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TWIN ZEROS AND TRIPLE ZEROS OF A HYPERLATTICE WITH RESPECT TO HYPERIDEALS

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Abstract. In this paper, we discuss the interrelations between 2-absorbing and weakly 2-absorbing hyperideals in join hyperlattices. We define twin zeros and triple zeros of a hyperideal and establish certain properties of triple zeros of weakly 2-absorbing hyperideal and twin zeros of weakly primary hyperideals in modular and distributive hyperlattices. As an application, we attempt to compute the probabilities of twin zeros and triple zeros in a lattice with respect to an ideal and provide examples.

1. Introduction

Algebraic hyperstructures are the classical generalizations of algebraic structures which has several applications in uncertainity theory [6], rough set theory [7], lattice based probability theories, analysis etc. Davvaz et.al.[5] extensively studied the chemical and biological applications of hyperstructures by exploring several inheritance examples of algebraic hyperstructures. This paper focusses on the occurences of twin zeros and triple zeros in Hyperlattices with respect to hyperideals. A lattice is a partially ordered set in which every pair of elements has a least upper bound (supremum or join) and a greatest lower bound (infimum or meet). Multilattice is a generalization of a lattice introduced by Benado [3]. They extended the concept of supremum and infimum to"multi" versions, allowing for the consideration of suprema and infima over multiple elements instead of just pairs. This provides a more flexible framework for dealing with larger collections of elements. A lattice can also be viewed as an algebraic structure with two binary operations: join (supremum) and meet (infimum). These operations are used to define the least upper bound and greatest lower bound of elements in the lattice, respectively. Konstantinidou [12], further generalized lattices by replacing the binary operations of join and meet with hyperoperations. However, with these generalizations some properties are not retained. Later, Konstantinidou [11] discussed the concept of distributivity of hyperlattices, particularly of P-hyperlattices. Rasouli and Davvaz [17] considered special relations on hyperlattices, called regular relations and showed that the quotient structure with respect to regular relations form a lattice. Rasouli and Davvaz [16] defined a topology on the spectrum of join hyperlattices and showed that it forms a T_0-space. Ameri [2] and others have explored the distributivity and dual distributivity of elements in a hyperlattice.

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Bideshki et. al. [4] studied prime hyperfilters in meet hyperlattices. Koguep and Lele [10] studied interrelation between the congruence relations, homomorphisms and hyperideals of a hyperlattice. Pallavi et al. [14] genralized prime hyperideals to 2-absorbing ideals and 2-absorbing primary and established some interrelations in meet hyperlattices. In [15], the authers studied hypervector spces. Section 2 of the paper deals with preliminary definitions related to join hyperlattices. Section 3 focuses on weakly 2 absorbing hyperideals and weakly primary ideals of a join hyperlattices. In section 4, as an application, we compute the probabilities of occurence of a twin zero and a triple zero in a lattice with respect to an ideal with suitable examples.

2. Preliminaries

Definition 2.1. [12] Let $\mathbb{H}$ be a non-empty set, and $\mathcal{P}^*(\mathbb{H}) = \{A \subseteq H : A \neq \emptyset\}$, $\bigvee : \mathbb{H} \times \mathbb{H} \to \mathcal{P}^*(\mathbb{H})$ be a hyperoperation, and $\wedge : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ be a binary operation. Then $(\mathbb{H}, \bigvee, \wedge)$ is a *join hyperlattice* if the following conditions hold:

(1) $l_1 \in l_1 \bigvee l_1$ and $l_1 = l_1 \wedge l_1$;
(2) $l_1 \bigvee (l_2 \bigvee l_3) = (l_1 \bigvee l_2) \bigvee l_3$ and $l_1 \wedge (l_2 \wedge l_3) = (l_1 \wedge l_2) \wedge l_3$;
(3) $l_1 \bigvee l_2 = l_2 \bigvee l_1$ and $l_1 \wedge l_2 = l_2 \wedge l_1$;
(4) $l_2 \in l_2 \wedge (l_1 \bigvee l_2) \cap l_2 \bigvee (l_1 \wedge l_2)$,

for all $l_1, l_2, l_3 \in \mathbb{H}$.

We define the relation '$\leq$' on $\mathbb{H}$ as follows:

$$l_1 \leq l_2 \text{ if and only if } l_1 \wedge l_2 = l_1.$$

Then $(\mathbb{H}, \leq)$ is a Poset.
Throughout, $(\mathbb{H}, \bigvee, \wedge)$ denotes a join hyperlattice.

Definition 2.2. [19] A non-empty subset J of $\mathbb{H}$ is called a *hyperideal* if

(1) $l_1, l_2 \in J$ implies $l_1 \bigvee l_2 \subseteq J$;
(2) $l_1 \in J, l_2 \in \mathbb{H}$ such that $l_2 \leq l_1$, then $l_2 \in J$, holds.

Definition 2.3. [19] A proper hyperideal J of $\mathbb{H}$ is said to be *prime* if $l_1, l_2 \in \mathbb{H}$ and $l_1 \wedge l_2 \in J$ implies $l_1 \in J$ or $l_2 \in J$.

Definition 2.4. [19] $\mathbb{H}$ is said to be

(1) *distributive* if $l_1 \wedge (l_2 \bigvee l_3) = (l_1 \wedge l_2) \bigvee (l_1 \wedge l_3)$;
(2) *s-distributive* if $l_1 \bigvee (l_2 \wedge l_3) = (l_1 \bigvee l_2) \wedge (l_1 \bigvee l_3)$,

for all $l_1, l_2, l_3 \in \mathbb{H}$, holds.

Theorem 2.5. [11] *Let $(L, \wedge, \vee)$ be a lattice and P a non-empty subset of L. We define a hyperoperation $\bigvee^P$ on L by*

$$l_1 \overset{P}{\bigvee} l_2 = l_1 \vee l_2 \vee P = \{l_1 \vee l_2 \vee p \,|\, p \in P\}.$$

Then $(L, \bigvee^P, \wedge)$ is a join hyperlattice if and only if for each $l_2 \in L$ there exists $p \in P$ such that $p \leq l_2$.

Example 2.6. Let $\mathbb{H} = D_{30} = \{1, 2, 3, 5, 6, 10, 15\}$, the set of all positive divisors of 30. The hyperoperation $\bigvee$ and the binary operation $\wedge$ on $\mathbb{H}$ are defined in Table 1. Then $(\mathbb{H}, \bigvee, \wedge)$ is a join hyperlattice.

$\bigvee$	1	2	3	5	6	10	15	1
1	$\{1\}$	$\{2\}$	$\{3\}$	$\{5\}$	$\{6\}$	$\{10\}$	$\{15\}$	$\{30\}$
2	$\{2\}$	$\{1,2\}$	$\{6\}$	$\{10\}$	$\{3,6\}$	$\{5,10\}$	$\{30\}$	$\{15,30\}$
3	$\{3\}$	$\{6\}$	$\{1,3\}$	$\{15\}$	$\{2,6\}$	$\{30\}$	$\{5,15\}$	$\{10,30\}$
5	$\{5\}$	$\{10\}$	$\{15\}$	$\{1,5\}$	$\{30\}$	$\{2,10\}$	$\{3,15\}$	$\{6,30\}$
6	$\{6\}$	$\{3,6\}$	$\{2,6\}$	$\{30\}$	$\{1,2,3,6\}$	$\{15,30\}$	$\{10,30\}$	$\{5,10,15,30\}$
10	$\{10\}$	$\{5,10\}$	$\{2,10\}$	$\{30\}$	$\{15,30\}$	$\{1,2,5,10\}$	$\{6,30\}$	$\{3,6,15,30\}$
15	$\{15\}$	$\{30\}$	$\{5,15\}$	$\{3,15\}$	$\{10,30\}$	$\{6,30\}$	$\{1,3,5,15\}$	$\{2,6,10,30\}$
30	$\{30\}$	$\{15,30\}$	$\{10,30\}$	$\{6,30\}$	$\{5,10,15,30\}$	$\{3,6,15,30\}$	$\{2,6,10,30\}$	$\mathbb{H}$

$\wedge$	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2	1	2	1	1	2	2	1	2
3	1	1	3	1	3	1	3	3
5	1	1	1	5	1	5	5	5
6	1	2	3	1	6	2	3	6
10	1	2	1	5	2	10	5	10
15	1	1	3	5	3	5	15	15
30	1	2	3	5	6	10	15	30

Table 1

Definition 2.7. [9] $\mathbb{H}$ is called *modular* if, for any $l_1, l_2, l_3 \in \mathbb{H}$, $l_1 \wedge l_3 = l_1$ implies $(l_1 \bigvee l_2) \wedge l_3 = l_1 \bigvee (l_2 \wedge l_3)$.

Inheritence examples for $\bigvee$-Hyperlattices: We construct the following $\bigvee$-hyperlattices from the inheritance examples of hypergroups given in [1]. Let "Parents" be denoted by p, "filial generation" be denoted by f and mating by $\times$.

Consider the monohybrid crossing of two varieties of pea plants, with two type of seed varieties viz round (RR genotype) and wrinkled (rr genotype). Then the offsprings obtained from this crossing will be represented as follows:

$$
\begin{array}{llcl}
P: & \text{Round} & \times & \text{Wrinkled} \\
 & (RR) & & (rr) \\
 & & \downarrow & \\
f_1: & & \text{Round} & \\
 & & (Rr) & \\
f_1 \times f_1: & \text{Round} & \times & \text{Round} \\
 & (Rr) & & (Rr) \\
 & & \downarrow & \\
f_2: & & (RR), (Rr), (Rr), (rr) &
\end{array}
$$

We construct the following hyperoperations given in the Table 2 on the set $L = \{R, W\}$ where R denotes the round genotype, and W denotes the wrinkled genotype. Then $(L, \bigvee, \wedge)$ is a hyperlattice.

$\bigvee$	R	W
R	$\{R, W\}$	$\{R\}$
W	$\{R\}$	$\{W\}$

$\wedge$	R	W
R	R	W
W	W	W

TABLE 2

3. Triple zeros and twin zeros of hyperideals

Definition 3.1. $J \in Id(\mathbb{H})$ is called *weakly prime* if whenever $0 \neq l_1 \wedge l_2$, $l_1 \wedge l_2 \in J$ implies $l_1 \in J$ or $l_2 \in J$.

Proposition 3.2. *Every prime hyperideal of* $\mathbb{H}$ *is weakly prime.*

Definition 3.3. [8] Let $J \in Id(\mathbb{H})$. J is called 2*-absorbing* if whenever $l_1, l_2, l_3 \in \mathbb{H}$ with $l_1 \wedge l_2 \wedge l_3 \in J$, then either $l_1 \wedge l_2 \in J$ or $l_2 \wedge l_3 \in J$ or $l_1 \wedge l_3 \in J$.

Definition 3.4. $J \in Id(\mathbb{H})$ is called *weakly* 2*-absorbing* if whenever $0 \neq l_1 \wedge l_2 \wedge l_3 \in J$, then $l_1 \wedge l_2 \in J$ or $l_2 \wedge l_3 \in J$ or $l_1 \wedge l_3 \in J$. We say that J is strictly weakly 2-absorbing if it is weakly 2-absorbing but not 2-absorbing.

Remark 3.5. Every 2-absorbing hyperideal of $\mathbb{H}$ is weakly 2-absorbing, and the converse need not be true.

Example 3.6. In Example 2.6, $J = \{1\}$ is a strictly weakly 2-absorbing hyperideal. In fact, $12 \wedge 10 \wedge 15 = 1$ but $12 \wedge 10 \notin I$, $15 \wedge 10 \notin I$ and $12 \wedge 15 \notin I$.

Definition 3.7. [8] For $J \in Id(\mathbb{H})$, we define the *radical* of J as the intersection of all prime hyperideals containing J and we denote it by $rad_{\vee}(J)$. If J is not contained in any prime hyperideal, then we take $rad_{\vee}(J) = \mathbb{H}$.

Definition 3.8. [8] $J \in Id(\mathbb{H})$ is called a *primary hyperideal* if whenever $l_1, l_2 \in \mathbb{H}$ and $l_1 \wedge l_2 \in J$, then $l_1 \in J$ or $l_2 \in rad_{\vee}(J)$.

Definition 3.9. $J \in Id(\mathbb{H})$ is said to be *weakly primary* if whenever $l_1, l_2 \in \mathbb{H}, 0 \neq l_1 \wedge l_2 \in J$ imply $l_1 \in J$ or $l_2 \in rad_{\vee}(J)$. We call J as strictly weakly primary if it is weakly primary but not primary.

Example 3.10. In Example 2.6, the hyperideal $I = \{1\}$ is a strictly weakly primary hyperideal.

Example 3.11. Let $\mathbb{H} = \{0, l_1, l_2, l_3, l_4, l_5, l_6, 1\}$. The hyperoperation $\bigvee$ and the binary operation $\wedge$ are represented by Figure 1.

Then $(\mathbb{H}, \bigvee, \wedge)$ is a join hyperlattice. Here, $J = \{0, l_1\}$ is a strictly weakly primary hyperideal with $rad_{\vee}(J) = \{0, l_1, l_2, l_4\}$. In fact, $l_2 \wedge l_3 \in J$, but $l_2 \notin J$ and $l_3 \notin rad_{\vee}(J)$.

Remark 3.12. Every primary hyperideal of $\mathbb{H}$ is weakly primary, and every weakly prime hyperideal of $\mathbb{H}$ is weakly primary.

Proposition 3.13. [19] *For every* $l_1, l_2 \in \mathbb{H}$*, there exists* $u \in l_1 \bigvee l_2$ *such that* $l_1 \leq u$.

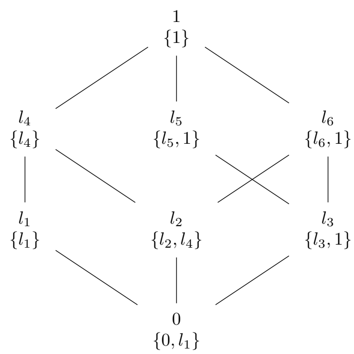

Figure 1

Definition 3.14. Let $J, J' \in Id(\mathbb{H})$ and $x \in \mathbb{H}$. We define,

$$[J : x] = \{l_1 \in \mathbb{H} : l_1 \wedge x \in J\}$$

$$[0 : x] = \{l_2 \in \mathbb{H} : x \wedge l_2 = 0\}$$

and

$$[J : J'] = \{l_1 \in \mathbb{H} : l_1 \wedge l_2 \in J \, \forall \, l_2 \in J'\}.$$

Proposition 3.15. *Let $J \in Id(\mathbb{H})$. Then $J \subsetneq \mathbb{H}$ is a weakly primary if and only if for any $x \in \mathbb{H} \setminus rad_{\vee}(J), [J : x] = J \cup [0 : x]$.*

Proof. Suppose that $J \in Id(\mathbb{H})$ is weakly primary. Assume that $0 \neq v \wedge x$. Since J is weakly primary, we have $v \in J$. Therefore, $[J : x] \subseteq J \cup [0 : x]$. On the other hand, by Proposition 3.13, $J \subseteq [J : x]$. Let $w \in [0 : x]$. Then $w \wedge x = 0$, and so $w \wedge x \in J$. Therefore, $[0 : x] \subseteq [J : x]$. Hence, $J \cup [0 : x] \subseteq [J : x]$.
Conversely, let $0 \neq x \wedge v$ and $(x \wedge v) \in J$ with $x \notin rad_{\vee}(J)$. Then $v \in [J : x] = J \cup [0 : x]$. Since $0 \neq v \wedge x$, $v \notin [0 : x]$. Therefore, $v \in J$, and hence, J is weakly primary. □

Definition 3.16. [2] An element $l \in \mathbb{H}$ is called *distributive*, if

$$l \wedge \left(l_1 \bigvee l_2\right) = (l \wedge l_1) \bigvee (l \wedge l_2),$$

for all $l_1, l_2 \in \mathbb{H}$ holds.

Proposition 3.17. *Let $J_1, J_2, J_3 \in Id(\mathbb{H})$. If $J_1 \subseteq J_1 \cup J_3$ then $J_1 \subset J_2$ or $J_1 \subseteq J_3$.*

Proposition 3.18. *Let $\emptyset \neq J \in Id(\mathbb{H})$ and $l \in \mathbb{H} \setminus rad_{\vee}(J)$ be a distributive element. If J is weakly primary then $[I : l] = I$ or $[I : l] = [0 : l]$.*

Proof. Suppose that J is a weakly primary hyperideal of $\mathbb{H}$, by Proposition 3.15, $[I : l] = I \cup [0 : l]$, and by Proposition 3.17, we get $[I : l] = I$ or $[I : l] = [0 : l]$. □

Definition 3.19. $J \in Id(\mathbb{H})$ is called *weakly 2-absorbing primary* if whenever $0 \neq l_1 \wedge l_2 \wedge l_3 \in J$ implies $l_1 \wedge l_2 \in J$ or $l_2 \wedge l_3 \in rad_{\vee}(J)$ or $l_1 \wedge l_3 \in rad_{\vee}(J)$.

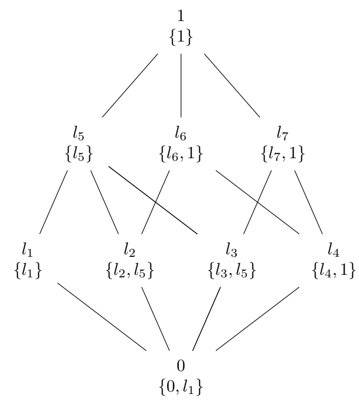

FIGURE 2

Proposition 3.20. *If $\mathbb{H}$ is distributive and $\emptyset \neq J \in Id(\mathbb{H})$, and $[J : l] = J$ or $[J : l] = [0 : l]$ for all $l \in \mathbb{H} \setminus rad_{\vee}(J)$, then J is weakly primary.*

Proof. Suppose that $0 \neq l \wedge l_1$ with $l \notin rad_{\vee}(J)$. Then $l_1 \in [J : l]$. Since $l \wedge l_1 \neq 0$, we get $[J : l] = J$, and so $l_1 \in J$. Hence, J is weakly primary. $\square$

Definition 3.21. Let $I \in Id(\mathbb{H})$. If there exist $l_1, l_2, l_3 \in \mathbb{H}$ with $l_1 \wedge l_2 \wedge l_3 = 0$ such that $l_1 \wedge l_2 \notin I$ and $l_2 \wedge l_3 \notin I$ and $l_1 \wedge l_3 \notin I$, then we call (l_1, l_2, l_3) as a *triple zero of* I.

Example 3.22. In Example 2.6, $(12, 10, 15)$ is a triple zero of the hyperideal $I = \{1\}$.

Example 3.23. Let $\mathbb{H} = \{0, l_1, l_2, l_3, l_4, l_5, l_6, l_7, 1\}$. The hyperoperation $\bigvee$ and the binary operation $\wedge$ are represented by Figure 2.

Then $(\mathbb{H}, \bigvee, \wedge)$ is a join hyperlattice. Here, $J = \{0, l_1\}$ is a strictly weakly 2-absorbing hyperideal. In fact, $l_5 \wedge l_6 \wedge l_7 \in J$, but $0 = l_5 \wedge l_6 \notin J$, $l_6 \wedge l_7 \notin J$ and $l_5 \wedge l_7 \notin J$, and so (l_5, l_6, l_7) is a triple zero of J.

Remark 3.24. If $I \in Id(\mathbb{H})$ is strictly weakly 2-absorbing, then I has a triple zero.

Remark 3.25. A hyperlattice $\mathbb{H}$ is modular if and only if $l_1 \wedge (l_2 \bigvee (l_1 \wedge l_3)) = (l_1 \wedge l_2) \bigvee (l_1 \wedge l_3)$.

Let $I_1, I_2, I_3 \in Id(\mathbb{H})$ and $l_1, l_2, l_3 \in \mathbb{H}$. Then we use the following notations:

(1) $I_1^2 = \{i_1 \wedge i_2 : i_1, i_2 \in I_1, i_1 \neq i_2\}$.
(2) $I_1^3 = \{i_1 \wedge i_2 \wedge i_3 : i_1, i_2, i_3 \in I_1, i_1 \neq i_2 \neq i_3\}$.
(3) $l_1 \wedge l_2 \wedge I_1 = \{l_1 \wedge l_2 \wedge i : i \in I_1\}$.
(4) $l_1 \wedge I_1^2 = \{l_1 \wedge i : i \in I_1^2\}$.
(5) $I_1^2 I_2 I_3 = \{i_1 \wedge i_2 \wedge i_3 : i_1 \in I_1^2, i_2 \in I_2, i_3 \in I_3\}$.
(6) $I_1^2 I_2^2 = \{i_1 \wedge i_2 : i_1 \in I_1^2, i_2 \in I_2^2\}$.

Theorem 3.26. *Let $\mathbb{H}$ be a modular join hyperlattice with $l \bigvee 0 = \{l\}$ $\forall\, l \in \mathbb{H}$, and I be a strictly weakly 2-absorbing hyperideal with a triple zero (l_1, l_2, l_3) in I. Then $l_1 \wedge l_2 \wedge I = l_2 \wedge l_3 \wedge I = l_1 \wedge l_3 \wedge I = \{0\}$.*

Proof. Suppose that $l_1 \wedge l_2 \wedge l_i \neq 0$ for some $l_i \in I$. Then $l_1 \wedge l_2 \wedge l_i \in I$. Now $\{l_1 \wedge l_2 \wedge l_i\} = 0 \bigwedge (l_1 \wedge l_2 \wedge l_i) = (l_1 \wedge l_2 \wedge l_3) \bigvee (l_1 \wedge l_2 \wedge l_i) \subseteq I$. Since $\mathbb{H}$ is modular, $(l_1 \wedge l_2 \wedge l_3) \bigvee (l_1 \wedge l_2 \wedge l_i) = (l_1 \wedge l_2) \wedge (l_3 \bigvee (l_1 \wedge l_2 \wedge l_i)) \subseteq I$. Now by Lemma 3.13, there exists $x \in l_3 \bigvee (l_1 \wedge l_2 \wedge l_i)$ such that $l_3 \leq x$. In particular, $0 \neq l_1 \wedge l_2 \wedge l_i = (l_1 \wedge l_2) \wedge x \in I$. As I is a weakly 2-absorbing hyperideal, and $l_1 \wedge l_2 \notin I$, we get $l_1 \wedge x \in I$ or $l_2 \wedge x \in I$. Since $l_3 \leq x$, we get $l_1 \wedge l_3 \leq l_1 \wedge x \in I$ or $l_2 \wedge l_3 \leq l_2 \wedge x \in I$, a contradiction. Thus, $l_1 \wedge l_2 \wedge I = \{0\}$. □

Proposition 3.27. *Let $\mathbb{H}$ be a distributive join hyperlattice with $l \bigvee 0 = \{l\}$ $\forall\, l \in \mathbb{H}$, I be a strictly weakly 2-absorbing hyperideal, and (l_1, l_2, l_3) be a triple zero of I. Then $l_1 \wedge I^2 = l_2 \wedge I^2 = l_3 \wedge I^2 = \{0\}$.*

Proof. Suppose that $l_1 \wedge l_i \wedge l_i' \neq 0$ for some $l_1 \neq l_i' \in I$. Now $\{0\} \neq l_1 \wedge l_i \wedge l_i' = (l_1 \wedge l_2 \wedge l_3) \bigvee l_1 \wedge l_i \wedge l_i'$. As $\mathbb{H}$ is distributive, it follows that

$$\begin{aligned}
[(l_1 \wedge l_2 \wedge l_3) \bigvee (l_1 \wedge l_2 \wedge l_i')] & \bigvee [(l_1 \wedge l_i \wedge l_3) \bigvee (l_1 \wedge l_i \wedge l_i')] \\
&= [(l_1 \wedge l_2) \wedge (l_3 \bigvee l_i')] \bigvee [(l_1 \wedge l_i) \wedge (l_3 \bigvee l_i')] \\
&= \{u \wedge l_1 \wedge l_2 : u \in l_3 \bigvee l_i'\} \bigvee \{v \wedge l_1 \wedge l_i : v \in l_3 \bigvee l_i'\} \\
&\supseteq \{x \wedge l_1 \wedge l_2 \bigvee u \wedge l_1 \wedge l_i : x \in l_3 \bigvee l_i'\} \\
&= \{x \wedge [(l_1 \wedge l_2) \bigvee (l_1 \wedge l_i)] : x \in l_3 \bigvee l_i'\} \\
&= \{x \wedge [l_1 \wedge (l_2 \bigvee l_i)] : x \in l_3 \bigvee l_i'\} \\
&= (l_3 \bigvee l_i') \wedge [l_1 \wedge (l_2 \bigvee l_i)]
\end{aligned}$$

Now, by Theorem 3.26, $l_1 \wedge l_2 \wedge l_i' = l_1 \wedge l_i \wedge l_3 = 0$, we get

$$[(l_1 \wedge l_2 \wedge l_3) \bigvee (l_1 \wedge l_2 \wedge l_i')] \bigvee [(l_1 \wedge l_i \wedge l_3) \bigvee (l_1 \wedge l_i \wedge l_i')] = \{l_1 \wedge l_i \wedge l_i'\}.$$

Therefore,

$$(l_3 \bigvee l_i') \wedge [l_1 \wedge (l_2 \bigvee l_i)] = \{l_1 \wedge l_i \wedge l_i'\}.$$

Further, by Lemma 3.13, there exists $x \in l_3 \bigvee l_i'$ and there exists $y \in l_2 \bigvee l_i$ such that $l_3 \leq x$ and $l_2 \leq y$. Now $x \wedge (l_1 \wedge y) = l_1 \wedge l_i \wedge l_i' \neq 0$ and as I is weakly 2-absorbing, we get either $x \wedge l_1 \in I$ or $x \wedge y \in I$ or $l_1 \wedge y \in I$. Since $l_1 \wedge l_2 \leq l_1 \wedge y, l_2 \wedge l_3 \leq x \wedge y$ and $l_1 \wedge l_3 \leq l_1 \wedge x$, we get $l_1 \wedge l_2 \in I$ or $l_1 \wedge l_3 \in I$ or $l_2 \wedge l_3 \in I$, a contradiction. Thus, $l_1 \wedge l_i \wedge l_i' = 0$. □

Proposition 3.28. *Let $\mathbb{H}$ be a distributive join hyperlattice with $l \bigvee 0 = \{l\}$ $\forall\, l \in \mathbb{H}$, I be a strictly weakly 2-absorbing hyperideal, and (l_1, l_2, l_3) a triple zero of I. Then $I^3 = \{0\}$.*

Proof. On a contrary, suppose that $l_i \wedge l_j \wedge l_k \neq 0$ for some $l_i, l_j, l_k \in I$. As $\mathbb{H}$ is distributive, we have

$$\begin{aligned}(l_1 \bigvee l_i) \wedge & (l_2 \bigvee l_j) \wedge (l_3 \bigvee l_k) \\ & \subseteq (l_1 \wedge l_2 \wedge l_3) \bigvee (l_1 \wedge l_3 \wedge l_j) \bigvee (l_2 \wedge l_3 \wedge l_i) \bigvee (l_1 \wedge l_2 \wedge l_k) \\ & \bigvee (l_2 \wedge l_i \wedge l_k) \bigvee (l_1 \wedge l_j \wedge l_k) \bigvee (l_3 \wedge l_i \wedge l_j) \bigvee (l_i \wedge l_j \wedge l_k).\end{aligned}$$

By Theorem 3.26, we have $l_1 \wedge l_3 \wedge l_j = l_2 \wedge l_3 \wedge l_i = l_1 \wedge l_2 \wedge l_k = 0$, and by Proposition 3.27, we get $l_2 \wedge l_i \wedge l_k = l_1 \wedge l_j \wedge l_k = l_3 \wedge l_i \wedge l_j = 0$. This gives $(l_1 \bigvee l_i) \wedge (l_2 \bigvee l_j) \wedge (l_3 \bigvee l_k) \subseteq \{l_i \wedge l_j \wedge l_k\}$. Then (by lemma 3.13) there exist $x \in l_1 \bigvee l_i, y \in l_2 \bigvee l_j, z \in l_3 \bigvee l_k$ such that $l_1 \leq x, l_2 \leq y, l_3 \leq z$ with $0 \neq l_i \wedge l_j \wedge l_k = x \wedge y \wedge z$. Since I is weakly 2-absorbing, we have $x \wedge y \in I$ or $y \wedge z \in I$ or $x \wedge z \in I$, and so $l_1 \wedge l_2 \in I$ or $l_2 \wedge l_3 \in I$ or $l_1 \wedge l_3 \in I$, a contradiction. □

Proposition 3.29. *Let I_1, I_2, I_3 be strictly weakly 2-absorbing ideals of a s-distributive hyperlattice $\mathbb{H}$ with $l \bigvee 0 = \{l\}$ $\forall\, l \in \mathbb{H}$. Then*

(1) $I_1^2 I_2 I_3 = I_1 I_2^2 I_3 = I_1 I_2 I_3^2 = \{0\}$,
(2) $I_1^2 I_2^2 = \{0\}$.

Proof. (1) We prove that $I_1^2 I_2 I_3 = \{0\}$ and the other cases are similar. On a contrary, suppose that $l_1 \wedge l_1' \wedge l_2 \wedge l_3 \neq 0$ for some $l_1, l_1' \in I_1, l_2 \in I_2, l_3 \in I_3$. Then $l_1 \wedge l_1' \neq 0$. As I_1 is strictly weakly 2-absorbing, by Remark 3.24, I_1 has a triple zero say (l_i, l_j, l_k). Now $(l_1 \wedge l_j \wedge l_k) \bigvee (l_1 \wedge l_1') = \{l_1 \wedge l_1'\}$. Since $\mathbb{H}$ is s-distributive,

$$\begin{aligned}\{l_1 \wedge l_1'\} &= (l_i \wedge l_j \wedge l_k) \bigvee (l_1 \wedge l_1') \\ &= [l_i \bigvee (l_1 \wedge l_1')] \wedge [l_j \bigvee (l_1 \wedge l_1')] \wedge [l_k \bigvee (l_1 \wedge l_1')].\end{aligned}$$

Then by Lemma 3.13, there exist $x \in l_i \bigvee (l_1 \wedge l_1'), y \in l_j \bigvee (l_1 \wedge l_1'), z \in l_k \bigvee (l_1 \wedge l_1')$ such that $l_i \leq x, l_j \leq y$ and $l_k \leq z$. Now $0 \neq x \wedge y \wedge z = l_1 \wedge l_1' \in I_1$ and I_1 is weakly 2-absorbing, either $x \wedge y \in I_1$ or $x \wedge z \in I_1$ or $y \wedge z \in I_1$. And so $l_i \wedge l_j \in I_1$ or $l_i \wedge l_k \in I_1$ or $l_j \wedge l_k \in I_1$, a contradiction.

(2) Suppose that $l_1 \wedge l_1' \wedge l_2 \wedge l_2' \neq 0$ for some $l_1, l_1' \in I_1, l_2, l_2' \in I_2$. Then $l_1 \wedge l_1' \neq 0$. As I_1 is strictly weakly 2-absorbing, it has a triple zero say (l_i, l_j, l_k). Now $(l_1 \wedge l_j \wedge l_k) \bigvee (l_1 \wedge l_1') = \{l_1 \wedge l_1'\}$. Since $\mathbb{H}$ is strongly distributive,

$$\begin{aligned}\{l_1 \wedge l_1'\} &= (l_i \wedge l_j \wedge l_k) \bigvee (l_1 \wedge l_1') \\ &= [l_i \bigvee (l_1 \wedge l_1')] \wedge [l_j \bigvee (l_1 \wedge l_1')] \wedge [l_k \bigvee (l_1 \wedge l_1')]\end{aligned}$$

Now by similar argument as in the first part we obtain $l_i \wedge l_j \in I_1$ or $l_i \wedge l_k \in I_1$ or $l_j \wedge l_k \in I_1$, a contradiction. Therefore, $l_1 \wedge l_1' \wedge l_2 \wedge l_3 = 0$. □

Definition 3.30. Let $J \in Id(\mathbb{H})$ and $l_1, l_2 \in \mathbb{H}$. We call (l_1, l_2) as twin zero of J if $l_1 \wedge l_2 = 0$ but $l_1 \notin J$ and $l_2 \notin rad_{\vee}(J)$.

Remark 3.31. A strictly weakly primary hyperideal has a twin-zero.

Example 3.32. In Example 2.6, $(2,5)$ is a twin zero of the hyperideal $I = \{1\}$.

Theorem 3.33. *Let J be a strictly weakly primary hyperideal of a modular hyperlattice $\mathbb{H}$ with $l \bigvee 0 = \{l\}$ for all $l \in \mathbb{H}$, and (l_1, l_2) a twin zero of J. Then $l_1 \wedge J = \{0\}$.*

Proof. Suppose that $l_1 \wedge J \neq \{0\}$. Then $l_1 \wedge j \neq 0$ for some $j \in J$. Now $l_1 \wedge j = (l_1 \wedge l_2) \bigvee (l_1 \wedge j) \subseteq J$. Since $\mathbb{H}$ is modular, $\{l_1 \wedge j\} = l_1 \wedge (l_2 \bigvee (l_1 \wedge j)) \subseteq J$. By Lemma 3.13, there exists $u \in l_2 \bigvee (l_1 \wedge j)$ with $l_2 \leq u$. Now $l_1 \wedge j = l_1 \wedge u \in J$. Since $0 \neq l_1 \wedge u \in J$, $l_1 \notin J$, we get $u \in rad_{\vee}(J)$, and hence, $l_2 \in rad_{\vee}(J)$, a contradiction. Thus, $l_1 \wedge J = \{0\}$. □

4. Computing probabilities of twin zeros and triple zeros

In this section, we compute the probabilities of occuring a twin zero and a triple zero in a lattice with respect to an ideal. Let $|L|$ denotes the number of elements in the lattice L.

Definition 4.1. [21] Let J be an ideal of a lattice L (with zero) and $l_1, l_2 \in L$. We call (l_1, l_2) as twin zero of J if $l_1 \wedge l_2 = 0$ but $l_1 \notin J$ and $l_2 \notin rad(J)$.

Definition 4.2. [13] Let J be an ideal of a lattice L (with zero) and $l_1, l_2, l_3 \in L$. Then (l_1, l_2, l_3) is s triple zero of J if $l_1 \wedge l_2 \wedge l_3 = 0$ but $l_1 \wedge l_2 = 0 \notin J$ and $l_2 \wedge l_3 = 0 \notin J$ and $l_1 \wedge l_3 = 0 \notin J$.

Definition 4.3. Let L be a lattice and J be an ideal of L. Then

(1) probability of getting a twin zero in L with respect to J is defined as

$$P_J(\text{twin zero}) = \frac{|\{(l_1, l_2) : l_1 \wedge l_2 = 0, l_1 \notin J, l_2 \notin rad(J)\}|}{|L \times L|}$$

(2) probability of getting a triple zero in L with respect to an ideal J is defined as

$$P_J(\text{triple zero}) = \frac{|\{(l_1, l_2, l_3) : l_1 \wedge l_2 \wedge l_3 = 0, l_1 \wedge l_2 \notin J, l_2 \wedge l_3 \notin J, l_1 \wedge l_3 \notin J\}|}{|L \times L \times L|}$$

Remark 4.4. If J is a primary ideal of a latttice, then $P_J(\text{twin tero}) = 0$.

The following remark can be observed from the definition. However, we illustrate it in Example 4.6.

Remark 4.5. $P_J(\text{twin zero}) \leq \dfrac{|J^c \times (rad(J))^c|}{|L \times L|}$, and so $P_J(\text{twin zero}) < 1$.

Example 4.6. Consider the lattice given in Figure 3. Here $(l_1, l_3), (l_1, l_5), (l_4, l_3)$ and (l_4, l_6) are twin zeros of L with respect to the ideal $J = \{0, l_2\}$, where $rad(J) = \{0, l_1, l_2, l_4\}$. Hence, $P_J(\text{twin zero}) = \frac{4}{64} < \dfrac{6 \times 4}{64} = \dfrac{|J^c \times (rad(J))^c|}{|L \times L|}$. For $I = \{0, l_1\}, (rad(I) = \{0, l_1, l_2, l_4\})$, twin zeros with respect to J are $(l_4, l_5), (l_4, l_6)$, (l_4, l_3), and so $P_I(\text{twin zero}) = \dfrac{3}{64}$. As $I \vee J = \{0, l_1, l_2, l_4\}$, a prime ideal,

$P_{I\vee J}(\text{twin zero}) = 0$. We have $rad(I \cap J) = \{0, l_1, l_2, l_4\}$, and twin zeros of $I \cap J$ are (l_6, l_4), (l_6, l_1), (l_3, l_4), (l_3, l_1). $P_{I\cap J}(\text{twin zero}) = \dfrac{4}{64}$. Thus we have $P_{I\vee J}(\text{twin zero}) \leq P_I(\text{twin zero}) + P_J(\text{twin zero}) - P_{I\cap J}(\text{twin zero})$.

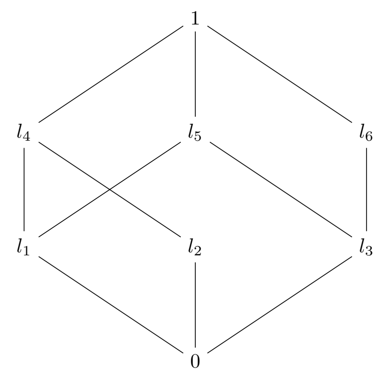

FIGURE 3

Example 4.7. Consider the lattice given in Figure 4. Here $\{(x_1, x_2, x_3) : x_i \in T,\ x_i \neq x_j \text{ for } i \neq j, \}$ (where $T = \{l_4, l_5, l_6\}$) are the triple zeros of L with respect to the ideal $J = \{0\}$. Therefore $P_J(\text{triple zero}) = \frac{6}{8^3}$.

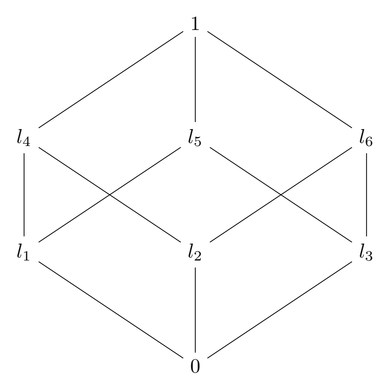

FIGURE 4

The following remark can be observed from the above examples.

Remark 4.8. Let L be a lattice and I, J are ideals of L such that $I \subseteq J$. Then

(1) $P_J(\text{twin zero}) \leq P_I(\text{twin zero})$
(2) $P_{I\vee J}(\text{twin zero}) \leq \dfrac{1}{2}\left(P_I(\text{twin zero}) + P_J(\text{twin zero})\right)$

Remark 4.9. If I is an ideal in a lattice L and the corresponding $P-$hyperlattice, then twin zeros in L and the $P-$hyperlattice with respect to I will coincide. Moreover, the probability of getting a twin zero with respect to I on both structures will be equal.

Open question

Let L be a lattice.

(1) Can we attain a bound for the probability of twin zeros and triple zeros in a distributive lattice or a modular lattice with respect to an ideal?
(2) In general, for any two ideals I and J of L, does it hold that

$$P_{I\vee J}(\text{twin zero}) \leq P_I(\text{twin zero}) + P_J(\text{twin zero}) - P_{I\wedge J}(\text{twin zero})?$$

Conclusion

In this paper, we have considered the generalization of lattices as join hyperlattices, in which the interrelations between 2-absorbing and weakly 2-absorbing hyperideals are established. Later, the concepts like twin zeros and triple zeros of a hyperideal are defined with suitable illustrations. The properties of triple zeros of weakly 2-absorbing hyperideal and twin zeros of weakly primary hyperideals in modular and distributive hyperlattices are proved. We have computed the probabilities for the occurence of twin zeros and triple zeros in a lattice with respect to an ideal. As a future scope one can extend the notions of essential elements and superflous elements discussed in [18, 20] to hyperlattices.

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