Global and Stochastic Analysis
Vol. 7 No. 1 (January-June, 2020)
MUK PUBLICATIONS
Open Access Publisher

EXAMPLES OF FINANCIAL MARKET MODELS OBTAINED BY EULER DISCRETIZATION OF CONTINUOUS MODELS

M. L. ESQUÍVEL* AND N. P. KRASII**

Abstract. We present a methodology to study discrete time financial models with one risky asset and a risk free asset that may thought to result as a discretization of a suitable continuous time model. In a numerical example we compare the pricing results, obtained with these models, with results obtained from the related continuous time models. Our approach relies on some known important results describing a particular class of discrete time models – the conditionally Gaussian models – a class that, regardless of its particular definition, contains many interesting instances. We aim at a better understanding of the implications of the discretization procedures which are inevitable, both at the parameter estimation and derivative price computation moments, by reason of the observational and computational limitations. We also present a preliminary study of a a model of stochastic differential equations for commodity spot and futures prices that may be studied with the proposed methodology. For that purpose we summarize a naive theory of Ito integration in Hilbert space.

The first author dedicates this work to Hans Föllmer to whom he owes warm encouragement in the beginning of his career and the most important initiatives that launched Financial Mathematics studies in Portugal.

1. Introduction

Given a financial market model with dynamics defined by stochastic differential equations (SDE) we have that, often, the estimation procedures rely in the observation of the processes at discrete times and so, it would be natural to consider discrete time models associated to the initially given continuous time models and with the estimating procedures and the estimates obtained. Important references for the study of discrete time models are [7] and [20]. The issue of completeness of discrete time models is a most decisive one and entails a limitation that must be taken into account, namely, the fact that the probability space of an arbitrage free and complete model of d risky assets with a finite set of dates $\{0, 1, 2, \ldots, T\}$ has a number of atoms bounded by $(d + 1)^T$ (see [7, p. 231] or, for a deeper analysis [10]). Of course when studying discrete models obtained by discretization of continuous models, which is the purpose of the present work, we may suppose – aiming at some convergence results – that we consider discretization intervals thiner and thiner and this corresponds to a set of dates with a larger and larger number of elements; we observe that, no matter what price unit is used, the set of possible price values of tradable assets is finite. The convergence of

2000 *Mathematics Subject Classification.* Primary 60H15 (Stochastic ordinary differential equations); 60H35 (Computational methods for stochastic equations); Secondary 97M30 (Financial and insurance mathematics).

Key words and phrases. Euler-Maruyama discretization, Girsanov change of probability in discrete time, commodity prices, coupled stochastic differential equations system, naive stochastic integration in Hilbert space.

* This work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UIDB/00297/2020 (Centro de Matemática e Aplicações, FCT Nova, UNL)..

** This work was done under partial financial support of RFBR (Grant n. 19-01-00451).

discrete models to continuous ones is a delicate subject (see, for instance, [15] and the important reference [18]).

As a first approach, we propose to consider the Euler-Maruyama discretization scheme (see [9, p. 62]) that allows us to consider models for which it is possible to retrieve a computable martingale measure. We illustrate the approach in the case of a SDE model with time varying deterministic coefficients, dealt in Section 2.2 below. In order to fully study models given by Euler discretization of a continuous time model we should establish the appropriate convergence results detailing the kind of approximation that we may get from the estimating procedures to be utilized to fit the discretized model to real data. In this preliminary work we restrict our presentation mostly to some computational aspects of an example.

Also, we only outline the approach for the case of the coefficients being allowed to be random, in Section 6.

2. The non random coefficients case

The case of discretization of stochastic differential equations with non random drift and volatility coefficients is treated in this section.

2.1. Black-Scholes model with varying coefficients. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(B_t)_{t\geq 0}$ a standard Brownian motion and $\mathbb{F} = (\mathcal{F}_n)_{n\geq 0}$ the Brownian filtration. For the financial market model we may consider a SDE of the type:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dB_t \ , \ t \in [0,T] \ , \ S_0 = s_0 \in \mathbb{R}_+^* \ ,$$

as a model for stock prices and admitting a strong solution. We suppose that the risk free rate process $(\rho_t)_{t\in[0,T]}$ of the market satisfies $d\rho_t = \rho_t dt$, $t \in [0,T]$, $\rho_0 = 1$ and that $(\mu_t)_{t\in[0,T]}$, $(\sigma_t)_{t\in[0,T]}$ and $(\rho_t)_{t\in[0,T]}$ are predictable with respect to $\mathbb{F}$. Then (see [5, pp. 160]), the discounted price process $(\widetilde{S}_t)_{t\in[0,T]}$ given by,

$$d\widetilde{S}_t = (\mu_t - \rho_t)\,\widetilde{S}_t dt + \sigma_t \widetilde{S}_t dB_t \ , \ t \in [0,T] \ , \ S_0 = s_0 \in \mathbb{R}_+^* \ , \tag{2.1}$$

has a solution given by:

$$\widetilde{S}_t = \exp\left(\int_0^t \left(\mu_s - \rho_s - \frac{\sigma_s^2}{2}\right) ds + \int_0^t \sigma_s dB_s\right) \ .$$

And so, we have that the process $(X_t)_{t\in[0,T]}$ defined by:

$$X_t = \log\left(\widetilde{S}_t\right) \ ,$$

verifies the SDE given by,

$$dX_t = \left(\mu_t - \rho_t - \frac{\sigma_t^2}{2}\right) dt + \sigma_t dB_t \ . \tag{2.2}$$

We now consider the Euler-Maruyama discretization of a stochastic differential equation having in mind applying it to an equation such as (2.2). Let $T < +\infty$ be the temporal time horizon,and for a given integer $N \geq 1$ let:

$$\Delta t := \Delta^N t = \frac{T}{N}.$$

Suppose that $(X_t)_{t\in[0,T]}$ is a continuous time stochastic process adapted to Brownian filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Consider the discretized stochastic process given by:

$$X_{k\Delta t} \ , \ k = 0, 1, \dots, N \ ,$$

that has a natural interpretation as the process of observations of the continuous time process made at the epochs $0, \Delta t, 2\Delta t, \dots, T$. Let us consider the discrete time stochastic process defined by:

$$\forall k \geq 0 \ , \ X_{(k+1)\Delta t} = X_{k\Delta t} + \mu\left((k+1)\Delta t\right)\Delta t + \sigma\left((k+1)\Delta t\right)\Delta t \cdot Z_{k+1} \ , \tag{2.3}$$

where we have that $(Z_k)_{k\geq 1}$ is a sequence of standardized normal random variables. In Section 2.3, using the results of Section 2.2 we will illustrate amethodology to the practical use of this Euler-Maruyama discretization in a simple market model.

2.2. The Girsanov change of probability. In the simpler case – for formula (2.3) – in which $(\mu_t)_{t\in[0,T]}$, $(\sigma_t)_{t\in[0,T]}$ are non random, we have that $\mathbb{F} = (\mathcal{F}_n)_{n\geq 0}$ may be the natural filtration generated by $(Z_k)_{k\geq 0}$ or, equivalently by $(X_{k\Delta})_{k\geq 0}$ – as the coefficients μ e σ are non random – and we have the following known result.

Proposition 2.1. *The sequence of random variables defined by,*

$$Y_k = \exp\left(-\sum_{n=0}^{k}\frac{\mu\left(n\Delta t\right)}{\sigma\left(n\Delta t\right)}Z_n - \sum_{n=0}^{k}\frac{\mu\left(n\Delta t\right)^2}{2\sigma\left(n\Delta t\right)^2}\right) \ ,$$

is an $\mathbb{F}$-martingale and the process $(X_{k\Delta t}Y_k)_{n\geq 0}$ is a $\mathbb{F}$-martingale.

Proof. For completeness we present a proof. Let us now show that

$$\mathbb{E}\left[X_{(k+1)\Delta t}Y_{k+1} \mid \mathcal{F}_k\right] = X_{k\Delta t}Y_k \ , \tag{2.4}$$

Consider for simplicity, that if we have,

$$D_k := \mu\left((k+1)\Delta t\right)\Delta t + \sigma\left((k+1)\Delta t\right)\Delta t \cdot Z_{k+1} \ ,$$

we then have $X_{(k+1)\Delta t} = X_{k\Delta t} + D_k$ and, if we have,

$$W_k := \exp\left(-\frac{\mu\left((k+1)\Delta t\right)}{\sigma\left((k+1)\Delta t\right)}Z_{k+1} - \frac{\mu\left((k+1)\Delta t\right)^2}{2\sigma\left((k+1)\Delta t\right)^2}\right) \ , \tag{2.5}$$

we also have that $Y_{k+1} = Y_k \cdot W_k$. As so, by the usual properties of conditional expectations, we have that:

$$\begin{aligned}
\mathbb{E}\left[X_{(k+1)\Delta t}Y_{k+1} \mid \mathcal{F}_k\right] &= \mathbb{E}\left[(X_{k\Delta t} + D_k)\cdot Y_k \cdot W_k \mid \mathcal{F}_k\right] = \\
&= \mathbb{E}\left[X_{k\Delta t}\cdot Y_k \cdot W_k \mid \mathcal{F}_k\right] + \mathbb{E}\left[D_k \cdot Y_k \cdot W_k \mid \mathcal{F}_k\right] = \\
&= X_{k\Delta t}\cdot Y_k \cdot \mathbb{E}\left[W_k \mid \mathcal{F}_k\right] + Y_k \cdot \mathbb{E}\left[D_k \cdot W_k \mid \mathcal{F}_k\right] = \\
&= X_{k\Delta t}\cdot Y_k \cdot \mathbb{E}\left[W_k\right] + Y_k \cdot \mathbb{E}\left[D_k \cdot W_k\right] \ ,
\end{aligned}$$

given that Z_{k+1} – which is the random variable that enters the composition of D_k and W_k – is independent of $\mathcal{F}_k$. Let us now observe that if we have $X \frown \mathcal{N}(0,1)$, then we must have that

$$\mathbb{E}\left[e^{uX}\right] = e^{\frac{u^2}{2}} \ , \tag{2.6}$$

and by derivation or direct computation,

$$\mathbb{E}\left[X \cdot e^{uX}\right] = u \cdot e^{\frac{u^2}{2}} \ . \tag{2.7}$$

As so, we have that:

$$X_{k\Delta t}\cdot Y_k \cdot \mathbb{E}\left[W_k\right] = X_{k\Delta t}\cdot Y_k \ ,$$

as by formula (2.6) we have that: $\mathbb{E}\left[W_k\right] = 1$. It now holds that:

$$\mathbb{E}\left[D_k \cdot W_k\right] = \mu\left((k+1)\Delta t\right)\Delta t + \sigma\left((k+1)\Delta t\right)\Delta t \cdot \mathbb{E}\left[Z_{k+1}\cdot W_k\right] = 0 \ ,$$

coming from the conjunction of formula (2.7), and the following:

$$\mathbb{E}\left[Z_{k+1}\cdot W_k\right]=\mathbb{E}\left[Z_{k+1}\cdot e^{-\frac{\mu((k+1)\Delta t)}{\sigma((k+1)\Delta t)}Z_{k+1}}\right]\cdot e^{-\frac{\mu((k+1)\Delta t)^2}{2\sigma((k+1)\Delta t)^2}}=$$
$$=-\frac{\mu\left((k+1)\Delta t\right)\Delta t}{\sigma\left((k+1)\Delta t\right)\Delta t}\ ,$$

thus ending the proof of the propostion. □

As a consequence, by considering the probability space $(\Omega,\mathcal{F},\mathbb{Q})$ with $\mathbb{Q}$ defined by:

$$\forall F\in\mathcal{F}\ \mathbb{Q}\left[F\right]=\int_F Y_N d\mathbb{P}\ ,$$

also represented by $d\mathbb{Q}=Y_N d\mathbb{P}$ – with Y_N the Radon-Nicodym density of $\mathbb{Q}$ relatively to a $\mathbb{P}$ – we have that the discretized process is a $\mathbb{Q}$ martingale with repect to $\mathbb{F}=(\mathcal{F}_n)_{n\geq 0}$ that is, the following known proposition proved for completness.

Proposition 2.2. *In the space $(\Omega,\mathcal{F},\mathbb{Q})$ the process $(X_{k\Delta t})_{k=0,1,\dots,N}$ is an $\mathbb{F}$ martingale.*

Proof. We are now going to show that:

$$\mathbb{E}^{\mathbb{Q}}\left[X_{(k+1)\Delta t}\mid\mathcal{F}_k\right]=X_{k\Delta t}\ .$$

Let U_{k+1} be a version of $\mathbb{E}^{\mathbb{Q}}\left[X_{(k+1)\Delta t}\mid\mathcal{F}_k\right]$. We have to show that:

$$\forall F\in\mathcal{F}_k\ ,\ \int_F U_{k+1}d\mathbb{Q}=\int_F X_k d\mathbb{Q}\ ,$$

which is equivalent to show that:

$$\forall F\in\mathcal{F}_k\ ,\ \int_F U_{k+1}Y_N d\mathbb{P}=\int_F X_k Y_N d\mathbb{P}\ ,$$

or equivalently, with the notations introduced in formula (2.5),

$$\forall F\in\mathcal{F}_k\ ,\ \int_F U_{k+1}Y_{k+1}\left(\prod_{n=k+1}^{N-1}W_n\right)d\mathbb{P}=\int_F X_kY_k\left(\prod_{n=k}^{N-1}W_n\right)d\mathbb{P}\ . \tag{2.8}$$

Let us now observe that $\prod_{n=k+1}^{N-1}W_n$ is independent of $\mathcal{F}_{k+1}$ and that U_{k+1} is measurable with respect to $\mathcal{F}_k$ and so, also, with respect to $\mathcal{F}_{k+1}$ and Y_{k+1} is measurable with respect to $\mathcal{F}_{k+1}$; being so we then have:

$$\int_F U_{k+1}Y_{k+1}\left(\prod_{n=k+1}^{N-1}W_n\right)d\mathbb{P}=\int_\Omega U_{k+1}Y_{k+1}\mathbb{1}_F\left(\prod_{n=k+1}^{N-1}W_n\right)d\mathbb{P}=$$
$$=\mathbb{E}^{\mathbb{P}}\left[U_{k+1}Y_{k+1}\mathbb{1}_F\left(\prod_{n=k+1}^{N-1}W_n\right)\right]=$$
$$=\mathbb{E}^{\mathbb{P}}\left[U_{k+1}Y_{k+1}\mathbb{1}_F\right]\mathbb{E}^{\mathbb{P}}\left[\prod_{n=k+1}^{N-1}W_n\right]=$$
$$=\mathbb{E}^{\mathbb{P}}\left[U_{k+1}Y_{k+1}\mathbb{1}_F\right]=\int_F U_{k+1}Y_{k+1}d\mathbb{P}\ ,$$

as by the independence, and again by formula (2.6),

$$\mathbb{E}^{\mathbb{P}}\left[\prod_{n=k+1}^{N-1} W_n\right] = 1 \ .$$

With a similar reasoning we have that:

$$\int_F X_k Y_k \left(\prod_{n=k}^{N-1} W_n\right) d\mathbb{P} = \int_F X_k Y_k d\mathbb{P}$$

that is, the equality (2.8), that we want to hold true, is equivalent to the equality:

$$\forall F \in \mathcal{F}_k \ , \ \int_F U_{k+1} Y_{k+1} d\mathbb{P} = \int_F X_k Y_k d\mathbb{P} \ ,$$

but this formula is precisely the formula (2.4) and so the theorem is proved. □

Remark 2.3. We observe that we may now compute arbitrage free prices of derivatives in a financial a market model with discounted prices prices given by formula (2.2) in the case where the coefficients are deterministic. For that it is required to estimate the coefficients (see for instance [19]). We present a first numerical study in Section 2.3 to illustrate this computational possibility.

The result in proposition 2.2 may be seen as a particular case of the two important results that we now also recall, for further reference, on the change of probability measure and its influence in the characterization of martingales.

Theorem 2.4 (Conditional expectation and change of probability)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mathbb{P}$ and $\mathbb{Q}$ be two probabilities over this space; let $\lambda : \Omega \mapsto \mathbb{R}_+$ be a measurable function over this space such that:*

$$\forall F \in \mathcal{F} \ , \ \mathbb{Q}\left[F\right] = \int_F \lambda d\mathbb{P} \ .$$

Consider $\mathcal{G} \subset \mathcal{F}$ a sub σ-algebra of $\mathcal{F}$and also X a random variable integrable with respect to $\mathbb{Q}$.then we have that λX is integrable with respect to $\mathbb{P}$ and we have that:

$$\mathbb{E}^{\mathbb{Q}}\left[X \mid \mathcal{G}\right] = \frac{\mathbb{E}^{\mathbb{P}}\left[\lambda X \mid \mathcal{G}\right]}{\mathbb{E}^{\mathbb{P}}\left[\lambda \mid \mathcal{G}\right]} \ .$$

Proof. See theorem 10.5 in [12, p. 240]). □

This second result which is a consequence of the first shows the altered structure of a martingale after a change of probability absolutely continous. (see theorem 10.6 in [12, p. 241]).

Theorem 2.5 (Martingales and change of probability)**.** *Let $(\lambda_t)_{t\in[0,T]}$ be a $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ non negative martingale in $(\Omega, \mathcal{F}, \mathbb{P})$ such that: $\mathbb{E}\left[\lambda_T\right] = 1$. Consider the probability $\mathbb{Q}$ defined over $(\Omega, \mathcal{F})$ by:*

$$\forall F \in \mathcal{F} \ , \ \mathbb{Q}\left[F\right] = \int_F \lambda_T d\mathbb{P} \ .$$

We then have that:

(1) *For $t \in [0, T]$ and sufficient integrability conditions on X,*

$$\mathbb{E}^{\mathbb{Q}}\left[X \mid \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{\lambda_T}{\lambda_t} X \mid \mathcal{F}_t\right] \ ,$$

(2) *and if X is a random variable with respect to $(\Omega, \mathcal{F}_t)$ then, for $s \leq t$:*

$$\mathbb{E}^{\mathbb{Q}}\left[X \mid \mathcal{F}_s\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{\lambda_t}{\lambda_s} X \mid \mathcal{F}_s\right] .$$

(3) *The stochastic process $\mathbb{X} = (X_t)_{t\in[0,T]}$ is a martingale with respect to $\mathbb{Q}$ if and only if the stochastic process $(\lambda_t \cdot X_t)_{t\in[0,T]}$ is a martingale with respect to $\mathbb{P}$.*

2.3. A first numerical study. In this section we illustrate the results of Sections 2.1 and 2.2 with a numerical application for the case of non random coefficients. We consider a discount rate given by the daily Libor interest rates r_{date}, with $\text{date} \in \{20100901, \ldots, 20190614\}$ and we have a range given by:

$$r_{20100901} = 0.22625\% \text{ and } r_{20190614} = 2.34663\% .$$

The discount factor, at each date, used is given by the exponentiated negative accumulated Libor rates, to that date:

$$d_{\text{date}} = \exp\left(-\sum_{k=1}^{\text{date}} \frac{r_k}{100}\right)$$

which is discrete form of the usual discount factor (see [5, pp. 160,161]). And we have:

$$d_{20100901} = 0.999977\% \text{ and } r_{20190614} = 0.881969\% .$$

The accumulated Libor rates are depicted in Figure 1. The chosen stock prices were Apple prices

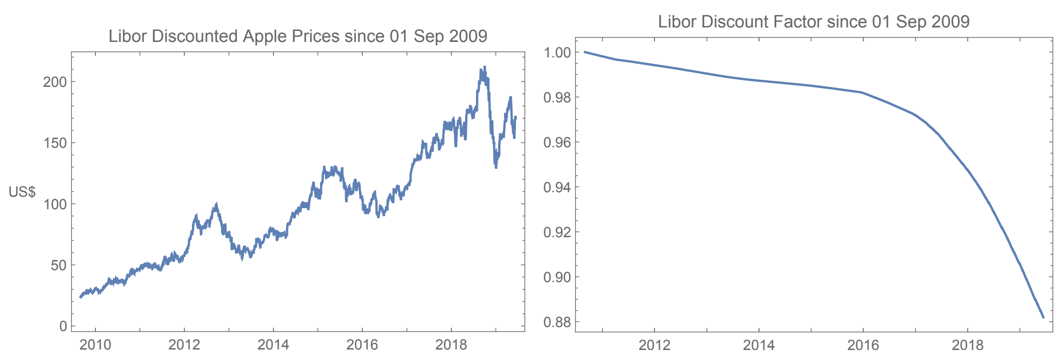


FIGURE 1. Libor discounted Apple prices and Libor discount factor since 2010

p_{date}, with $\text{date} \in \{20090901, \ldots, 20190614\}$, with a range given by:

$$p_{20090901} = 23.614\% \text{ and } p_{20190614} = 192.74\% .$$

The Apple discounted prices with the Libor rates – $p^d_{\text{date}} = p_{\text{date}} \times d_{\text{date}}$ – are shown in Figure 1. In order to proceed to the estimation of the coefficients of the Black-Scholes model with time varying coefficients we consider the usual logarithmic returns given by:

$$R_{\text{date}+\Delta t} := \frac{1}{\Delta t} \ln\left(\frac{p^d_{\text{date}+\Delta t}}{pd_{\text{date}}}\right) .$$

In general the non parametric estimation of the coefficients of the Black-Scholes model with time varying drift and diffusion is delicate see [17, pp. 257–282]. In the example under scrutiny we will

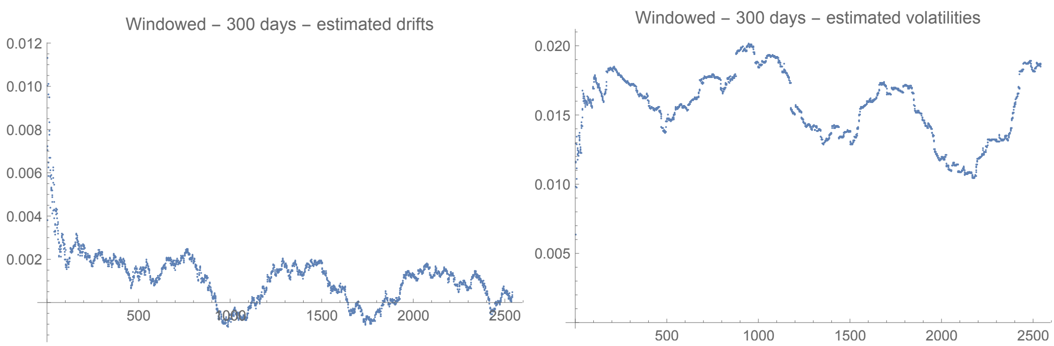


FIGURE 2. Windowed estimated drifts (left) and volatilities (right) with window $M = 300$

use the following observation to obtain sequences of parameters coherent with the coefficients μ and σ in the Euler-Maruyama discretization formula (2.3). As μ and σ are supposed to be non random, we may assume the normality of the returns R_{date} calculated daily – that is with $\Delta t = 1$ as we have daily observations In order to have time varying coefficients we may determine the coefficients – in the usual way, as independent observations of a Gaussian random variable – by considering a moving window of M days and for $\text{date} \geq M + 1$:

$$\sigma^2_{\text{date},M,\Delta t} := \frac{1}{\Delta t}\mathbb{V}\left[R_{\text{date}}, R_{\text{date}-1}, \ldots R_{\text{date}-M}\right] \ , \tag{2.9}$$

and

$$\mu_{\text{date},M,\Delta t} := \frac{1}{\Delta t}\mathbb{E}\left[R_{\text{date}}, R_{\text{date}-1}, \ldots R_{\text{date}-M}\right] + \frac{1}{2}\sigma_{\text{date},M,\Delta t} \ . \tag{2.10}$$

In Figure 2 we show the estimated quantities to be used as coefficients with a window $M = 300$ and $\Delta t = 1$. We do not claim that these moving averages are bona fide estimators of the functions μ_t and σ_t; the detailed study of these quantities - possibly as estimators of μ_t and σ_t with good properties - is postponed to future work. Nevertheless we will use these sequences to illustrate the methodology proposed. With the estimated drifts and volatilities we may now determine the Girsanov density for the martingale measure, given by

$$Y_N := \exp\left(-\sum_{n=1}^{N}\frac{\mu\left(n\Delta t\right)}{\sigma\left(n\Delta t\right)}Z_n - \sum_{n=1}^{N}\frac{\mu\left(n\Delta t\right)^2}{2\sigma\left(n\Delta t\right)^2}\right) \ , \tag{2.11}$$

In order to have a perception of this density we simulated a sample of values of this density; the graphical representation is given in Figure 3.

We may now proceed to the computation of the price of a call option both in two different ways: first by the known extension of the Black-Scholes formula to the case of time varying non random coefficients $\pi_{\text{BS}}(K)$ and then, by Monte Carlo simulation of the discounted cash-flow of the option with the martingale measure, $\pi_{\mathbb{Q}}(K)$, which is given by $d\mathbb{Q} = Y_N d\mathbb{P}$. Let us recall that, for a derivative with cash flow given by X, we have by theorem 2.4:

$$\pi_{\mathbb{Q}}(K) = \mathbb{E}^{\mathbb{Q}}\left[X d_{\text{exercise date}}\right] = \mathbb{E}^{\mathbb{P}}\left[Y_N X d_{\text{exercise date}}\right] \ .$$

In Table 1 we present two instances of computations of these two prices and in Remark 2.6 we discuss the results obtained.

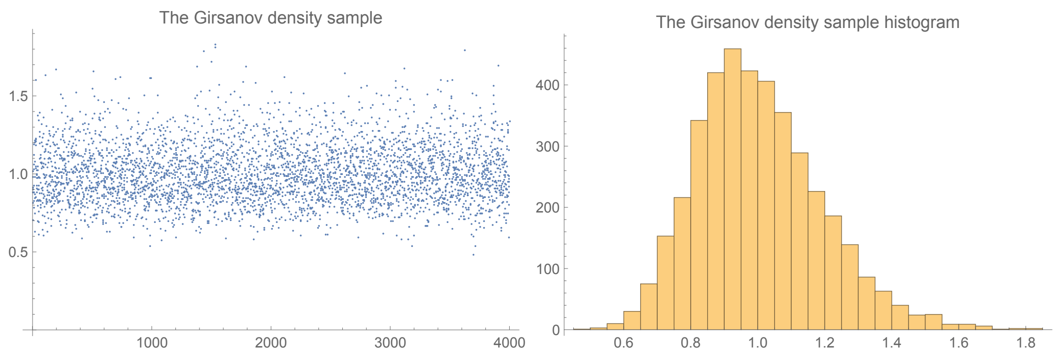


FIGURE 3. Monte Carlo simulation of the Girsanov density and correspondent histogram

In Figure 4 we show a simulated price trajectory and a sample of the discounted call cash flow. We now detail some remarks about the computational procedures.

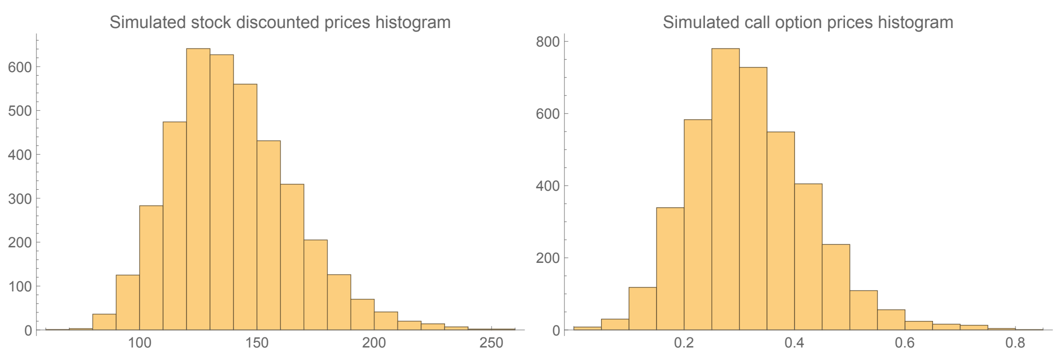


FIGURE 4. A trajectory of a simulated price and the simulated option price

(1) We observe that in formula (2.11) giving the Girsanov density we have that,

$$\ln(Y_N) = -\sum_{n=1}^{N} \frac{\mu(n\Delta t)}{\sigma(n\Delta t)} Z_n - \sum_{n=1}^{N} \frac{\mu(n\Delta t)^2}{2\sigma(n\Delta t)^2} \frown \mathcal{N}\left(-\sum_{n=1}^{N} \frac{\mu(n\Delta t)^2}{2\sigma(n\Delta t)^2}, \sum_{n=1}^{N} \frac{\mu(n\Delta t)^2}{\sigma(n\Delta t)^2}\right) ,$$

as the variables Z_n for $n = 1, 2, \ldots, N$ are independent. In this particular case of the Black-Scholes model with time varying coefficients, in order to simulate Y_N, we may replace an iterative procedure by a direct simulation of a sample of the Gaussian random variable $\ln(Y_N)$.

(2) The computation of Black-Scholes prices was performed, according to [16, pp. 101–102] or [5, p. 162] by replacing, in the usual Black-Scholes formula, the appropriate quantities related to time varying interest rate and volatility. More precisely we have the arbitrage

free price, at time $t = t_0$, of a call option with strike price K at maturity $t = T$ is given by:

$$\pi_{\mathrm{BS}}(K) = S_{t_0}\Phi\left(\frac{\ln\left(\frac{S_{t_0}}{K}\right) + \left(\int_{t_0}^{T} r(u)du + \frac{1}{2}\int_{t_0}^{T}\sigma^2(u)du\right)}{\sqrt{\int_{t_0}^{T}\sigma^2(u)du}}\right) - \\ - Ke^{-\int_{t_0}^{T} r(u)du}\Phi\left(\frac{\ln\left(\frac{S_{t_0}}{K}\right) + \left(\int_{t_0}^{T} r(u)du - \frac{1}{2}\int_{t_0}^{T}\sigma^2(u)du\right)}{\sqrt{\int_{t_0}^{T}\sigma^2(u)du}}\right), \tag{2.12}$$

with as usually, Φ being the cumulative distribution function of a standard normal random variable.

(3) In order to be able to use the estimated quantities for the drift and volatilities we computed both the Monte Carlo simulated price and the Black-Scholes price at times $t = T - 100$ – respectively $t = T - 1600$ – with the maturity date being the last available date in our data, to wit $T = 20190614$ – respectively, in the second example $t = T - 1400$. Considering the Girsanov density parameters, for a general interval $[T-\alpha, T-\beta]$ by:

$$\lambda_{T-\alpha}^{T-\beta} = \sum_{k=T-\alpha}^{T-\beta}\left(\frac{\mu(k\Delta t)}{\sigma(k\Delta t)}\right)^2$$

the algorithm was the following.

INPUT S_{t_0}, K, Δt, $\lambda_{T-\alpha}^{T-\beta}$
$M =$ Integer part $(T/\Delta t)$
FOR [$i = 1, i =$ number of repetitions $, i{+}{+},$
Sample $:= \{$ **Random Reals** $\in [0,1]\}$ (dimension M)
$S_i = S_{t_0} \times \prod_{k=T-100}^{T}\left(1 + \mu(k\Delta t)\Delta t + \sigma(k\Delta t)\sqrt{\Delta t}\text{ Sample }[[i]]\right)$
$W = \exp\left(\textbf{Random Real} \frown \mathcal{N}\left(\frac{-\lambda_{T-\alpha}^{T-\beta}}{2}, \sqrt{\lambda_{T-\alpha}^{T-\beta}}\right)\right)$
$\widetilde{S}_i = S_i \times W \times e^{-\int_{T-100}^{T} r(u)du}$
$\widetilde{P}_i = (S_i - K)^+ \times W \times e^{-\int_{T-100}^{T} r(u)du}$
]
PRINT $S :=$ Mean $(\widetilde{S}_i), P =$ Mean $(\widetilde{P}_i)$

Some typical results of the application of this algorithm are presented in Table 1. We used interpolation polynomials of degree 5 in order to have a regular functions to compute $\mu(k\Delta t)$ and $\sigma(k\Delta t)$ in the algorithm above; this is necessary because the computation procedure in formulas (2.9) and (2.10) was carried for $\Delta t = 1$ and so an interpolation is needed when we consider $\mu(k\Delta t)$ and $\sigma(k\Delta t)$ for $\Delta t < 1$ as we did.

Remark 2.6 (Discussion of the results). As a measure of control, by the definition of the martingale measure, we should have $P = 139.597 \approx S_{t_0} = 142.318$ and $P = 58.2951 \approx S_{t_0} = 58.9574$ for the second interval of dates $[T-1600, T-1400]$; these may be considered reasonable approximations. As a consequence we have credible correspondent arbitrage free prices for the call option of $\pi_{\mathbb{Q}}(142) = 0.319551$ and, for the second set of dates, $\pi_{\mathbb{Q}}(50) = 8.86026$. We observe that these prices are not similar to the Black-Scholes prices $\pi_{\mathrm{BS}}(K) = 12.2958$ and respectively $\pi_{\mathrm{BS}}(K) = 11.4588$, computed with the formula (2.12); there are several possible reasons for this discrepancy: the possibility of more than one arbitrage free price given that the discretized model was not shown to

Call option Monte Carlo simulation and Black-Scholes prices							
Monte Carlo	Δt	S_{t_0}	K	$P = \mathbb{E}_{\mathbb{Q}}[S_i]$	Repetitions	Dates	$\pi_{\mathbb{Q}}(K)$
	10^{-4}	**142.318**	142	**139.597**	$4 \cdot 10^3$	$[T-100, T]$	0.319551
	10^{-4}	**58.9574**	50	**58.2951**	$4 \cdot 10^3$	$[T-1600, T-1400]$	8.86026
Black-Scholes	Δt	S_{t_0}	K	$\int_{t_0}^{T} r(u)du$	$\int_{t_0}^{T} \sigma^2(u)du$	Dates	$\pi_{\mathrm{BS}}(K)$
	$--$	142.318	142	0.0225872	0.0348842	$[T-100, T]$	12.2958
	$--$	58.9574	50	0.0021455	0.0731667	$[T-1600, T-1400]$	11.4588

TABLE 1. With 5 degree polynomial order of interpolation for the coefficients

be complete; the convergence of the discretized model to the continuous model was not established; the quantities used as estimates for $\int_{t_0}^{T} \sigma^2(u)du$ in the Black-Scholes formula that may not have the adequate properties.

3. A coupled system of SDE for commodities

Another example of a continuous time model – with constant coefficients – that can be of interest to study in the perspective of comparing results with the discretized model is the one presented in this section. This model was first introduced in [1] in a very condensed form. The motivation for the renewed study of this model comes from the possibility of considering a discretized model, the need to develop the full proofs of the results announced and also a need of verification if the previously observed relationships between futures and spot prices were modified after a decade. The literature on models for future pricing has a variety of approaches (for instance [8]).

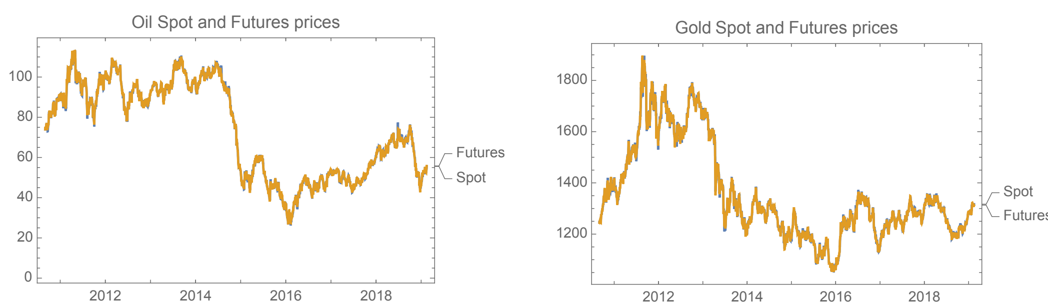


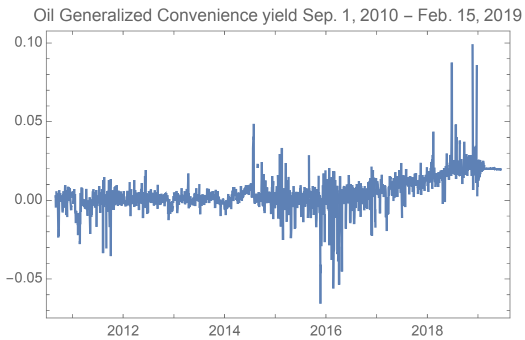
FIGURE 5. Oil and Gold futures and spot prices

The data presented comes from NYMEX (Oil futures [1]), COMEX (Gold futures), EIA - US Energy Information Administration (WTI oil spot price); gold spot price is available at a multiplicity of sites and was obtained with Mathematica ®; LIBOR overnight rates were obtained at the Federal Reserve Bank of St. Louis [2]. The period was September 1, 2010 to February 15, 2019.

Remark 3.1. Some observations:

[1]NYMEX WTI Light Sweet Crude Oil futures (ticker symbol CL), the world's most liquid and actively traded crude oil contract.

[2]https://fred.stlouisfed.org/series/USDONTD156N



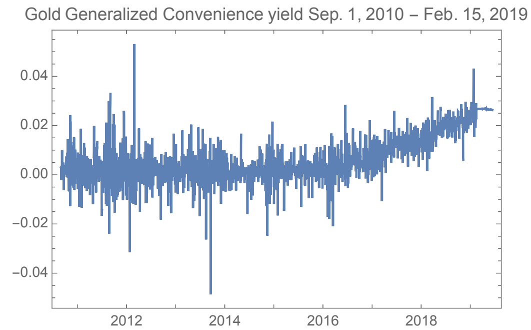


FIGURE 6. Oil and Gold convenience yield computed with the overnight LIBOR rate

(1) We observe that the generalized convenience yield process is the stochastic process $(y_t)_{t\geq 0}$ such that, $(r_t)_{t\geq 0}$ being the spot interest rate process we have $F_t^T = S_t\ e^{(r_t - y_t)(T-t)}$. Being so, the formal definition of the convenience yield is, r_t being the Libor interest rate,

$$y_t := r_t + \frac{1}{T-t}\ln(\frac{S_t}{F_t^T})\ .$$

(2) There is a remarkable superposition between the futures a and spot prices; these spot and futures prices are not identical as can be seen in the convenience yield figures.
(3) The observed growth tendency in the generalized convenience yield is due, for the most part, to the growth in the Libor rates; the observed structure in the logarithm of the ratios of the spot and future prices seems to be of a white noise type.

4. The system of coupled SDE model

Theorem 4.1. *Let the spot and futures prices be coupled by a system of SDE's,*

$$\begin{cases} \dfrac{dS_t}{S_t} = k^S(\theta^S - \log(F_t))dt + \sigma^S dB_t,\ S_0 \in \mathbb{R}^+ \\ \dfrac{dF_t}{F_t} = k^F(\theta^F - \log(S_t))dt + \sigma^F dB_t,\ F_0 \in \mathbb{R}^+\ , \end{cases} \tag{4.1}$$

where the process $(B_t)_{t\geq 0}$ is a unidimensional Brownian process. A solution of this EDE system is given by:

$$\begin{aligned} S_t = \exp\Bigg(& \cosh(\sqrt{k^S k^F}t)(((k^S\theta^S - \frac{\sigma^S}{2}) - k^S\log(F_0)) - \\ & - \sqrt{\frac{k^S}{k^F}}\sinh(\sqrt{k^S k^F}t)((k^F\theta^F - \frac{\sigma^F}{2}) - k^F\log(S_0)) - \\ & - \cosh(\sqrt{k^S k^F}t)\int_0^t (\sqrt{\frac{k^S}{k^F}}\sinh(\sqrt{k^S k^F}s)k^F\sigma^S + \cosh(\sqrt{k^S k^F}s)k^S\sigma^F)dB_t + \\ & + \sqrt{\frac{k^S}{k^F}}\sinh(\sqrt{k^S k^F}t)\int_0^t (\sqrt{\frac{k^F}{k^S}}\sinh(\sqrt{k^S k^F}s)k^F\sigma^S + \cosh(\sqrt{k^S k^F}s)k^S\sigma^F)dB_t\Bigg) \end{aligned} \tag{4.2}$$

and

$$
\begin{aligned}
F_t = \exp \Bigg(& \cosh(\sqrt{k^S k^F} t)(((k^S \theta^S - \frac{\sigma^S}{2}) - k^S \log(F_0)) - \\
& - \sqrt{\frac{k^F}{k^S}} \sinh(\sqrt{k^S k^F} t)((k^F \theta^F - \frac{\sigma^F}{2}) - k^F \log(S_0)) + \\
& + \sqrt{\frac{k^F}{k^S}} \sinh(\sqrt{k^S k^F} t) \int_0^t (\sqrt{\frac{k^S}{k^F}} \sinh(\sqrt{k^S k^F} s) k^F \sigma^S + \cosh(\sqrt{k^S k^F} s) k^S \sigma^F) dB_t - \\
& - \cosh(\sqrt{k^S k^F} t) \int_0^t (\sqrt{\frac{k^F}{k^S}} \sinh(\sqrt{k^S k^F} s) k^F \sigma^S + \cosh(\sqrt{k^S k^F} s) k^S \sigma^F) dB_t \Bigg) \ .
\end{aligned}
\tag{4.3}
$$

Proof. By repeated use of Ito's formula, this system of SDE can be written as a multidimensional Ornstein-Uhlenbeck SDE.

$$
dZ_t = AZ_t dt + A\Sigma dB_t \tag{4.4}
$$

with $Z_t = (Z_t^1, Z_t^2)'$, $\Sigma = (\sigma^S, \sigma^F)'$, the prime denoting the transposed vector or matrix, and the matrix

$$
A = \begin{bmatrix} 0 & -k^S \\ -k^F & 0 \end{bmatrix} .
$$

The interpretation of equation (4.4) is of a vectorial Ito process as in [21] or by the results of Section 5. By similarity with the unidimensional case we have the following. The process:

$$
\widetilde{Z}_t = e^{At} Z_0 + e^{At} \left(\int_0^t e^{-As} A\Sigma \, dB_s \right) \tag{4.5}
$$

is a solution for (4.4) SDE that is,

$$
\int_0^t A\widetilde{Z}_s ds = \widetilde{Z}_t - Z_0 + \int_0^t A\Sigma \, dB_s \ . \tag{4.6}
$$

We have, by a Fubini stochastic theorem (see [3, p.531], [4, p. 86] and [14]) that,

$$
\begin{aligned}
\int_0^t A\widetilde{Z}_s ds &= \int_0^t Ae^{As} Z_0 ds + \int_0^t Ae^{As} \left(\int_0^s e^{-Au} A\Sigma \, dB_u \right) ds = \\
&= \left[e^{As} \right]_0^t Z_0 + \int_0^t \left(\int_u^t Ae^{As} ds \right) e^{-Au} A\Sigma \, dB_u = \\
&= e^{At} Z_0 - Z_0 + \int_u^t \left(e^{At} - e^{Au} \right) e^{-Au} A\Sigma \, dB_u = \\
&= e^{At} Z_0 - Z_0 + \int_u^t e^{At} e^{-Au} A\Sigma \, dB_u - \int_u^t e^{Au} e^{-Au} A\Sigma \, dB_u = \\
&= e^{At} Z_0 + e^{At} \int_u^t e^{-Au} A\Sigma \, dB_u - Z_0 - \int_u^t A\Sigma \, dB_u \ ,
\end{aligned}
$$

that is, formula (4.6). We notice that e^{At} is a linear operator,

$$
\int_u^t e^{At} e^{-Au} A\Sigma \, dB_u = e^{At} \int_u^t e^{-Au} A\Sigma \, dB_u \ .
$$

□

Remark 4.2. The parameters of this model, k^S, k^F, θ^S, θ^F, σ^S, and σ^F do depend on the maturity T of the futures contract.

4.1. Model Estimation. The estimation was performed using a quasi-likelihood estimation method adapted from [9, p. 122], which in turn, follows [6] and [22]. The main idea is to consider a Euler discretization of each of the SDE's and observe that an approximation of the transition density of $\mathbb{E}[S_{t+\Delta t} \mid S_t = x]$ – and of $\mathbb{E}[F_{t+\Delta t} \mid F_t = x]$ – may be written explicitly. Let us detail the procedure. We consider model in (4.1) in the form:

$$\begin{cases} S_t = k^S(\theta^S - \log(F_t))S_t dt + \sigma^S S_t dB_t, \ S_0 \in \mathbb{R}^+ \\ F_t = k^F(\theta^F - \log(S_t))F_t dt + \sigma^F F_t dB_t, \ F_0 \in \mathbb{R}^+ \ . \end{cases} \tag{4.7}$$

The Euler discretization of the system in (4.7) is given by:

$$\begin{cases} S_{t+\Delta t} - S_t = k^S(\theta^S - \log(F_t))S_t \Delta t + \sigma^S S_t \left(B_{t+\Delta t} - B_t\right), \ S_0 \in \mathbb{R}^+ \\ F_{t+\Delta t} - F_t = k^F(\theta^F - \log(S_t))F_t \Delta t + \sigma^F F_t \left(B_{t+\Delta t} - B_t\right), \ F_0 \in \mathbb{R}^+ \ . \end{cases} \tag{4.8}$$

By supposing in (4.8) that S_t and F_t are constant in intervals of length Δt around regularly spaced times t, we may consider that the law of $S_{t+\Delta t} - S_t$ is approximately Gaussian given by $\mathcal{L}(S_{t+\Delta t} - S_t) \approx \mathcal{N}(k^S(\theta^S - \log(F_t))S_t\Delta t, (\sigma^S S_t)^2\Delta t)$ with a correspondent result for the law of $F_{t+\Delta t} - F_t$ yielding $\mathcal{L}(F_{t+\Delta t} - F_t) \approx \mathcal{N}(k^F(\theta^F - \log(S_t))F_t\Delta t, (\sigma^F F_t)^2\Delta t)$. As a consequence we have that approximations to the coupled transition densities of $\mathbb{E}[S_{t+\Delta t} \mid S_t = x]$ and of $\mathbb{E}[F_{t+\Delta t} \mid F_t = x]$ are given by the coupled expressions $p^S(t, y_S \mid x_S)$ and $p^F(t, y_F \mid x_F)$ in formula (4.9).

$$\begin{cases} p^S(t, y_S \mid x_S) = \dfrac{1}{\sqrt{2\pi t(\sigma^S x_S)^2}} \exp\left(-\dfrac{1}{2}\dfrac{(y_S - x_S - k^S(\theta^S - \log(x_F))x_S t)^2}{t(\sigma^S x_S)^2}\right) \\ p^F(t, y_F \mid x_F) = \dfrac{1}{\sqrt{2\pi t(\sigma^F x_F)^2}} \exp\left(-\dfrac{1}{2}\dfrac{(y_F - x_F - k^F(\theta^F - \log(x_S))x_F t)^2}{t(\sigma^F x_F)^2}\right) \end{cases} \tag{4.9}$$

Now, considering sequences of observations $(S_i)_{i\in\{0,1,\dots,N\}}$ and $(F_i)_{i\in\{0,1,\dots,N\}}$ at regularly spaced time intervals – in our case, daily – the discretized system in (4.8) now corresponds to:

$$\begin{cases} S_{i+1} - S_i = k^S(\theta^S - \log(F_i))S_i \Delta t + \sigma^S S_i \left(B_{i+1} - B_i\right), \ S_0 \in \mathbb{R}^+ \\ F_{i+1} - F_i = k^F(\theta^F - \log(S_i))F_i \Delta t + \sigma^F F_i \left(B_{i+1} - B_i\right), \ F_0 \in \mathbb{R}^+ \ , \end{cases} \tag{4.10}$$

and so, by using the approximated coupled transition densities in (4.9) we have the log-likelihoods – the so called *locally Gaussian approximation*– given by:

$$\begin{cases} \log(\mathcal{L}_N^S) = -\dfrac{1}{2}\left(N \ln(2\pi\Delta t(\sigma^S S_i)^2 + \displaystyle\sum_{i=0}^{N-1} \dfrac{(S_{i+1} - S_i - k^S(\theta^S - \log(F_i))S_i\Delta t)^2}{2(\sigma^S S_i)^2\Delta t}\right) \\ \log(\mathcal{L}_N^F) = -\dfrac{1}{2}\left(N \ln(2\pi\Delta t(\sigma^F F_i)^2 + \displaystyle\sum_{i=0}^{N-1} \dfrac{(F_{i+1} - F_i - k^F(\theta^F - \log(S_i))F_i\Delta t)^2}{2(\sigma^F F_i)^2\Delta t}\right) \ , \end{cases} \tag{4.11}$$

The method implemented – using Mathematica ® – consists on maximizing $\log(\mathcal{L}_N^S)$ and $\log(\mathcal{L}_N^F)$, as functions of all the parameters, using the observed daily prices, in small intervals around values determined by other estimators of the parameters. We could get, as usual, a robust estimation for σ^S and σ^F.

Data from Sep. 1, 2010 to Feb. 15, 2019						
Gold	k^S	θ^S	σ^S	k^F	θ^F	σ^F
	-0.336745	7.19141	0.150781	0.302309	7.2997	0.161795
Oil	k^S	θ^S	σ^S	k^F	θ^F	σ^F
	0.0100322	6.	0.339887	0.4	4.2817	0.329276

Table 2. Estimated parameters

Remark 4.3. The estimated values in Table 2 suggest the following remarks. The model proposed achieves a good replication of the overall behavior of the coupling between spot and futures prices. The estimated generalized convenience yield, for both commodities, has a tendency to increase. For Oil, the estimated long term returns verify: $\theta_F - \sigma^F/2k^F = 4.14617 > \theta_S - \sigma^S/2k^S = 0.242381$, thus anticipating a possible – to be seen – disconnection between the value of the futures and of the spot prices. For Gold, no such disconnection may be inferred as: $\theta_F - \sigma^F/2k^F = 7.2564 \approx \theta_S - \sigma^S/2k^S = 7.22517$

Remark 4.4. Being able to estimate the parameters of the coupled model it is possible to find, a martingale measure by means of Theorem 5.8 in Section 5. We will present in a future work a numerical exploration of the Euler-Maruyama discretization methodology applied to this model.

5. Naive stochastic integration in Hilbert space

We propose next a streamlined approach to stochastic integration in Hilbert space that is suitable for the example in Section 3. Complete studies of this subject are developed, for instance, in [13] and [21]. Consider $(Z_t)_{t\in[0,T]}$ a process taking values in H, that is such that $Z_t \in H$ for $t \in [0,T]$. Given $(\alpha_k)_{k\geq 0}$ an orthonormal system we have the representation, for all $t \in [0,T]$

$$Z_s = \sum_{k=0}^{+\infty} \langle Z_s, \alpha_k \rangle \alpha_k \ ,$$

with convergence in H. We note that in the example of Section 3 we have only two dimensions and so the above sum is a finite one. With $(B_t)_{t\geq 0}$ being a standard Brownian process we have the natural definitions for the integrals in H as superposition of usual integrals:

$$\int_0^t Z_s ds = \sum_{k=0}^{+\infty} \left(\int_0^t \langle Z_s, \alpha_k \rangle ds \right) \alpha_k \text{ and } \int_0^t Z_s dB_s = \sum_{k=0}^{+\infty} \left(\int_0^t \langle Z_s, \alpha_k \rangle dB_s \right) \alpha_k \ .$$

under sufficient conditions for these integrals to exist.

Theorem 5.1 (Linear operators and stochastic integration)**.** *Let $\mathbf{\Psi} \in \mathcal{L}(H)$ be a bounded linear operator on H. We then have that:*

$$\mathbf{\Psi}\left(\int_0^t Z_s dB_s \right) = \int_0^t \mathbf{\Psi}(Z_s)\, dB_s$$

if both integrals exist.

Definition 5.2 (Ito's type process)**.** An Ito's type process is defined by

$$X_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s = \sum_{k=0}^{+\infty} \left(\int_0^t \langle \mu_s, \alpha_k \rangle ds + \int_0^t \langle \sigma_s, \alpha_k \rangle dB_s \right) \alpha_k \ , \tag{5.1}$$

with conditions such that the scalar integrals with the definitions above converge.

Of special interest are the operators that have a decomposable range.

Definition 5.3 (Decomposable range operators)**.** Consider an operator $\mathbf{\Phi} : H \mapsto H$, such that there exists a sequence $(\varphi_k)_{k\geq 0}$ of functions $\varphi_k : \mathbb{R} \times [0,T] \mapsto \mathbb{R}$ such that with

$$\forall x \in \mathbb{R} \ \ \forall t \in [0,T] \ \ \sum_{k=0}^{+\infty} |\varphi_k(x,t)|^2 < +\infty$$

such that for $\boldsymbol{x} \in H$

$$\boldsymbol{x} = \sum_{k=0}^{+\infty} \langle \boldsymbol{x}, \alpha_k \rangle \alpha_k \text{ and } \mathbf{\Phi}(\boldsymbol{x}) := \sum_{k=0}^{+\infty} \varphi_k \left(\langle \boldsymbol{x}, \alpha_k \rangle, \cdot \right) \alpha_k \ .$$

Remark 5.4 (Linear operators)*.* If for some map $\varphi : \mathbb{R} \times [0,T] \mapsto \mathbb{R}$ – linear in the first coordinate for all values of the second coordinate – we have that $\varphi_k \equiv \varphi$ for all $k \geq 0$ then $\mathbf{\Phi}$ is a linear operator.

Definition 5.5 (Regular decomposable range operators)**.** Let $\mathbf{\Phi} : H \mapsto H$ be a decomposable range operator. We say that $\mathbf{\Phi}$ is in the space $\mathcal{O}^{2,1}$ if for all $k \geq 0$, $\varphi_k \in \mathcal{C}^{2,1}(\mathbb{R} \times [0,T])$.

Theorem 5.6 (Ito's type formula)**.** *Denote the usual Ito's scalar integrals in formula* (5.1) *by,*

$$x_t := \int_0^t \langle \mu_s, \alpha_k \rangle ds + \int_0^t \langle \sigma_s, \alpha_k \rangle dB_s \ . \tag{5.2}$$

We then have for $\mathbf{\Phi} \in \mathcal{O}^{2,1}$*,*

$$\mathbf{\Phi}(X_t) = \sum_{k=0}^{+\infty} \varphi_k \left(\int_0^t \langle \mu_s, \alpha_k \rangle ds + \int_0^t \langle \sigma_s, \alpha_k \rangle dB_s, \cdot \right) \alpha_k =$$

$$= \sum_{k=0}^{+\infty} \left(\int_0^t \left[\frac{\partial \varphi_k}{\partial t} + \langle \mu_s, \alpha_k \rangle \frac{\partial \varphi_k}{\partial x} + \frac{1}{2} \langle \sigma_s, \alpha_k \rangle^2 \frac{\partial^2 \varphi_k}{\partial x^2} \right] (x_s, s)\, ds + \int_0^t \langle \sigma_s, \alpha_k \rangle \frac{\partial \varphi_k}{\partial x} (x_s, s)\, dB_s \right) \alpha_k \tag{5.3}$$

that is a superposition of integrals like in (5.1)*,*

Remark 5.7*.* With the operators

$$\mathcal{L}_k := \frac{\partial}{\partial t} + \langle \mu_{\cdot}, \alpha_k \rangle \frac{\partial}{\partial x} + \frac{1}{2} \langle \sigma_{\cdot}, \alpha_k \rangle^2 \frac{\partial^2}{\partial x^2} \text{ and } \mathcal{K}_k := \langle \sigma_{\cdot}, \alpha_k \rangle \frac{\partial}{\partial x} \ ,$$

formula (5.3) can be written,

$$\mathbf{\Phi}(X_t) = \sum_{k=0}^{+\infty} \left(\int_0^t \mathcal{L}_k(\varphi_k)\, (x_s, s)\, ds + \int_0^t \mathcal{K}_k(\varphi_k)\, (x_s, s)\, dB_s \right) \alpha_k \ .$$

Theorem 5.8 (Change of measure)**.** *Consider an Ito's type process as in* (5.1)*and the scalar processes* $x_t^k := x_t$ *as in* (5.2)*. Let the sufficient conditions be verified for* $H_s^k := \langle \mu_s, \alpha_k \rangle / \langle \sigma_s, \alpha_k \rangle$ *to be well defined and consider:*

$$\Lambda_k := \exp\left(- \int_0^T H_s^k dB_s - \frac{1}{2} \int_0^T \left(H_s^k \right)^2 ds \right) \ .$$

Then, with $\Lambda := \prod_{k=0}^{+\infty} \Lambda_k$ *and* $d\mathbb{Q} = \Lambda d\mathbb{P}$*, if*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \left(H_s^k\right)^2 ds\right)\right] < +\infty \text{ and } \mathbb{E}\left[\Lambda\right] = 1 ,$$

then, with:

$$\widetilde{B}_t := B_t + \int_0^t \left(\sum_{k=0}^{+\infty} H_s^k\right) ds$$

being a $\mathbb{Q}$ *Brownian process, we have that,*

$$dX_t = \sum_{k=0}^{+\infty}\left(\int_0^t \langle \sigma_s, \alpha_k\rangle d\widetilde{B}_s\right)\alpha_k \ .$$

is a martingale in the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$*.*

Remark 5.9. Theorem 5.8 may be applied to build the adequate martingale measures for models such as the ones described in Section 3.

6. The random coefficients case

For the case where the coefficients of both the price SDE model and the risk free rate are random we defer to future work the presentation of relevant examples. In this context it is important to recall both the caveats to the use of discrete time models as in [11] and the general complete theory for these models as explained in [10]. For the present work let us just outline the proposed method; similarly to the non random case, we may also consider the discrete market model obtained by the Euler-Maruyama discretization of the continuous market model by using what we may call the Girsanov theorem in discrete time in the conditionally Gaussian case.

The main result may to be used in the context of random coefficients and random risk free rate may be found in [20, pp. 433–446] or in [2, pp. 123–125] although not in this synthetic form.

Theorem 6.1. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space and* $\mathbb{F} = (\mathcal{F}_n)_{n\geq 0}$ *a filtration ove rthis space such as* $\mathcal{F}_0 = \{\emptyset, \Omega\}$*. Let the process* $\mathbb{X} = (X_n)_{n\geq 1}$ *verify the following decomposition:*

$$\forall n \geq 0 \ , \ X_{n+1} = X_n + \mu_{n+1} + \sigma_{n+1} Z_{n+1} \ ,$$

with $(\mu_n)_{n\geq 1}$ *and* $(\sigma_n)_{n\geq 1}$ *previsible processes,* $(Z_n)_{n\geq 1}$ *an* $\mathbb{F}$*- adapted process of independent random variables identically distributed such that the law of* Z_n *conditioned to* $\mathcal{F}_{n-1}$ *is a standardized normal random variable. Let the Girsanov process be given by:*

$$Y_n := \exp\left(-\sum_{k=1}^{n}\frac{\mu_k}{\sigma_k} - \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\mu_k}{\sigma_k}\right)^2\right) \ , \ n \geq 1$$

such that the following Novikov condition is verified:

$$\forall n \geq 1 \ , \ \mathbb{E}\left[\frac{1}{2}\sum_{k=1}^{n}\left(\frac{\mu_k}{\sigma_k}\right)^2\right] < +\infty \ . \tag{6.1}$$

We then have that:

(1) *the Girsanov process* $\mathbb{Y} = (Y_n)_{n\geq 1}$ *is a* $\mathbb{F}$ *uniformly integrable martingale which converges, almost surely to a random variable* Y_∞ *verifying:*

$$\mathbb{P}\left[Y_\infty \geq 0\right] = 1 \ e \ \mathbb{E}\left[Y_\infty\right] = 1;$$

(2) *considering the pprobability measure* $\mathbb{Q}$ *defined obver* $(\Omega, \mathcal{F})$ *by:*

$$\forall F \in \mathcal{F} \ , \ \mathbb{Q}\left[F\right] = \int_F Y_\infty d\mathbb{P} \ ,$$

then, in the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$*, the probability law of* $(X_{n+1} - X_n)_{n \geq 1}$ *conditioned by* $\mathbb{F}$*, coincides, term by term, with the probability law of* $(\sigma_n Z_n)_{n \geq 1}$ *;*

(3) *in the probability space* $(\Omega, \mathcal{F}, \mathbb{Q})$*, the process* $(X_{n+1} - X_n)_{n \geq 1}$ *is a local martingale.*

We observe that the hypothesis made that Z_n condicioned to $\mathcal{F}_{n-1}$ is a standardized Gaussian implies that the probability law of $X_{n+1} - X_n$ conditioned to $\mathcal{F}_{n-1}$ is given for $F \in \mathcal{F}$, by:

$$\mathbb{P}\left[(X_{n+1} - X_n) \in F \mid \mathcal{F}_{n-1}\right] := \mathbb{E}\left[\mathbb{1}_{(X_{n+1} - X_n) \in F} \mid \mathcal{F}_{n-1}\right] = \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_F e^{-\frac{(y-\mu_n)^2}{2\sigma_n^2}} dy \ . \tag{6.2}$$

Following [20, pp. 62, 103] this formula may be interpreted by saying that the probability law of $X_{n+1} - X_n$ conditioned to $\mathcal{F}_{n-1}$is a mixture of Gaussian random variables with parameters μ_n, and σ_n^2 that in themselves are random variables

Remark 6.2. The second result in the theorem 6.1 is that in the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ the law of $X_{n+1} - X_n$ conditioned to $\mathcal{F}_{n-1}$ is given for $F \in \mathcal{F}$, by:

$$\mathbb{Q}\left[(X_{n+1} - X_n) \in F \mid \mathcal{F}_{n-1}\right] := \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{(X_{n+1} - X_n) \in F} \mid \mathcal{F}_{n-1}\right] = \frac{1}{\sqrt{2\pi\sigma_n^2}} \int_F e^{-\frac{y^2}{2\sigma_n^2}} dy \ ,$$

that is, with the new probability $\mathbb{Q}$, the coefficient μ_n is suppressed and we now have a local martingale.

6.1. Discrete conditionally Gaussian models. The general methodology in the context of the random coefficients case may now be described as follows. We have discrete time observations of a discounted price process and we want to model it as a discrete conditionally Gaussian model to which we may apply theorem 6.1 in order to obtain a martingale measure and to price derivatives. We may consider following [20, pp. 89, 104–108], a discrete model in which the discounted price process $(S_n)_{n \geq 1}$, defined in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ and adapted to $\mathbb{F} = (\mathcal{F}_n)_{n \geq 0}$ satisfying,

$$S_n = S_0 e^{h_1 + h_2 + \ldots h_n} \text{ with } \mathbb{E}\left[|h_n|\right] < +\infty \ , \tag{6.3}$$

with $h_n = \mu_n + \sigma_n \epsilon_n$ where $(\epsilon_n)_{n \geq 1}$ is a sequence of independent standardized Gaussian random variables adapted to $\mathbb{F}$ and,

$$\text{Law}\left(h_n \mid \mathcal{F}_{n-1}\right) = \mathcal{N}(\mu_n, \sigma_n^2) \ , \tag{6.4}$$

with the processes $(\mu_n)_{n \geq 1}$ and $(\sigma_n)_{n \geq 1}$ predictable with respect to $\mathbb{F}$ and formula (6.4) having an interpretation similar to the one given for formula (6.2). This class of models for $(h_n)_{n \geq 1}$ is quite rich as it encompasses, among others, the ARMA(p,q), the ARCH(p) and the GARCH(p,q) models. For instance, for the ARMA(p,q) model we have $\mathcal{F}_n = \sigma(\epsilon_1, \ldots, \epsilon_n)$, the prescribed initial conditions $h_{1-p}, \ldots, h_{-1}, h_0$ and $\epsilon_{1-q}, \ldots, \epsilon_{-1}, \epsilon_0$ and,

$$\mu_n = a_0 + a_1 h_{n-1} + \ldots a_p h_{n-p} + b_1 \epsilon_{n-1} + \ldots b_p \epsilon_{n-q} \ ,$$

with $\sigma_n \equiv \sigma$ constant. Estimating the adequate time series model we may get under suitable hypothesis a martingale measure and the possibility of computing derivative prices. There are nevertheless important questions to be dealt with, namely,

- under what conditions, the model given by (6.3) and (6.4), is a Euler-Maruyama discretized model of a continuous time model?
- under what conditions does the discretized model converges – see [15] – to some continuous time market model with good properties?
- under what conditions is an arbitrage free and complete model – see [10] and [11] – for the whole sequence of discretizations considered?

Remark 6.3. The weak convergence of the Euler-Maruyama sequence of approximations to the solution of the discretized SDE has been studied in many works. For instance, in [23] the usual regularity conditions ensuring weak convergence are relaxed in a way that may useful to our purposes.

7. Conclusion

The study of discrete time models is justified by the fact that all data that we may collect is discrete and finite – although, in some cases, of large dimensionality; as so, the interplay between discrete – closer to the data collected – and continuous related models – having very important conceptual properties – deserves attention. In this work we developed the computation of a martingale measure by means of the Girsanov density in two examples obtained from the Euler-Maruyama discretization of two models: the Black-Scholes model with time dependent – although non random – coefficients and a vectorial model with constant coefficients for futures and spot prices of commodities. We also present the general lines of a systematic approach to the case of random coefficients to be developed in future work.

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M. L. Esquível, Department of Mathematics and CMA FCT UNL, FCT NOVA, Quinta da Torre, 2829-516 Caparica, Portugal.

E-mail address: mle@fct.unl.pt

URL: http://ferrari.dmat.fct.unl.pt/personal/mle/

N. P. Krasii, Don State Technical University, Gagarin Square 1, Rostov-on-Don 344000, Russian Federation

E-mail address: krasnad@yandex.ru

URL: https://donstu.ru/structure/cadre/krasiy-nadezhda-pavlovna