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NEW SUBCLASS F ANALYTIC FUNCTIONS ASSOCIATED WITH RAPID OPERATOR

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ABSTRACT. The object of the present paper is to investigate a new subclass of analytic functions which are defined by means of a Rapid operator. Some results connected to coefficient estimates, growth and distortion theorems, radii of starlikeness, convexity close-to-convexity and integral means inequalities related to the subclass is obtained.

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1. Introduction

Let A denote the class of functions u of the form

$$u(z) = z + \sum_{\eta=2}^{\infty} a_n z^{\eta} \tag{1.1}$$

which are analytic in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$

A function u in the class A is said to be in the class $ST(\alpha)$ of starlike functions of order α in E, if it satisfy the inequality

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \alpha, \ (0 \le \alpha < 1), z \in E \tag{1.2}$$

Note that $ST(0) = ST$ is the class of Starlike functions.
Denote by T the subclass of A consisting of functions u of the form

$$u(z) = z - \sum_{\eta=2}^{\infty} a_{\eta} z^{\eta}, \ (a_{\eta} \ge 0) \tag{1.3}$$

This subclass was introduced and extensively studied by Silvermann[4].
Let u be a function in the class A. Recently, Atshan and Buti [1] introduced a Rapid operator of $f \in \Re$ for $0 \le \lambda < 1$ and $0 \le \nu < 1$. It is denoted by $G_{\lambda}^{\nu}u(z)$ and defines follows.

$$G_{\lambda}^{\nu}u(z) = \frac{1}{(1-\lambda)^{\nu+1}\Gamma(\nu+1)} \int_0^{\infty} t^{\lambda-1} e^{-(\frac{t}{1-\lambda})} u(zt)dt \tag{1.4}$$

Thus $u(z)$ has of the form(1.1),then it follows from (1.4)

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$$G^{\nu}_{\lambda}u(z) = z + \sum_{\eta=2}^{\infty} \chi_{\eta}(\lambda,\nu) a_{\eta} z^{\eta} \tag{1.5}$$

where

$$\chi_{\eta}(\lambda,\nu) = (1-\lambda)^{\nu-1} \frac{\Gamma(\eta+\nu)}{\Gamma(\nu+1)}$$

Now we define the following new subclass motivated by Murugusunderamoorthy and Magesh [3]

Definition 1.1. The function u(z) of the form (1.1) is in the class $S^{\nu}_{\lambda}(\mu,\gamma,\varrho)$, if it satisfies the inequality

$$\Re\left\{\frac{z(G^{\nu}_{\lambda}u(z))'}{(1-\mu)z+\mu G^{\nu}_{\lambda}u(z)} - \gamma\right\} > \varrho\left|\frac{z(G^{\nu}_{\lambda}u(z))'}{(1-\mu)z+\mu G^{\nu}_{\lambda}u(z)} - 1\right|$$

for $0 \leq \lambda \leq 1$, $0 \leq \gamma \leq 1$ and $\varrho \geq 0$.

Further we define $TS^{\nu}_{\lambda}(\mu,\gamma,\varrho) = S^{\nu}_{\lambda}(\mu,\gamma,\varrho) \cap T$

The aim of present paper is to study the coefficient bounds, radii of close-to-convex and starlikeness convex linear combinations and integral means inequalities of the $TS^{\nu}_{\lambda}(\mu,\gamma,\varrho)$

2. Coefficient bounds

Theorem 2.1. *A function u(z) of the form* (1.1) *is in* $S^{\nu}_{\lambda}(\mu,\gamma,\varrho)$, *then*

$$\sum_{\eta=2}^{\infty}[(1+\varsigma)\eta - \mu(\gamma+\varsigma)]\chi_{\eta}(\lambda,\nu)|a_{\eta}| \leq 1-\gamma \tag{2.1}$$

where $0 \leq \mu \leq 1$, $0 \leq \gamma \leq 1$,$\varrho \geq 0$ *and* $\chi_{\eta}(\lambda,\nu)$ *is given by (1.5).*

Proof. It suffices to show that

$$\varrho\left|\frac{z(G^{\nu}_{\lambda}u(z))'}{(1-\mu)z+\mu G^{\nu}_{\lambda}u(z)} - 1\right| - \Re\left\{\frac{z(G^{\nu}_{\lambda}u(z))'}{(1-\mu)z+\mu G^{\nu}_{\lambda}u(z)} - 1\right\}$$
$$\leq 1-\gamma$$

We have

$$\varrho\left|\frac{z(G_\lambda^\nu u(z))'}{(1-\mu)z+\mu G_\lambda^\nu u(z)}-1\right|-\Re\left\{\frac{z(G_\lambda^\nu u(z))'}{(1-\mu)z+\mu G_\lambda^\nu u(z)}-1\right\}$$
$$\leq(1+\varrho)\left|\frac{z(G_\lambda^\nu u(z))'}{(1-\mu)z+\mu G_\lambda^\nu u(z)}-1\right|$$
$$\leq\frac{(1+\varrho)\sum\limits_{\eta=2}^{\infty}(\eta-\mu)\chi_\eta(\lambda,\nu)|a_\eta||z|^{\eta-1}}{1-\sum\limits_{\eta=2}^{\infty}\mu\chi_\eta(\lambda,\nu)|a_\eta||z|^{\eta-1}}$$
$$\leq\frac{(1+\varrho)\sum\limits_{\eta=2}^{\infty}(\eta-\mu)\chi_\eta(\lambda,\nu)|a_\eta|}{1-\sum\limits_{\eta=2}^{\infty}\mu\chi_\eta(\lambda,\nu)|a_\eta|}$$

The last expression is bounded above by $(1-\gamma)$, if

$$\sum_{\eta=2}^{\infty}[(1+\varrho)\eta-\mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)|a_\eta|\leq 1-\gamma$$

and the proof is complete. □

Theorem 2.2. *Let $0\leq\mu\leq 1$, $0\leq\gamma\leq 1$, and $\varrho\geq 0$ then a function u of the form (1.3) to be in the class $TS_\lambda^\nu(\mu,\gamma,\varrho)$ if and only if*

$$\sum_{\eta=2}^{\infty}[(1+\varrho)\eta-\mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)|a_\eta|\leq 1-\gamma \tag{2.2}$$

where $\chi_\eta(\lambda,\nu)$ is given by (1.5).

Proof. In view of Theorem (2.1) we need only to prove the necssity. If $u\in TS_\lambda^\nu(\mu,\gamma,\varrho)$ and z is real, then

$$\Re\left\{\frac{1-\sum\limits_{\eta=2}^{\infty}\eta\chi_\eta(\lambda,\nu)a_\eta z^{\eta-1}}{1-\sum\limits_{\eta=2}^{\infty}\mu\chi_\eta(\lambda,\nu)a_\eta z^{\eta-1}}-\gamma\right\}>\left|\frac{\sum\limits_{\eta=2}^{\infty}(\eta-\mu)\chi_\eta(\lambda,\nu)a_\eta z^{\eta-1}}{1-\sum\limits_{\eta=2}^{\infty}\mu\chi_\eta(\lambda,\nu)a_\eta z^{\eta-1}}\right|$$

Letting $z\to 1$ along the real axis, we obtain the desired inequality

$$\sum_{\eta=2}^{\infty}[(1+\varrho)\eta-\mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)|a_\eta|\leq 1-\gamma$$

where $0\leq\mu\leq 1$, $0\leq\gamma\leq 1$, $\varrho\geq 0$ and $\chi_\eta(\lambda,\nu)$ is given by (1.5). □

Corollary 2.3. *If $u(z)\in TS_\lambda^\nu(\mu,\gamma,\varrho)$, then*

$$|a_\eta|\leq\frac{1-\gamma}{[(1+\varrho)\eta-\mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)} \tag{2.3}$$

where $0 \leq \mu \leq 1$, $0 \leq \gamma \leq 1$, $\varrho \geq 0$ and $\chi_\eta(\lambda,\nu)$ is given by (1.5). Equality holds for the function

$$u(z) = z - \frac{1-\gamma}{[(1+\varrho)\eta - \mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)} z^\eta \tag{2.4}$$

Theorem 2.4. *Let $u_1(z) = z$ and*

$$u_\eta(z) = z - \frac{1-\gamma}{[(1+\varrho)\eta - \mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)} z^\eta, \ \eta \geq 2 \tag{2.5}$$

Then $u(z) \in TS_\lambda^\nu(\mu,\gamma,\varrho)$, if and only if, it can be expressed in the form

$$u(z) = \sum_{\eta=1}^{\infty} w_\eta u_\eta(z), \ w_\eta \geq 0, \ \sum_{\eta=1}^{\infty} w_\eta = 1 \tag{2.6}$$

Proof. Suppose $u(z)$ can be written as in (2.6), then

$$u(z) = z - \sum_{\eta=2}^{\infty} w_\eta \frac{1-\gamma}{[(1+\varrho)\eta - \mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)} z^\eta$$

Now,

$$\sum_{\eta=2}^{\infty} w_\eta \frac{(1-\gamma)[(1+\varrho)\eta - \mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)}{(1-\gamma)[(1+\varrho)\eta - \mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)} = \sum_{\eta=2}^{\infty} w_\eta = 1 - w_1 \leq 1$$

Thus $u(z) \in TS_\lambda^\nu(\mu,\gamma,\varrho)$.
Conversely, let $u(z) \in TS_\lambda^\nu(\mu,\gamma,\varrho)$, then by using (2.3), we get

$$w_\eta = \frac{[(1+\varrho)\eta - \mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)}{(1-\gamma)} a_\eta, \ \eta \geq 2$$

and $w_1 = 1 - \sum\limits_{\eta=2}^{\infty} w_\eta$. Then we have $u(z) = \sum\limits_{\eta=1}^{\infty} w_\eta u_\eta(z)$ and hence this completes the proof of Theorem. □

Theorem 2.5. *The class $TS_\lambda^\nu(\mu,\gamma,\varrho)$ is a convex set.*

Proof. Let the function

$$u_j(z) = z - \sum_{\eta=2}^{\infty} a_{\eta,j} z^\eta, \ a_{\eta,j} \geq 0, \ j = 1,2 \tag{2.7}$$

be in the class $TS_\lambda^\nu(\mu,\gamma,\varrho)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \xi u_1(z) + (1-\xi) u_2(z), \ 0 \leq \xi < 1,$$

in the class $TS_\lambda^\nu(\mu,\gamma,\varrho)$. Since

$$h(z) = z - \sum_{\eta=2}^{\infty} [\xi a_{\eta,1} + (1-\xi) a_{\eta,2}] z^\eta,$$

An easy computation with the aid of Theorem (2.2) gives

$$\sum_{\eta=2}^{\infty}[(1+\varrho)\eta-\mu(\gamma+\varrho)]\xi\chi_\eta(\lambda,\nu)a_{\eta,1}+$$

$$\sum_{\eta=2}^{\infty}[(1+\varrho)\eta-\mu(\gamma+\varrho)](1-\xi)\chi_\eta(\lambda,\nu)a_{\eta,2}$$
$$\leq \xi(1-\gamma)+(1-\xi)(1-\gamma)$$
$$\leq (1-\gamma)$$

which implies that $h \in TS_\lambda^\nu(\mu,\gamma,\varrho)$
Hence $TS_\lambda^\nu(\mu,\gamma,\varrho)$ is convex.

□

3. Radii of Close-to-Convexity, Starlikeness and Convexity

In this section, we obtain the radii of close-to-convexity, starlikeness and convexity for the class $TS_\lambda^\nu(\mu,\gamma,\varrho)$.

Theorem 3.1. *Let the function $u(z)$ defined by (1.3) belong to the class $TS_\lambda^\nu(\mu,\gamma,\varrho)$, then $u(z)$ is close-to-convex of order $\delta(0\leq\delta<1)$ in the disc $|z|<r_1$, where*

$$r_1=\inf_{\eta\geq 2}\left[\frac{(1-\delta)\sum\limits_{\eta=2}^{\infty}[(1+\varrho)\eta-\mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)}{\eta(1-\gamma)}\right]^{1/\eta-1},\eta\geq 2 \tag{3.1}$$

The result is sharp, with the external function $u(z)$ is given by (2.5)

Proof. Given $u\in T$ and u is close-to-convex of order δ, we have

$$|f'(z)-1|<1-\delta \tag{3.2}$$

For the left hand side of (3.2), we have

$$|u'(z)-1|\leq\sum_{\eta=2}^{\infty}\eta a_\eta|z|^{\eta-1}$$

The last expression is less than $1-\delta$

$$\sum_{\eta=2}^{\infty}\frac{\eta}{1-\delta}a_\eta|z|^{\eta-1}\leq 1$$

Using the fact, that $u(z)\in TS_\lambda^\nu(\mu,\gamma,\varrho)$ if and only if

$$\sum_{\eta=2}^{\infty}\frac{[(1+\varrho)\eta-\mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)}{1-\gamma}a_\eta\leq 1$$

We can see that (3.2) is true, if

$$\frac{\eta}{1-\delta}|z|^{\eta-1}\leq\frac{[(1+\varrho)\eta-\mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)}{1-\gamma}$$

or, equivalently

$$|z| \leq \left\{ \frac{(1-\delta)[(1+\varrho)\eta - \mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)}{\eta(1-\gamma)} \right\}^{1/\eta-1}$$

which completes the proof. □

Theorem 3.2. *Let the function $u(z)$ defined by (1.3) belong to the class $TS^{\nu}_{\lambda}(\mu,\gamma,\varrho)$. Then $u(z)$ is starlike of order $\delta (0 \leq \delta < 1)$ in the disc $|z| < r_2$, where*

$$r_2 = \inf_{\eta \geq 2} \left[\frac{(1-\delta) \sum_{\eta=2}^{\infty} [(1+\varrho)\eta - \mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)}{(\eta-\delta)(1-\gamma)} \right]^{1/\eta-1} \tag{3.3}$$

The result is sharp, with external function $u(z)$ is given by (2.5)

Proof. Given $u \in T$ and u is starlike of order δ, we have

$$\left| \frac{zu'(z)}{u(z)} - 1 \right| < 1 - \delta \tag{3.4}$$

For the left hand side of (3.4), we have

$$\left| \frac{zu'(z)}{u(z)} - 1 \right| \leq \sum_{\eta=2}^{\infty} \frac{(\eta-1)a_\eta |z|^{\eta-1}}{1 - \sum_{\eta=2}^{\infty} a_\eta |z|^{\eta-1}}$$

The last expression is less than $1-\delta$ if

$$\sum_{\eta=2}^{\infty} \frac{\eta-\delta}{1-\delta} a_\eta |z|^{\eta-1} < 1$$

Using the fact that $u(z) \in TS^{\nu}_{\lambda}(\mu,\gamma,\varrho)$ if and only if

$$\sum_{\eta=2}^{\infty} \frac{[(1+\varrho)\eta - \mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)}{1-\gamma} a_\eta \leq 1$$

We can say (3.4) is true, if

$$\sum_{\eta=2}^{\infty} \frac{\eta-\delta}{1-\delta} |z|^{\eta-1} \leq \frac{[(1+\varrho)\eta - \mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)}{1-\gamma}$$

or equivalently

$$|z|^{\eta-1} \leq \frac{(1-\delta)[(1+\varrho)\eta - \mu(\gamma+\varrho)]\chi_\eta(\lambda,\nu)}{(\eta-\delta)(1-\gamma)}$$

which yields the starlikeness of the family. □

4. Integral Means Inequalities

In [4], Silverman found that the function $u_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality conjunctured [5] and setteled in [6], that

$$\int_0^{2\pi} |u(re^{i\varphi})|^\tau d\varphi \leq \int_0^{2\pi} |u_2(re^{i\varphi})^\eta|^\tau d\varphi$$

for all $u \in T$, $\tau > 0$ and $0 < r < 1$. In [6], he also proved his conjuncture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T.

Now, we prove Silverman's conjecture for the class of functions $TS_\lambda^\nu(\mu, \gamma, \varrho)$
We need the concept of subordination between analytic functions and a subordination theorem of Littlewood [2].

Two functions u and v, which are analytic in E, the function u is said to be subordinate to v in E, if there exists a function w analytic in E with $w(0) = 0$, $|w(z) < 1$, $(z \in E)$ such that $u(z) = v(w(z))$, $(z \in E)$. We denote this subordination by $u(z) \prec v(z)$. ($\prec$ denote subordination)

Lemma 4.1. *If the function u and v are analytic in E with $u(z) \prec v(z)$, then for $\tau > 0$ and $z = re^{i\varphi}$, $0 < r < 1$*

$$\int_0^{2\pi} |v(re^{i\varphi})|^\tau d\varphi \leq \int_0^{2\pi} |u(re^{i\varphi})|^\tau d\varphi$$

Now, we discuss the integral means inequalities for functions u in $TS_\lambda^\nu(\mu, \gamma, \varrho)$

Theorem 4.2. *$u \in TS_\lambda^\nu(\mu, \gamma, \varrho)$, $0 \leq \mu < 1$, $0 \leq \gamma < 1$, $\varrho \geq 0$ and $u_2(z)$ be defined by*

$$u_2(z) = z - \frac{1-\gamma}{\phi_2(\lambda, \gamma, \varrho)} z^2 \tag{4.1}$$

Proof. For $u(z) = z - \sum_{\eta=2}^{\infty} a_\eta z^\eta$, (4.1) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{\eta=2}^{\infty} a_\eta z^{\eta-1}\right|^\tau d\varphi \leq \int_0^{2\pi} \left|1 - \frac{1-\gamma}{\varphi_2(\lambda, \gamma, \varrho)} z\right|^\tau d\varphi$$

By Lemma (4.1), it is enough to prove that

$$1 - \sum_{\eta=2}^{\infty} a_\eta z^{\eta-1} \prec 1 - \frac{1-\gamma}{\varphi_2(\lambda, \gamma, \varrho)} z,$$

Assuming

$$1 - \sum_{\eta=2}^{\infty} a_\eta z^{\eta-1} \prec 1 - \frac{1-\gamma}{\varphi_2(\lambda, \gamma, \varrho)} w(z),$$

and using (2.2), we obtain

$$|w(z)| = \left|\sum_{\eta=2}^{\infty} \frac{\varphi_2(\lambda,\gamma,\varrho)}{1-\gamma} a_\eta z^{\eta-1}\right| \leq |z| \sum_{\eta=2}^{\infty} \frac{\varphi_2(\lambda,\gamma,\varrho)}{1-\gamma} a_\eta \leq |z|$$

where

$$\varphi_\eta(\lambda,\gamma,\varrho) = [(1+\varrho)\eta - \mu(\gamma+\varrho)\chi_\eta(\lambda,\nu)$$

This completes the proof □

5. Conclusion

This research has introduced a new linear operator related to Ananlytic function and studied some basic properties of geometric function theory . Accordingly, some results related to closure theorems have also been considered, inviting future research for this field of study.

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