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OPTIMAL WATER DISTRIBUTION IN LARGE MAIN CANALS OF IRRIGATION SYSTEM

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ABSTRACT. The article presents the optimality of the problem of the optimal distribution of water in the channels of irrigation systems under conditions of discreteness of water supply in the case of using as a mathematical model of unsteady water movement in the channel sections the system of complete differential equations of Saint-Venant, and also the method of numerical solution by the finite method the difference of the differential system of the Saint-Venant equation.

1. Introduction

The optimal distribution of water in the channels of irrigation systems in the conditions of discreteness of water supply to water consumers is formulated as the problem of optimal control of systems with distributed parameters [**?**]. In the article [3, 4], mathematical models and criteria for the quality of water distribution in the channels of irrigation systems in the conditions of discreteness of water supply are developed. When it comes to the optimality of water distribution, the functions on the right-hand sides of the Saint-Venant equations are discrete in time, and the coefficients in the differential equations of continuity and momentum for systems with distributed parameters are no longer continuous functions in time, therefore, it is required additional necessary condition. The necessary conditions for the optimality of water distribution in the channels of irrigation systems were studied in [1, 2].

2. Methods and models

Let each object with distributed parameters be described by one-dimensional differential equations of the first order in time and second order in the spatial variable [1, 2, 7].

$$\frac{\partial Z_i}{\partial t} = f_i^1\left(Z_i, \frac{\partial Z_i}{\partial x_i}, q_i\right),$$

$$\frac{\partial Q_i}{\partial t} = f_i^2\left(Z_i, Q_i, \frac{\partial Z_i}{\partial x_i}, \frac{\partial Q_i}{\partial x_i}\right), \tag{2.1}$$

$$i = 1, 2; 0 < x_1 < l_1; l_1 < x_2 < l_2; 0 < t < T.$$

Key words and phrases. mathematical model, unsteady flow of water, main canals, optimal control problems, fundamental solution, differential equation, hydraulic structures.

Here

$$
\begin{gathered}
f_i^1\left(Z_i, \frac{\partial Z_i}{\partial x_i}, q_i\right) = -\frac{1}{B_i}\left(\frac{\partial Q_i}{\partial x} - q_i\right), \\
f_i^2\left(Z_i, Q_i, \frac{\partial Z_i}{\partial x_i}, \frac{\partial Z_i}{\partial x_i}\right) = \\
= \frac{g\omega_{i0}Q_i^2}{(\omega_i c_i)^2} - \frac{g\omega_i Q_i|Q_i|}{K^2} - \frac{2Q_i}{\omega_i}\frac{\partial Q_i}{\partial x_i} - \\
-g\omega_i\left(1 - \frac{Q_i^2}{(\omega_i c_i)^2}\right)\frac{Z_i}{\partial x_i},
\end{gathered}
\tag{2.2}
$$

where $Q_i = Q_i(x_i, t), Z_i = Z_i(x_i, t)$ respectively, the flow rate and the ordinate of the free surface of the stream; i - of the irrigation canal section; $B_i = B_i(Z_i)$ - flow width on top; $\omega_i = \omega_i(Z_i)$ - living area; $c_i = c_i(Z_i)$ is the propagation velocity of small waves; $K_i = K_i(Z_i)$ - flow module. The last four values are determined by the morphometric and hydraulic parameters of the channel section. The directional inflow (outflow) $q_i = q_i(x_i, t)$ calculated per unit length of the i-th channel section is a distributed disturbance [3, 4, 5].

Given initial conditions

$$
Q_i(x, 0) = Q_{i0}(x_i), Z_i(x_i, 0) = Z_{i0}(x_i) \tag{2.3}
$$

Boundary conditions in the points $x_i = 0$ and $x_2 = l_2$ are written as the following form

$$
\begin{gathered}
Q_1(0, t) = g_1(Z_1(0, t), u_1(t)), \\
Q_2(l_2, t) = g_2(Z_2(l_2, t), u_2(t), u_3(t)),
\end{gathered}
\tag{2.4}
$$

where

$$
\begin{gathered}
g_1 = \mu_1 b_1 u_1(t)\sqrt{2g\left(Z_1^1 - Z_1(0, t)\right)}, \\
g_2 = \mu_2 b_2 u_2(t)\sqrt{2g\left(Z_2(l_2, t) - \varepsilon_2 u_2(t)\right)},
\end{gathered}
\tag{2.5}
$$

$u_i = u_i(t_i), i = 1, 2, 3$ - control functions applied at the boundary points (the height of the openings of the gates), $b_i, i = 1, 2$ - the width of the openings of the gates, z_1^1 - the ordinate of the free surface of the water flow of the headwater of the first shutter.

The conjugation conditions at the point $x_1 = x_2 = l_1$ are written as follows

$$
\begin{gathered}
Q_1(l_1, t) = g_1^c(Z_1(l_1, t), Z_2(l_1, t), u_1^c(l_1, t), u_2^c(t)), \\
Q_2(l_1, t) = g_2^c(Z_1(l_1, t), Z_2(l_1, t), u_2^c(t)),
\end{gathered}
\tag{2.6}
$$

where

$$
\begin{gathered}
g_1^c = \mu_1^c b_1^c u_1^c(t)\sqrt{2g\left(Z_1(l_1, t) - \varepsilon_1^c u_1^c(t)\right)} + g_2^c \\
g_2^c = \mu_2^c b_2^c u_2^c(t)\sqrt{2g\left(Z_1(l_1, t) - Z_1(l, t)\right)}
\end{gathered}
\tag{2.7}
$$

where $u_i^c = u_i^c(t_i), i = 1, 2$ are the control functions applied at the connection point of the channel sections (the height of the openings of the gates), $b_i^c, 1, 2$ is the width of the openings of the gates.

For the numerical solution of boundary value problems (2.1)–(2.3), we use the finite difference method [9, 11].

In the $\Omega = \{0 \le x \le l, 0 \le t \le T\}$ area, we introduce the

$$\bar{\omega}_{h\tau} = \{(x_i, t_j): \quad x_i = ih; \quad t_j = j\tau; \quad i = \overline{0, N}; \\ j = \overline{0, M}; \quad h = l/N; \quad \tau = T/M\}$$

grid with the steps h in x and τ on T.

Approximating the system of equations (2.1) using an absolutely stable implicit difference scheme having a second order of approximation in x, and a first order of approximation in t, we get

$$S_i^k \tfrac{Z_i^{k+1} - Z_i^k}{\tau} + (\Delta S)_i^k \tfrac{Z_{i+1}^{k+1} - Z_{i-1}^{k+1}}{2h} = F_n^k + \left(\tfrac{\partial F}{\partial Z}\right)_n^k Z_i^k, \\ i = \overline{1, n-1}, \ k = \overline{1, m}$$

here $[Z_i^k = \{Z\left(x_i,\, t_k\right),\ z\left(x_i,\ t_k\right)\}$ is the difference vector function of the variable with respect to x_i, t_k the right-hand side of equations (2.1)-(2.2) is linearized by the quasi-linearization method, expanding it in Newton's series, leaving only the first approximation terms in the vicinity of the point F_n^k, we obtain algebraic equations.

After simple transformations, we obtain the following system of three diagonal matrix difference equations for internal grid points

$$\mathbf{P}_n^k \cdot Z_{n-1}^{k-1} + \mathbf{R}_n^k \cdot Z_n^{k+1} - \mathbf{P}_n^k \cdot Z_{n+1}^{k+1} = \varpi_n^k, \\ n = \overline{1, N-1} \tag{2.8}$$

Here:

$$\begin{aligned} &\mathbf{P}_n^k = \tfrac{\tau}{2h} (\Delta \mathbf{S})_n^k; \\ &\mathbf{R}_n^k = \mathbf{S}_n^k - \tau \left(\tfrac{\partial \mathbf{F}}{\partial Z}\right)_n^k \\ &Z_n^{k+1} = Z(x_n, t_{k+1}); \\ &\mathbf{w}_n^k = \left[\mathbf{S}_n^k - \tau \left(\tfrac{\partial \mathbf{F}}{\partial Z}\right)_n^k\right] Z_n^k + \tau \mathbf{F}_n^k; \\ &\tfrac{\partial \mathbf{F}}{\partial Z} = \begin{bmatrix} \partial \mathbf{F}/\partial Z \\ \partial \mathbf{F}/\partial z \end{bmatrix}. \end{aligned} \tag{2.9}$$

The boundary conditions for the channel section bounded by the partitioning structures are linearized by the Newton method in the vicinity of the previous time step, then we obtain in a discrete form

$$\begin{aligned}
Z_{i0}^{k+1} &= Z_{i0}^{k} + \left(\frac{\partial G_{i1}}{\partial u_1}\right)_0^k \left(u_1^{k+1} - u_1^k\right) + \\
&+ \left(\frac{\partial G_{i1}}{\partial z_{up}}\right)_0^k \left(z_{vb}^{k+1} - z_{vb}^{k+1}\right) + \\
&+ \left(\frac{\partial G_{i1}}{\partial z}\right)_0^k \left(z_{i0}^{k+1} - z_{i0}^{k}\right), \\
Z_{iN}^{k+1} &= Z_{iN}^{k} + \left(\frac{\partial G_{i2}}{\partial u_2}\right)_0^k \left(u_2^{k+1} - u_2^{k+1}\right) + \\
&+ \left(\frac{\partial G_{i2}}{\partial z}\right)_0^k \left(z_{iN}^{k+1} - z_{iN}^{k}\right) + \\
&+ \left(\frac{\partial G_{i2}}{\partial z_{lp}}\right)_0^k \left(z_{lp}^{k+1} - z_{lp}^{k}\right).
\end{aligned} \tag{2.10}$$

Here are the partial derivatives $\left(\frac{\partial G_{1i}}{\partial u_1}\right)_0^k$, $\left(\frac{\partial G_{1i}}{\partial z_{up}}\right)_0^k$, $\left(\frac{\partial G_{1i}}{\partial z}\right)_0^k$, $\left(\frac{\partial G_{2i}}{\partial u_2}\right)_0^k$, $\left(\frac{\partial G_{2i}}{\partial z}\right)_0^k$ and $\left(\frac{\partial G_{2i}}{\partial z_{lp}}\right)_0^k$ mean the derivative functions of the flow rate of the hydraulic structure located at the beginning and at the end of the channel section with respect to the arguments u_1, z_{up} and z_{lp}.

If the channel section is limited between the partitioning structures, then the partial derivatives, taking into account the expressions of the flow of water flowing through the partitioning structures, have the following form

$$\begin{aligned}
&\left(\frac{\partial G_{1i}}{\partial a_{1i}}\right)_0^k = \mu_{1i} b_{1i}^k \sqrt{2g\left(z_{upi}^k - z_{0i}^k\right)}; \\
&\left(\frac{\partial G_{1i}}{\partial z_{vbi}}\right)_0^k = \mu_{1i} a_{1i}^k b_{1i}^k \frac{g}{\sqrt{2g\left(z_{upi}^k - z_{0i}^k\right)}}; \\
&\left(\frac{\partial G_{1i}}{\partial z_i}\right)_0^k = -\mu_{1i} a_{1i}^k b_{1i}^k \frac{g}{\sqrt{2g\left(z_{upi}^k - z_{0i}^k\right)}}; \\
&\left(\frac{\partial G_{2i}}{\partial a_{2i}}\right)_0^k = \mu_{2i} b_{2i}^k \sqrt{2g\left(z_{0i}^k - z_{lpi}^k\right)};; \\
&\left(\frac{\partial G_{2i}}{\partial z_i}\right)_0^k = \mu_{2i} a_{2i}^k b_{2i}^k \frac{g}{\sqrt{2g\left(z_{lpi} - z_{0i}^k\right)}}; \\
&\left(\frac{\partial G_{2i}}{\partial z_{nbi}}\right)_0^k = -\mu_{2i} a_{2i}^k b_{2i}^k \frac{g}{\sqrt{2g\left(z_{0i}^k - z_{lpi}^k\right)}}.
\end{aligned} \tag{2.11}$$

We transform the boundary conditions (2.11) to the form

$$\begin{aligned}
&Q_{0i}^{k+1} + \alpha_{0i}^{k} z_{0i}^{k+1} = \beta_{0i}^{k+1}, \\
&Q_{Ni}^{k+1} + \alpha_{Ni}^{k+1} z_{Ni}^{k+1} = \beta_{Ni}^{k+1},
\end{aligned} \tag{2.12}$$

where

$$\alpha_{0i}^{k} = -\left(\frac{\partial G_{1i}}{\partial z_i}\right)_0^k; \alpha_{Ni}^{k} = -\left(\frac{\partial G_{2i}}{\partial z_i}\right)_0^k;$$

$$\begin{aligned} \beta_{0i}^{k+1} &= Z_{0i}^{k} + \left(\frac{\partial G_{1i}}{\partial u_{1i}}\right)_0^k \left(u_{1i}^{k+1} - u_{1i}^{k}\right) + \\ &+ \left(\frac{\partial G_{1i}}{\partial z_{vbi}}\right)_0^k \left(z_{upi}^{k+1} - z_{upi}^{k}\right) - \left(\frac{\partial G_{1i}}{\partial z_i}\right)_0^k z_{0i}^{k}; \\ \beta_{Ni}^{k+1} &= Z_{Ni}^{k} + \left(\frac{\partial G_{2i}}{\partial u_{2i}}\right)_0^k \left(u_{2i}^{k+1} - u_{2i}^{k}\right) - \\ &- \left(\frac{\partial G_{2i}}{\partial z_i}\right)_0^k z_{Ni}^{k} + \left(\frac{\partial G_{2i}}{\partial z_{nbi}}\right)_0^k \left(z_{lpi}^{k+1} - z_{lpi}^{k}\right) \end{aligned} \tag{2.13}$$

Here

$$Z_{0i}^{k} = \mu_{1i} a_{1i}^{k} b_{1i}^{k} \sqrt{2g\left(z_{vbi}^{k} - z_{0i}^{k}\right)};$$

$$Z_{Ni}^{k} = \mu_{2i} a_{2i}^{k} b_{2i}^{k} \sqrt{2g\left(z_{Ni}^{k} - z_{nbi}^{k}\right)}.$$

Further, using the system of difference equations (2.8) and boundary conditions (2.11), we obtain the difference boundary conditions:

$$\begin{aligned} \mathbf{P}_0^k Z_0^{k+1} + \mathbf{R}_0^k Z_1^{k+1} &= \mathbf{w}_0^k, \\ \mathbf{R}_N^k Z_{N-1}^{k+1} - \mathbf{P}_N^k Z_N^{k+1} &= \mathbf{w}_N^k, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} \mathbf{P}_0^k &= \begin{bmatrix} b_{110}^k & b_{120}^k \\ 1 & \alpha_0^k \end{bmatrix}, \qquad \mathbf{R}_0^k = \begin{bmatrix} c_{110}^k & c_{120}^k \\ c_{210}^k & c_{220}^k \end{bmatrix}, \\ \mathbf{R}_N^k &= \begin{bmatrix} a_{11N}^k & a_{12N}^k \\ a_{21N}^k & a_{22N}^k \end{bmatrix}, \quad \mathbf{P}_N^k = \begin{bmatrix} 1 & \alpha_N^k \\ b_{21N}^k & b_{22N}^k \end{bmatrix}, \\ \mathbf{w}_0^k &= \begin{bmatrix} d_{10}^k \\ \beta_0^{k+1} \end{bmatrix}, \qquad \mathbf{w}_N^k = \begin{bmatrix} \beta_N^{k+1} \\ d_{2N}^k \end{bmatrix}. \end{aligned} \tag{2.15}$$

Here, for the initial alignment of the channel section, the coefficients of the first equation are taken as the coefficients of the first equation for the boundary conditions, and the coefficients of the second equation in the system of equations of the characteristic form of the unsteady motion of water are taken for the final alignment.

Equations (2.8) and (2.11) are a closed three-diagonal system of equations. They are written as follows

$$\begin{gathered}
\mathbf{P}_0^k Z_0^{k+1} + \mathbf{R}_0^k Z_1^{k+1} = \mathbf{w}_0^k, \\
\mathbf{P}_n^k \cdot Z_{n-1}^{k-1} + \mathbf{R}_n^k \cdot Z_n^{k+1} - \mathbf{P}_n^k \cdot Z_{n+1}^{k+1} = \mathbf{w}_n^k, \\
n = \overline{1, n-1}, k = 0, 1, ..., \\
\mathbf{P}_N^k Z_{N-1}^{k+1} - \mathbf{R}_N^k Z_N^{k+1} = \mathbf{w}_N^k,
\end{gathered} \tag{2.16}$$

This system of equations can also be written in general form

$$\sum_{j=0}^{N} \mathbf{A}_{ij}^k \cdot Z_j^{k+1} = \mathbf{w}_i^k, i = \overline{0, n}, k = 0, 1, ... \tag{2.17}$$

Here A_{ij} is an element of the matrix A, which itself is a matrix of dimension 2×2.

$$Z_i^{k+1} = \sum_{j=0}^{N} \mathbf{G}_{ij}^k \cdot \mathbf{w}_j^k, i = \overline{0, N}, k = 0, 1, ... \tag{2.18}$$

where G_{ij} is an element of the matrix G, inverse to matrix A.

Formula (2.18) can be written as

$$Z_{ni}^{k+1} = \sum_{j=0}^{N} \sum_{m=1}^{2} G_{nmij}^k \cdot w_{mj}^k, n = 1, 2, i = \overline{0, n}, k = 0, 1, ... \tag{2.19}$$

Here Z_{ij}^{k+1} and w_{mj}^k $n, m = 1, 2, i, j = \overline{1, n}$ when fixed are rectangular matrices of dimension $2 \times n$.

Expression (2.19) is conveniently written in tensor form

$$Q_{ni}^{k+1} = G_{nmij}^k \cdot w_{mj}^k, n = 1, 2, i = \overline{0, N}, k = 0, 1, ... \tag{2.20}$$

that is, the sum sign is released and the summation is performed on the paired indices.

To calculate the matrix G, it is necessary to calculate the inverse of the matrix A. One of the effective ways to calculate G_{ij}, the element of the matrix G is a matrix sweep that takes into account the three-diagonal structure of the system of equations (2.16).

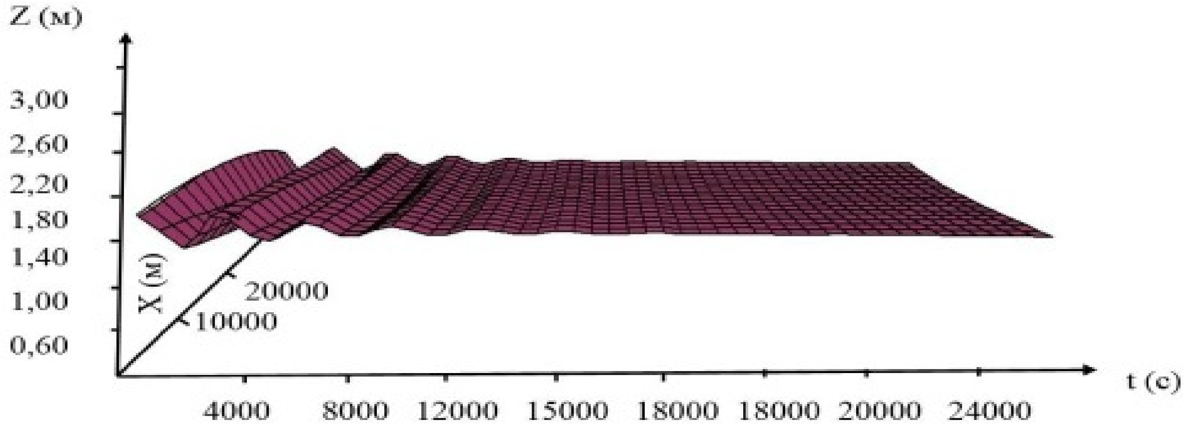

FIGURE 1. Changes in water levels over time and along the length of the main canal section.

For each fixed j $\left(j=\overline{1,n}\right)$, starting from the last point $i=n$ running coefficients are calculated

$$\mathbf{V}_N^k=\left(\mathbf{R}_N^k\right)^{-1}\cdot\mathbf{P}_N^k,$$

$$\mathbf{V}_n^k=\left(\mathbf{P}_n^k-\mathbf{R}_n^k\cdot\mathbf{V}_{n+1}^k\right)^{-1}\cdot\mathbf{R}_n^k,$$

$$\mathbf{W}_N^k=\left(\mathbf{R}_N^k\right)^{-1}\cdot\delta_{Nj},\tag{2.21}$$

$$\begin{gathered}\mathbf{W}_n^k=\left(\mathbf{P}_n^k+\mathbf{R}_n^k\cdot\mathbf{V}_{n+1}^k\right)^{-1}\times\\ \times\left(\delta_{1j}+\mathbf{R}_n^k\cdot\mathbf{W}_{n+1}^k\right),\end{gathered}$$

$$j=\overline{n-1,0}$$

After calculating V_i and W_i, we calculate G_{ij} on the formula

$$\begin{gathered}\mathbf{G}_{1j}^k=\left(\mathbf{R}_1^k+\mathbf{P}_1^k\mathbf{V}_2^k\right)^{-1}\cdot\left(\delta_{1j}-\mathbf{R}_1^k\cdot\mathbf{W}_2^k\right),\\ \mathbf{G}_{ij}^{k+1}=\mathbf{V}_i^k\cdot\mathbf{G}_{i-1j}^{k+1}+\mathbf{W}_i^k,\\ i=\overline{1,N}\end{gathered}\tag{2.22}$$

where $\delta_{ij}=\delta_{mn}\cdot\delta_{ij},\quad n,m=1,2;\ \ i,j=\overline{1,N}$.

Further, this procedure is repeated with a different j value. Thus, the calculated matrix G is remembered.

3. Results

Hydraulic and morphometris parameters of the $PC-145+00\div623+86$ main channel $Q_0=230\frac{m^3}{s}$; $H_0=6,59m$; $b_0=18m$; $y=\frac{1}{6}$; $l=224km$; $i=0,00005$; $B=66,9m$; $v_0=0,91\frac{m}{s}$; $Efficiency=0,9$.

Fig. 1, 2, shows the results of numerical experiments to determine changes in the level and flow of water in the channel section, which is equal to $22.4km$.

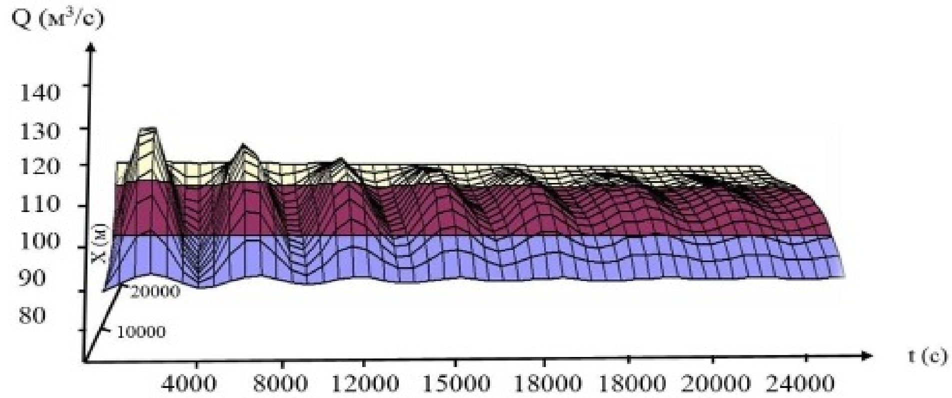

FIGURE 2. Changes in water consumption over time and along the length of the main channel section.

The figures show that, after opening the gates, the increased flow rate at the beginning of the channel section allows you to increase the water level along the length of the specified channel section. During $t = 24,615s$ ($41min$), the water level at the end of the section increases by $1.6m$.

The obtained results of numerical experiments show that the level and flow of water at the end of the channel section is stabilized, which is necessary for the water intake from the channel located there.

Conclusion

Thus, a unified algorithm for modeling channel sections using a system of differential equations of unsteady water motion was obtained.

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