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THE INVERSE PROBLEM FOR DETERMINING THE SOURCE FUNCTION IN THE EQUATION WITH THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE

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Abstract. In this paper we consider an inverse problem for determining the source function in fractional equation with Riemann-Liouville derivative. Using the classical Fourier method, we prove the uniqueness and the existence theorem for this inverse problem.

1. Introduction

In recent years, due to the application of fractional equations in physics, biology and engineering, there is a significant interest in studying them. Fractional equations have been studied by numerous mathematicians. More data about that can be found in the works ([1] - [9]).

In this work the existence and inverse problems are studied for the equation of fractional order by time and the elliptical part with an abstract operator.

Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $||\cdot||$ and $A : H \to H$ be an arbitrary unbounded positive selfadjoint operator in H. Suppose that A has a complete in H system of orthonormal eigenfunctions $\{v_k\}$ and a countable set of nonnegative eigenvalues λ_k. It is convenient to assume that the eigenvalues do not decrease as their number increases, i.e. $0 < \lambda_1 \le \lambda_2 \cdots \to +\infty$.

Using the definitions of a strong integral and a strong derivative, fractional analogues of integrals and derivatives can be determined for vector-valued functions (or simply functions) $h : \mathbb{R}_+ \to H$, while the well-known formulae and properties are preserved (see, for example, [1]). Recall that the fractional integration of order $\sigma < 0$ of the function $h(t)$ defined on $[0,\infty)$ has the form

$$\partial_t^\sigma h(t) = \frac{1}{\Gamma(-\sigma)} \int\limits_0^t \frac{h(\xi)}{(t-\xi)^{\sigma+1}} d\xi, \quad t > 0, \tag{1.1}$$

provided the right-hand side exists. Here $\Gamma(\sigma)$ is Euler's gamma function. Using this definition one can define the Riemann - Liouville fractional derivative of order

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$\rho,\quad m-1<\rho<m$, as

$$\partial_t^\rho h(t) = \frac{d^m}{dt^m}\partial_t^{\rho-m}h(t).$$

Note that if $\rho = m$, then fractional derivatives coincides with the ordinary classical derivative of the m order.

Let $\rho \in (m-1, m)$ be a fixed number and let $C((a,b);H)$ stand for a set of continuous functions $u(t)$ of $t \in (a,b)$ with values in H.

Let τ be an arbitrary real number. We introduce the power of operator A, acting in H according to the rule

$$A^\tau h = \sum_{k=1}^{\infty} \lambda_k^\tau h_k v_k,$$

where h_k is the Fourier coefficients of a function $h \in H$: $h_k = (h, v_k)$. Obviously, the domain of this operator has the form

$$D(A^\tau) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau}|h_k|^2 < \infty\}.$$

For elements of $D(A^\tau)$ we introduce the norm

$$||h||_\tau^2 = \sum_{k=1}^{\infty} \lambda_k^{2\tau}|h_k|^2 = ||A^\tau h||^2,$$

and together with this norm $D(A^\tau)$ turns into a Hilbert space.

For ρ and an arbitrary complex number μ, by $E_{\rho,\mu}(z)$ we denote the Mittag-Leffler function with two parameters:

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \mu)}. \tag{1.2}$$

If the parameter $\mu = 1$, then we have the classical Mittag-Leffler function: $E_\rho(z) = E_{\rho,1}(z)$.

We also need some estimates for the Mittag-Leffler function. For sufficiently large t one has the asymptotic estimate (see, examples, [4], p. 13, [2], p. 75)

$$E_{\rho,\rho+1}(-t) = \frac{1}{t}\left(1 + O\left(\frac{1}{t}\right)\right), \quad t > 1, \tag{1.3}$$

and for any complex number μ one has

$$0 < |E_{\rho,\mu}(-t)| \le \frac{C}{1+t}, \quad t > 0. \tag{1.4}$$

Proposition 1.1. *Let $m-1<\rho<m$ and $\lambda>0$. Then for all positive t one has*

$$\partial_t^{\rho-j}\left(t^{\rho-j}E_{\rho,\rho-j+1}(-\lambda t^\rho)\right) = E_\rho(-\lambda t^\rho), \quad j = 1,2,...,m. \tag{1.5}$$

Proof. If $j = 1,2,...,m-1$ the equation (1.5) follows from the formula (4.10.14) in ([2]). if $j = m$ by definition of the fractional integration (1.1) we have

$$\partial_t^{\rho-m}\left(t^{\rho-m}E_{\rho,\rho-m+1}(-\lambda t^\rho)\right) = \frac{1}{\Gamma(m-\rho)}\int_0^t \frac{\xi^{\rho-m}E_{\rho,\rho-m+1}(-\lambda\xi^\rho)}{(t-\xi)^{\rho-m+1}}d\xi =$$

$$= \frac{1}{\Gamma(m-\rho)} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(\rho k+\rho-m+1)} \int_0^t \frac{\xi^{\rho-m+\rho k}}{(t-\xi)^{\rho-m+1}} d\xi =$$

$$= \frac{1}{\Gamma(m-\rho)} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(\rho k+\rho-m+1)} t^{\rho k} \int_0^1 s^{\rho-m+\rho k}(1-s)^{-\rho+m-1} ds.$$

On the other hand, using the properties of Euler's beta function $B(a, b)$, we obtain

$$\int_0^1 s^{\rho-m+\rho k}(1-s)^{-\rho+m-1} ds = B(\rho-m+\rho k+1, m-\rho) =$$

$$= \frac{\Gamma(\rho-m+\rho k+1)\Gamma(m-\rho)}{\Gamma(\rho k+1)}.$$

By virtue of the definition of the Mittag-Leffler function $E_\rho(z)$ this implies the statement of the proposition. □

Proposition 1.2. *The Mittag-Leffler function of negative argument $E_\rho(-x)$ is monotonically decreasing function for all $0 < \rho < 1$ and*

$$0 < E_\rho(-x) < 1. \tag{1.6}$$

Consider the following problem

$$\begin{cases} \partial_t^\rho u(t) + Au(t) = f, & t > 0; \\ \lim\limits_{t\to 0} \partial_t^{\rho-j} u(t) = \varphi_j, & j = 1, 2, ..., m \end{cases} \tag{1.7}$$

where functions $f(t) \in C((0, \infty); H)$ and $\varphi_j \in H$. These problems are also called *the forward problems.*

Definition 1.3. A function $u(t) \in C((0, \infty); H)$ with the properties $\partial_t^\rho u(t)$, $Au(t) \in C((0, \infty); H)$ and satisfying conditions (1.7) is called **the solution** of the problem (1.7).

In the present paper we prove the existence and uniqueness theorems for solutions of problems (1.7).

Theorem 1.4. *Let functions φ_j and $f \in H$. Then the problem (1.7) has a unique solution and this solution has the following form*

$$u(t) = \sum_{k=1}^{\infty} \left[\sum_{j=1}^{m} \varphi_{jk} t^{\rho-j} E_{\rho,\rho-j+1}(-\lambda_k t^\rho) + f_k\, t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho) \right] v_k. \tag{1.8}$$

where are f_k, φ_{jk} – the Fourier coefficients of the functions f and φ_j respectively.

Proof. **Existence**. In the section we will prove existence and uniqueness solution of problem (1.7). It is not hard to verify that the series (1.8) is a formal solution to problem (1.7) (see, for example, [2], p. 173). In order to prove that function (1.8) is actually a solution to the problem, it remains to substantiate this formal statement, i.e. to show that the operators A and ∂_t^ρ can be applied term-by-term to series (1.8).

Let $S_n(t)$ be the partial sum of series (1.8). First, we prove that series (1.8) are converges. Due to the Parseval equality we may write

$$||S_n(t)||^2 = \sum_{k=1}^{n} \left| \sum_{j=1}^{m} \varphi_{jk} t^{\rho-j} E_{\rho,\rho-j+1}(-\lambda_k t^\rho) + f_k\, t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho) \right|^2 .$$

Then, we have

$$||S_n(t)||^2 \leq \sum_{j=1}^{m} \sum_{k=1}^{n} |\varphi_{jk} t^{\rho-j} E_{\rho,\rho-j+1}(-\lambda_k t^\rho)|^2 +$$

$$+ \sum_{k=1}^{n} |f_k\, t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho)|^2 = \sum_{j=1}^{m} S_{nj}^1 + S_n^2.$$

where

$$S_{nj}^1 = \sum_{k=1}^{n} |\varphi_{jk} t^{\rho-j} E_{\rho,\rho-j+1}(-\lambda_k t^\rho)|^2,$$

$$S_n^2 = \sum_{k=1}^{n} |f_k\, t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho)|^2.$$

Using inequality (1.4) estimate each sum

$$S_{nj}^1 \leq \sum_{k=1}^{n} |\varphi_{jk}|^2 t^{2\rho-2j} \left| \frac{1}{1+\lambda_k t^\rho} \right|^2 \leq \frac{1}{\lambda_1^2 t^{2j}} \sum_{k=1}^{n} |\varphi_{jk}|^2.$$

and

$$S_n^2 \leq \sum_{k=1}^{n} |f_k|^2 t^{2\rho} \left| \frac{1}{1+\lambda_k t^\rho} \right|^2 \leq \frac{1}{\lambda_1^2} \sum_{k=1}^{n} |f_k|^2$$

If $\varphi_j, f \in H$ then sum (1.8) is converges and $u(t) \in C((0,\infty); H)$.

Now let's estimate $Au(t)$

$$AS_n(t) = \sum_{k=1}^{n} \left[\sum_{j=1}^{m} \varphi_{jk} t^{\rho-j} E_{\rho,\rho-j+1}(-\lambda_k t^\rho) + f_k\, t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho) \right] \lambda_k v_k. \quad (1.9)$$

Due to the Parseval equality we may write

$$||AS_n(t)||^2 = \sum_{k=1}^{n} \lambda_k^2 \left| \sum_{j=1}^{m} \varphi_{jk} t^{\rho-j} E_{\rho,\rho-j+1}(-\lambda_k t^\rho) + f_k\, t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho) \right|^2 .$$

Then, we have

$$||AS_n(t)||^2 \leq \sum_{j=1}^{m} \sum_{k=1}^{n} \lambda_k^2 |\varphi_{jk} t^{\rho-j} E_{\rho,\rho-j+1}(-\lambda_k t^\rho)|^2 +$$

$$+ \sum_{k=1}^{n} \lambda_k^2 |f_k\, t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho)|^2 = \sum_{j=1}^{m} AS_{nj}^1 + AS_n^2.$$

Using inequality (1.4) estimate each sum

$$AS_{nj}^1 = \sum_{k=1}^{n} \lambda_k^2 |\varphi_{jk} t^{\rho-j} E_{\rho,\rho-j+1}(-\lambda_k t^\rho)|^2 \leq$$

$$\leq \sum_{k=1}^{n} \lambda_k^2 |\varphi_{jk}|^2 t^{2\rho-2j} \left| \frac{1}{1+\lambda_k t^\rho} \right|^2 \leq \frac{1}{t^{2j}} \sum_{k=1}^{n} |\varphi_{jk}|^2.$$

and

$$AS_n^2 = \sum_{k=1}^{n} \lambda_k^2 |f_k\, t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho)|^2 \leq \sum_{k=1}^{n} \lambda_k^2 |f_k|^2 t^{2\rho} \left| \frac{1}{1+\lambda_k t^\rho} \right|^2 \leq \sum_{k=1}^{n} |f_k|^2.$$

Hence, if $\varphi_j, f \in H$ we obtain $Au(t) \in C((0,\infty); H)$.

Further, from equation (1.7) one has $\partial_t^\rho S_n(t) = -AS_n(t) + \sum\limits_{k=1}^{n} f_k(t) v_k$, $t > 0$. Therefore, from the above reasoning, we have $\partial_t^\rho u(t) \in C((0,\infty); H)$.

Now, let's estimate $\partial_t^{\rho-j} u(t)$, $j = 1, 2, ..., m$ we use (1.5) to create the following equation

$$\partial_t^{\rho-j} S_n(t) = \sum_{k=1}^{n} \left[\sum_{j=1}^{m} \varphi_{jk} E_\rho(-\lambda_k t^\rho) + f_k\, t^j E_{\rho,j+1}(-\lambda_k t^\rho) \right] v_k,\ j = 1, 2, ..., m. \tag{1.10}$$

Due to the Parseval equality we may write

$$||\partial_t^{\rho-j} S_n(t)||^2 = \sum_{k=1}^{n} \left| \sum_{j=1}^{m} \varphi_{jk} E_\rho(-\lambda_k t^\rho) + f_k\, t^j E_{\rho,j+1}(-\lambda_k t^\rho) \right|^2 \leq$$

$$\leq \sum_{j=1}^{m} \sum_{k=1}^{n} |\varphi_{jk} E_\rho(-\lambda_k t^\rho)|^2 + \sum_{k=1}^{n} |f_k\, t^j E_{\rho,j+1}(-\lambda_k t^\rho)|^2 = \sum_{j=1}^{m} I_{1j} + I_2.$$

Using inequality (1.6) estimate each sum

$$I_{1j} \leq \sum_{k=1}^{n} |\varphi_{jk}|^2$$

and

$$I_2 \leq \sum_{k=1}^{n} |f_k|^2\, t^{2j}.$$

Therefore, if $\varphi_j, f \in H$, then (1.10) are converges. Thus, we have completed the rationale that (1.8) is a solution to the problem (1.7).

Uniqueness. The uniqueness of the solution can be proved by the standard technique based on completeness of the set of eigenfunctions $\{v_k\}$ in H (see, example [5]).

Let us prove that, if $u(t)$ is a solution to the homogeneous problem:

$$\partial_t^\rho u(t) + Au(t) = 0, \quad t > 0; \tag{1.11}$$

$$\lim_{t\to 0} \partial_t^{\rho-j} u(t) = 0 \quad j = 1, 2, ..., m, \tag{1.12}$$

then $u(t) \equiv 0$.

Let $u(t)$ be a solution to this problem and $u_k(t) = (u(t), v_k)$. Then, by virtue of equation (1.13) and the selfadjointness of operator A,

$$\partial_t^\rho u_k(t) = (\partial_t^\rho u(t), v_k) = -(Au(t), v_k) = -(u(t), Av_k) = \tag{1.13}$$

$$-(u(t), \lambda_k v_k) = -\lambda_k(u(t), v_k) = -\lambda_k u_k(t), \quad t > 0.$$

Thus, we have the following problem

$$\partial_t^\rho u_k(t) + \lambda_k u_k(t) = 0, \quad t > 0; \quad \lim_{t\to 0} \partial_t^{\rho-j} u(t) = 0, \quad j = 1, 2, ..., m.$$

Therefore, it follows that $u_k(t) \equiv 0$ for all k (see, examples [2], p.173, [4], p. 16 and [28]). Consequently, due to the completeness of the system of eigenfunctions $\{v_k\}$, we have $u(t) \equiv 0$, as required. □

2. Inverse problem of determining the heat source density

The inverse problems of determining the right-hand side (the heat source density) of various subdiffusion equations have been considered by a number of authors (see, e.g. [10] - [21] and the bibliography therein). You can completely be informed about "Inverse problems " in the work [7]. The recent article [22] - [23] is devoted to the inverse problem for the subdiffusion equation with Riemann-Liouville derivatives.

In [25] the authors of this paper considered an inverse problem for the simultaneous determination of the order of the Riemann-Liouville fractional derivative and the source function in the subdiffusion equations. Using the classical Fourier method, the authors proved the uniqueness and existence theorem for this inverse problem.

In [26] - [27], the authors investigated the inverse problem of determining the order of the fractional derivative in the subdiffusion equation and in the wave equation, respectively.

Let us consider *the inverse problem*

$$\begin{cases} \partial_t^\rho u(t) + Au(t) = f, \quad t > 0; \\ \lim\limits_{t\to 0} \partial_t^{\rho-j} u(t) = \varphi_j, \quad j = 1, 2, ..., k \end{cases} \tag{2.1}$$

with the additional condition

$$u(\tau) = \Psi, \quad 0 < \tau < T, \tag{2.2}$$

in which the unknown element $f \in H$, characterizing the action of heat sources, does not depend on t and $\Psi, \varphi \in H$ are given elements, $T > 0$ is constant.

Definition 2.1. A pair $\{u(t), f\}$ of function $u(t) \in C((0,\infty); H)$ and $f \in H$ with the properties $\partial_t^\rho u(t), Au(t) \in C((0,\infty); H)$ and satisfying conditions (2.1), (2.2) is called **the solution** of the inverse problem (2.1), (2.2).

In this section we will prove next theorem.

Theorem 2.2. *Let $\varphi, \Psi \in D(A)$. Then the inverse problem (2.1), (2.2) has a unique solution $\{u(t), f\}$ and this solution has the following form*

$$u(t) = \sum_{k=1}^{\infty} \left[\sum_{j=1}^{m} \varphi_{jk} t^{\rho-j} E_{\rho,\rho-j+1}(-\lambda_k t^\rho) + f_k\, t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho) \right] v_k. \tag{2.3}$$

where are the numbers

$$f_k = \frac{\Psi_k}{\tau^\rho E_{\rho,\rho+1}(-\lambda_k \tau^\rho)} - \sum_{j=1}^{m} \frac{\varphi_{jk} E_{\rho,\rho-j+1}(-\lambda_k \tau^\rho)}{\tau^j E_{\rho,\rho+1}(-\lambda_k \tau^\rho)}, \tag{2.4}$$

and

$$f(x) = \sum_{k=1}^{\infty} \frac{\Psi_k}{\tau^\rho E_{\rho,\rho+1}(-\lambda_k \tau^\rho)} v_k - \sum_{k=1}^{\infty} \sum_{j=1}^{m} \frac{\varphi_{jk} E_{\rho,\rho-j+1}(-\lambda_k \tau^\rho)}{\tau^j E_{\rho,\rho+1}(-\lambda_k \tau^\rho)} v_k. \tag{2.5}$$

Proof. **Existence.** We indicated above, that if f is known and since f does not depend on t, then the unique solution of the problem (2.1) has the form (2.3).

By virtue of an additional condition (2.2) and completeness of the system $\{v_k\}$ we obtain:

$$\sum_{j=1}^{m} \varphi_{jk} \tau^{\rho-j} E_{\rho,\rho-j+1}(-\lambda_k \tau^\rho) + f_k\, \tau^\rho E_{\rho,\rho+1}(-\lambda_k \tau^\rho) = \Psi_k.$$

After simple calculations, we get

$$f_k = \frac{\Psi_k}{\tau^\rho E_{\rho,\rho+1}(-\lambda_k \tau^\rho)} - \sum_{j=1}^{m} \frac{\varphi_{jk} E_{\rho,\rho-j+1}(-\lambda_k \tau^\rho)}{\tau^j E_{\rho,\rho+1}(-\lambda_k \tau^\rho)} \equiv f_{k,1} + \sum_{j=1}^{m} f_{jk,2}. \tag{2.6}$$

With these Fourier coefficients we have the above formal series (2.5) for the unknown function f: $f = \sum\limits_{k=1}^{\infty} \left(f_{k,1} + \sum\limits_{j=1}^{m} f_{jk,2} \right) v_k$.

Let us reveal the convergence of series (2.5). If F_j the partial sums of series (2.5), then by virtue of the Parseval equality we may write

$$||F_n||^2 = \sum_{k=1}^{n} \left[f_{k,1} + \sum_{j=1}^{m} f_{jk,2} \right]^2 \le C \sum_{k=1}^{n} f_{k,1}^2 + C \sum_{j=1}^{m} \sum_{k=1}^{n} f_{jk,2}^2 \equiv C I_{1,n} + C \sum_{j=1}^{m} I_{2j,n}. \tag{2.7}$$

where $C > 0$. Then for $I_{1,n}$ we have following estimation

$$I_{1,n} \le \sum_{k=1}^{n} \frac{|\Psi_k|^2}{|\tau^\rho E_{\rho,\rho+1}(-\lambda_k \tau^\rho)|^2}.$$

Using the asymptotic estimate (see, sample, [9], p. 134):

$$E_{\rho,\rho+1}(-t) = t^{-1} + O(t^{-2}), \tag{2.8}$$

we get

$$I_{1,n} \le \sum_{k=1}^{n} \frac{\lambda_k^2 |\Psi_k|^2}{\big(1 + O\big((-\lambda_k \tau^\rho)^{-1}\big)\big)^2} \le C \sum_{k=1}^{n} \lambda_k^2 |\Psi_k|^2 \le C ||\Psi||_1^2.$$

Therefore, using $|E_{\rho,\rho-j+1}(-\lambda_k\tau^\rho)|\le 1$ we have

$$I_{2j,n}\le\sum_{k=1}^{n}\left|\frac{\varphi_{jk}E_{\rho,\rho-j+1}(-\lambda_k\tau^\rho)}{\tau^jE_{\rho,\rho+1}(-\lambda_k\tau^\rho)}\right|^2\le\sum_{k=1}^{n}\frac{|\varphi_{jk}|^2}{\tau^{2j}|E_{\rho,\rho+1}(-\lambda_k\tau^\rho)|^2}.$$

By virtue of (2.8),

$$I_{2j,n}\le\sum_{k=1}^{n}\frac{\lambda_k^2|\varphi_k|^2}{\tau^{2j-2\rho}\left(1+O\big((-\lambda_k\tau^\rho)^{-1}\big)\right)^2}\le C\sum_{k=1}^{n}\lambda_k^2|\varphi_k|^2\le C||\varphi||_1^2.$$

Thus, if $\varphi,\Psi\in D(A)$, then from estimates of $I_{1,n}$, $I_{2j,n}$ and (2.7) we obtain $f\in H$.

After finding the unknown function $f\in H$, the fulfillment of the conditions of Definition 2.1 for function $u(t)$, defined by the series (2.3) is proved in exactly the same way as with Theorem 1.4.

Uniqueness. Suppose we have two solutions: $\{u_1(t),f_1\}$ and $\{u_2(t),f_2\}$. It is required to prove $u(t)\equiv u_1(t)-u_2(t)\equiv 0$ and $f\equiv f_1-f_2=0$. Since the problem is linear, to determine $u(t)$ and f we have the problem:

$$\partial_t^\rho u(t)+Au(t)=f,\quad t>0; \tag{2.9}$$

$$\lim_{t\to 0}\partial_t^{\rho-j}u(t)=0,\quad j=1,2,...,m, \tag{2.10}$$

$$u(\tau)=0. \tag{2.11}$$

Let $u(t)$ be a solution to this problem and $u_k(t)=(u(t),v_k)$. Then, by virtue of equation (2.9) and the selfadjointness of operator A,

$$\partial_t^\rho u_k(t)=(\partial_t^\rho u(t),v_k)=-(Au(t),v_k)+(f,v_k)=-(u(t),Av_k)+(f,v_k)= \tag{2.12}$$

$$-(u(t),\lambda_kv_k)+f_k=-\lambda_k(u(t),v_k)+f_k=-\lambda_ku_k(t)+f_k,\quad t>0.$$

Thus, taking into account (2.10), we have the following problem

$$\partial_t^\rho u_k(t)+\lambda_ku_k(t)+f_k=0,\quad t>0;\quad \lim_{t\to 0}\partial_t^{\rho-j}u(t)=0.$$

Then the solution to this problem has the form (see,example, [2], p.174, [3], [4], p. 17)

$$u_k(t)=f_k\int_0^t\eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^\rho)d\eta=f_k\cdot t^\rho\,E_{\rho,\rho+1}(-\lambda_kt^\rho).$$

Using (2.11), we have

$$u_k(\tau)=f_k\cdot\tau^\rho\,E_{\rho,\rho+1}(-\lambda_k\tau^\rho)=0.$$

Hence, due to the properties of the Mittag-Leffler function $E_{\rho,\rho+1}(-\lambda\tau^\rho)\ne 0$. It follows from here $f_k=0$, for all $k\ge 1$. In consequence, from the completeness of the system of eigenfunctions $\{v_k\}$, we finally obtain $f=0$ and $u(t)\equiv 0$, as required.

□

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