^2\left[\left(\tau_p - s_{p-1}\right)\mu(s) + 1\right]\mathbf{E}\|z(s)\|_{\mathbb{R}^n}^2 ds.$$

Then, applying Gronwall Inequality (see [11, 16, 17, 20]), we obtain that

$$\mathbf{E}\|z(t)\|_{\mathbb{R}^n}^2 \leq 3M^2\Bigg[L\|z(t_{p-1}^-)\|_{\mathbb{R}^n}^2$$
$$+ \left((\tau_p - s_{p-1}) + 1\right)\int_{s_{p-1}}^{\tau_p} \mu(s)ds\Bigg] \exp\left\{\int_{s_{p-1}}^{t} 3M^2\left[(\tau_p - s_{p-1})\mu(s) + 1\right]ds\right\}.$$

The two cases imply that $\|z(t)\|_{\mathbb{R}^n}$ remains bounded as $t \to \tau_1$, and applying Theorem 3.7 we get the result. $\square$

4. An Example

This section is devoted to present an example to illustrate our result. Consider the following stochastic non-instantaneous impulsive differential equations with delay and non local condition: For all $k = 1.2, 3, \ldots$ we have

$$\begin{cases} dz(t) = -z(t) + e^{-\dfrac{z_t(-1)}{10(t+5)^3}} + \dfrac{z_t(-1)}{10(t+5)^3}dW(t), \quad t \in I_k, \\ z(s) + \left(1 + \dfrac{\sin z}{30^2}\right)(s) = \phi(s), \quad s \in (-\infty, 0], \\ z(t) = \dfrac{\sin(z(t_k^-))}{4(t_k+8)^4} \cdot \cos(t - t_k), \quad t \in J_k. \end{cases} \tag{4.1}$$

In this case, we have that the terms involving system (1.1) are given by: $A(t) = -1$, $f(t,z) = \exp\{-\dfrac{z}{10(t+5)^3}\}$, $\sigma(t,z) = \dfrac{z}{10(t+5)^3}$, $g(z) = 1 + \dfrac{\sin(z)}{30^2}$ and $\mathcal{G}_k(t,z) = \dfrac{\sin(z)}{4(t_k+8^4)} \cdot \cos(t - t_k)$. Then we have,

$$\mathbf{E}\left|f(t,z)-f(t,x)\right|^2=\mathbf{E}\left|e^{-\dfrac{z}{10(t+5)^3}}-e^{-\dfrac{x}{10(t+5)^3}}\right|^2\le\frac{1}{10^2\cdot 5^6}\mathbf{E}\left|z-x\right|^2,$$

$$\mathbf{E}\left|\sigma(t,z)-\sigma(t,x)\right|^2=\mathbf{E}\left|\frac{z}{10(t+5)^3}-\frac{x}{10(t+5)^3}\right|^2\le\frac{1}{10^2\cdot 5^6}\mathbf{E}|z-x|^2, \quad (4.2)$$

$$\mathbf{E}\left|\mathcal{G}_k(t,z)-\mathcal{G}_k(t,x)\right|^2\le\frac{1}{4^2(t+8^4)^2}\mathbf{E}\left|\sin(z)-\sin(x)\right|^2\le\frac{1}{4^2\cdot 8^8}\mathbf{E}\left|z-x\right|^2,$$

$$\mathbf{E}\left|g(z)-g(x)\right|^2=\frac{1}{30^4}\mathbf{E}\left|\sin(z)-\sin(x)\right|^2\le\frac{1}{30^4}\mathbf{E}\left|z-x\right|^2,$$

In this case, we have that

$$q=1,\qquad \mathcal{S}(t,s)=e^{-(t-s)},\qquad M=1,\qquad \mathcal{K}=\frac{2}{10^2\cdot 5^6},$$

and

$$\begin{aligned}&\mathbf{E}\left|f(t,z)\right|^2+\mathbf{E}\left|\sigma(t,z)\right|^2\\&\le 2\mathbf{E}\left|f(t,z)-f(t,0)\right|^2+2\mathbf{E}\left|f(t,0)\right|^2+2\mathbf{E}|\sigma(t,z)-\sigma(t,0)|^2+2\mathbf{E}|\sigma(t,0)|^2\\&\le\frac{2}{10^2\cdot 5^6}\mathbf{E}\left|z\right|^2+2+\frac{2}{10^2\cdot 5^6}\mathbf{E}\left|z\right|^2=\frac{4}{10^2\cdot 5^6}\mathbf{E}\left|z\right|^2+2.\end{aligned} \quad (4.3)$$

Therefore, if we put $\tilde{\psi}(\xi)=\dfrac{4}{10^2\cdot 5^6}\xi+2$, with $\xi\ge 0$, then

$$\mathbf{E}\left|f(t,z)\right|^2+\mathbf{E}\left|\sigma(t,z)\right|^2\le\tilde{\psi}(\mathbf{E}|z|^2).$$

Now, for $\varepsilon>0$ small enough, let's take as the initial function

$$\phi(s)=\varepsilon cos(s),\qquad s\in\mathbb{R},$$

and define

$$\tilde{\phi}(t)=\begin{cases}e^{-t}\phi(0) & ,t\ge 0\\ \phi(t) & t\in(-\infty,0].\end{cases}$$

Then we have that

$$\begin{aligned}\tilde{\psi}\left(\left(\left\|\tilde{\phi}\right\|+\rho\right)^2\right)&=\frac{\left(\left\|\tilde{\phi}\right\|+\rho\right)^2}{10^2\cdot 5^6}+2\\&\le\frac{(\varepsilon+\rho)^2}{10^2\cdot 5^6}+2.\end{aligned}$$

Therefore, the last condition of the hypothesis H2-ii) is satisfied. In fact,

$$\begin{aligned}M^2L+d_q&=L+d_q\\&=\left(\frac{1}{4^2\cdot 8^8}+\frac{1}{30^4}\right)\\&=0,000001238<0,5.\end{aligned} \quad (4.4)$$

Since $\mathcal{G}_k(t,0)=0$ then, the condition of the hypothesis H3) is satisfied. In fact, the following inequality

$$
\begin{aligned}
&3M^2\Big\{ (d_q+L)\,(\|\tilde{\phi}\|+\rho)^2+\tau(\tau+1)\tilde{\psi}(\|\tilde{\phi}\|+\rho)^2\Big\}\\
&=3\Big\{ (d_q+L)\,(\|\tilde{\phi}\|+\rho)^2+\tau(\tau+1)\tilde{\psi}((\|\tilde{\phi}\|+\rho)^2)\Big\}\\
&\le 3\Big\{ \left(\frac{1}{4^2\cdot 8^8}+\frac{1}{30^4}\right)(\epsilon+\rho)^2+\tau(\tau+1)(\frac{(\varepsilon+\rho)^2}{10^2\cdot 5^6}+2)\Big\},
\end{aligned}
$$

holds for infinitely many values of τ, ρ and ε. In particular, we can take for example, $\tau=\frac{1}{16}$, $\rho=1$ and $\varepsilon=1$, and we get

$$
\begin{aligned}
&3\Big\{ \left(\frac{1}{4^2\cdot 8^8}+\frac{1}{30^4}\right)(\epsilon+\rho)^2+\tau(\tau+1)(\frac{(\varepsilon+\rho)^2}{10^2\cdot 5^6}+2)\Big\}\\
&=3\Big\{ \left(\frac{1}{4^2\cdot 8^8}+\frac{1}{30^4}\right)4+\frac{5}{16}\left(\frac{4}{10^2\cdot 5^6}+2\right)\Big\}\\
&=3\Big\{ \left(\frac{4}{1024}+\frac{4}{810000}\right)+\frac{17}{256}\left(\frac{4}{1562500}+2\right)\Big\}\\
&=\frac{9732288}{829440000}+\frac{204}{400000000}+\frac{102}{256}\\
&=0,012+0,00000051+0,398\\
&\le\tfrac{1}{2}
\end{aligned}
\tag{4.5}
$$

Thus, by (4.3),(4.2),(4.4),(4.5), we have that H1)-H3) holds. So, Theorem 3.4 ensures the existence of solutions for problem (4.1).

5. Conclusion and Final Remark

In this work, we study the existence and uniqueness of solutions for retarded stochastic semilinear equations with infinite delay, infinitely many non-instantaneous impulses, and nonlocal conditions. First, We set the problem in a natural Banach phase space satisfying Hale-Kato axiomatic Theory about the phase space for retarded ordinary differential equations with unbounded delay. Second, we assume that the nonlinear terms are locally Lipschitz, and to achieve the existence of solutions, Karakosta's Fixed Point Theorem is applied, which is an Extension of Krasnosel'skii's Fixed Point Theorem. After that, under some additional conditions, the uniqueness is proved as well. Next, assuming some bound on the nonlinear terms the global existence is proved by applying the Gronwall inequality. Finally, we present an example as an application of our method.

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