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MAXIMUM REVERSE DEGREE ENERGY OF SUBGRAPH COMPLEMENT OF GRAPHS

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Abstract. The subgraph complement of a graph G with respect to a set S is the graph obtained from G by removing the edges of induced subgraph $\langle S\rangle$ and adding edges which are not in $\langle S\rangle$ of G. In this paper we introduce the concept of maximum reverse degree energy of connected subgraph complements of a graph. Few properties on maximum reverse degree eigenvalues and bounds for maximum reverse degree energy of connected subgraph complement of a graph are achieved. Further maximum reverse degree energy of connected subgraph complement of some families of graphs are computed.

1. Introduction

Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \dots, v_n\}$ as its vertex set and $E = \{e_1, e_2, \dots, e_n\}$ as its edge set. Let $A = (a_{ij})$ be the adjacency matrix of G. Then $|A - \lambda I| = 0$ is called characteristic equation of G. $\lambda_1, \lambda_2, \dots, \lambda_n$ of A, are called eigenvalues of G which are assumed to be in non increasing order. As A is real symmetric matrix, the eigenvalues of G are real with sum equal to zero. The energy of G is defined to be sum of absolute values of the eigenvalues of G. i.,e $E(G) = \sum_{i=1}^{n} |\lambda_i|$.

Fedor V. Fomin et al.[2] introduced subgraph complements of a graph. Let $G = (V, E)$ be a graph and $S \subseteq V$. The subgraph complement of a graph G with respect to S, denoted by $G \oplus S$, is a graph (V, E_S), where for any two vertices u, $v \in V$, $uv \in E_S$ if and only if one of the following conditions hold good :

(1) $u \notin S$ or $v \notin S$ and $uv \in E$.
(2) $u, v \in S$ and $uv \notin E$.

Definition 1.1. Let $G \oplus S$ be subgraph complement of a graph G with respect to S. The subgraph complement adjacency matrix of $G \oplus S$ is an $n \times n$ matrix defined by $A_p(G \oplus S) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent with } i \neq j \\ 1, & \text{if } i = j \text{ and } v_i \in S \\ 0, & \text{otherwise.} \end{cases}$$

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Definition 1.2. [3] Let G be a simple graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ and d_i be the degree of v_i for $i = 1, 2, \ldots, n$. Then maximum degree matrix $M(G) = (d_{ij})$, is defined as

$$d_{ij} = \begin{cases} max\{d_i, d_j\}, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\Delta(G)$ denote the maximum degree among the vertices of G. The reverse vertex degree of a vertex v_i in G is defined as $c_{v_i} = \Delta(G) - d(v_i) + 1$, where $d(v_i)$ is degree of vertex v_i.

Definition 1.3. [5] Let G be a simple graph with n vertices and size m. Let c_{v_i} be the reverse vertex degree of the vertex v_i. Then maximum reverse degree matrix is defined as,
$M_R(G) = (r_{ij})$, where

$$r_{ij} = \begin{cases} max\{c_{v_i}, c_{v_j}\}, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

For more information on energy and subgraph complement of graphs, refer[1, 4, 6, 7, 8, 10].

In this paper, we have introduced maximum reverse degree energy of subgraph complement of graphs which is defined as follows:

Definition 1.4. Let $G \oplus S$ be a connected subgraph complement of a graph G with respect to S. Then maximum reverse degree subgraph complement matrix of the graph $G \oplus S$ is $n \times n$ matrix defined by $M_R(G \oplus S) = (r_{ij})$, where

$$r_{ij} = \begin{cases} 1, & \text{if } i = j, v_i \in S \\ max\{c_{v_i}, c_{v_j}\}, & \text{if } v_i \sim v_j \in E(G \oplus S) \\ 0, & \text{otherwise.} \end{cases}$$

The characteristic polynomial of maximum reverse degree subgraph complement of a graph G is defined by $\phi_p\{M_R(G \oplus S)\} = |\lambda I - M_R(G \oplus S)|$ and maximum reverse degree subgraph complement energy of $G \oplus S$ is denoted by $EM_R(G \oplus S)$, is defined as $\sum\limits_{i=1}^{n} |\lambda_i|$ where $\lambda_i' s$ are maximum reverse degree subgraph complement eigenvalues of $G \oplus S$.

Throughout this paper, x_i refers to the number of vertices in the neighbourhood of v_i whose reverse vertex degree is less then c_{v_i} and y_i refers to the number of vertices $v_j (j > i)$ in the neighbourhood of v_i whose reverse vertex degree is equal to c_{v_i}.

This paper is organised as follows. In section 2, the properties of maximum reverse degree energy of subgraph complement graphs are studied. In section 3, bounds for maximum reverse degree energy of subgraph complementary graphs are established. In section 4, maximum reverse degree energy of subgraph complement of some families of graphs are computed.

1.1. Preliminary Definitions.

Definition 1.5. [9] A double star is the graph denoted by $S(l,m)$, consisting of union of two stars $K_{1,l}$ and $K_{1,m}$ together with the line joining their centers. Let $V = \{u_i, v_j | i = 0,1,\dots,l, j = 0,1,\dots,m\}$ be the vertex set of the double star $S(l,m)$ with u_0 and v_0 as its centers.

Definition 1.6. Triangular book graph B_n^3 is a planar undirected graph with $n+2$ vertices and $2n+1$ edges constructed by n triangles sharing a common edge.

Definition 1.7. [9] Let $\{G^i | i \in 1,2,3,\dots,m\}$ for $m \in N$ and $m \geq 2$ be a collection of finite graphs and v_{oi} be a fixed vertex of each G^i, called terminal. Vertex Amalgamation $Amal(G^i, v_{oi})$ is a graph formed by taking all vertices and edges of G^i where $v_{oi} = v_{oj}$, for all $i \neq j$. If $G^i = G^j = G$ and $|G| = n$, then we write $Amal(G^i, v_{oi})$ with $Amal(G_n)_m$.

2. Properties of maximum reverse degree eigenvalues of connected subgraph complementary graphs

Theorem 2.1. *Let G be a simple graph with n vertices and m edges.*
If $\lambda_1, \lambda_2, \dots, \lambda_n$ represent maximum reverse degree eigenvalues of $G \oplus S$, then

(1) $\sum\limits_{i=1}^{n} \lambda_i = |S|$.

(2) $\sum\limits_{i=1}^{n} \lambda_i^2 = |S| + 2\sum\limits_{i=1}^{n}(x_i + y_i)c_{v_i}^2$.

Proof. (1) Sum of eigenvalues of $M_R(G \oplus S)$ is equal to trace of $M_R(G \oplus S)$, $\sum\limits_{i=1}^{n} \lambda_i = \sum\limits_{i=1}^{n} r_{ii} = |S|$.

(2) The sum of squares of eigenvalues of $M_R(G\oplus S)$ is the trace of $M_R^2(G\oplus S)$.

$$\begin{aligned} i.e., \sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{i=1}^{n} r_{ij} r_{ji} \\ &= \sum_{i=1}^{n} r_{ii}^2 + \sum_{i \neq j} r_{ij} r_{ji} \\ \sum_{i=1}^{n} \lambda_i^2 &= |S| + 2\sum_{i=1}^{n}(x_i + y_i)c_{v_i}^2. \end{aligned}$$

□

Theorem 2.2. *Let $G\oplus S = (V, E_S)$ be a connected subgraph complement of a graph $G = (V,E)$. Let $\phi\{M_R(G \oplus S), \lambda\} = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + \dots + a_n$ be the maximum reverse degree characteristic polynomial of graph $G \oplus S$. Then,*

(1) $a_0 = 1$.

(2) $a_1 = -|S|$.

(3) $a_2 = \binom{|S|}{2} - \sum\limits_{i=1}^{n}(x_i + y_i)c_{v_i}^2$.

Proof. (1) From the definition of $\phi\{M_R(G \oplus S), \lambda\}$, it follows that $a_0 = 1$.

(2) Sum of diagonal elements of $M_R(G \oplus S)$ is equal to cardinality of the set S. Hence, $(-1)a_1 = trace\{M_R(G \oplus S)\} = -|S|$.
(3) We have

$$\begin{aligned}(-1)^2 a_2 &= \sum_{1\le i<j\le n} \begin{vmatrix} r_{ii} & r_{ij} \\ r_{ji} & r_{jj} \end{vmatrix} \\ &= \sum_{1\le i<j\le n} r_{ii}r_{jj} - \sum_{1\le i<j\le n} r_{ji}r_{ij} \\ a_2 &= \binom{|S|}{2} - \sum_{i=1}^{n}(x_i+y_i)c_{v_i}^2.\end{aligned}$$

□

3. Bounds for maximum reverse degree energy of connected subgraph complementary graphs

Theorem 3.1. *Let $G \oplus S$ be connected subgraph complement of a graph G with $|S| = k$. Then $\sqrt{|S|+\beta} \le EM_R(G \oplus S) \le \sqrt{n(|S|+\beta)}$.*

Proof. Taking $a_i = 1$, $b_i = |\lambda_i|$ in Cauchy Schwarz inequality, we get

$$\left(\sum_{i=1}^{n}\lambda_i\right)^2 \le \left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}\lambda_i^2\right)$$

$$(EM_R(G \oplus S))^2 \le n\left(|S| + 2\sum_{i=1}^{n}(x_i+y_i)c_{v_i}^2\right).$$

Let

$$2\sum_{i=1}^{n}(x_i+y_i)c_{v_i}^2 = \beta.$$

$$EM_R(G \oplus S) \le \sqrt{n(|S|+\beta)}.$$

Equality holds if $G = K_n$ with $|S| = n$.
Also,

$$\left(\sum_{i=1}^{n}\lambda_i\right)^2 \ge \sum_{i=1}^{n}\lambda_i^2$$

$$(EM_R(G \oplus S))^2 \ge |S| + 2\sum_{i=1}^{n}(x_i+y_i)c_{v_i}^2$$

$$EM_R(G \oplus S) \ge \sqrt{(|S|+\beta)}.$$

Equality holds if $G = \overline{K_n}$ with $|S| = n$. □

Theorem 3.2. *Let $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$ represent maximum reverse degree eigenvalues of $G \oplus S$. Then $EM_R(G \oplus S) \le |\lambda_1| + \sqrt{(n-1)(|S|+\beta-|\lambda_1|^2)}$.*

Proof. Applying Cauchy Schwarz inequality for $(n-1)$ terms,

$$\left(\sum_{i=2}^{n}\lambda_i\right)^2 \leq \left(\sum_{i=2}^{n}1\right)\left(\sum_{i=2}^{n}\lambda_i^2\right).$$

$$(EM_R(G\oplus S)-|\lambda_1|)^2 \leq (n-1)(|S|+\beta-|\lambda_1|^2)$$

$$(EM_R(G\oplus S)-|\lambda_1|) \leq \sqrt{(n-1)(|S|+\beta-|\lambda_1|^2)}$$

$$EM_R(G\oplus S) \leq |\lambda_1|+\sqrt{(n-1)(|S|+\beta-|\lambda_1|^2)}.$$

Equality holds if $G=K_n$ with $|S|=n$. □

Theorem 3.3. *Let $G\oplus S$ be a connected subgraph complement of a graph G on n vertices with induced set S of order k. Then*
$EM_R(G\oplus S) \geq \sqrt{|S|+\beta+n(n-1)P^{\frac{2}{n}}}$, *where* $P=|M_{R_p}(G\oplus S)|$.

Proof. Using arithmetic and geometric mean inequality,

$$\begin{aligned}
\frac{1}{n(n-1)}\sum_{i\neq j}|\lambda_i||\lambda_j| &\geq \left(\prod_{i\neq j}|\lambda_i||\lambda_j|\right)^{\dfrac{1}{n(n-1)}}\\
&= \left(\prod_{i=1}^{n}|\lambda_i|^{2(n-1)}\right)^{\dfrac{1}{n(n-1)}}\\
&= \left(\prod_{i=1}^{n}|\lambda_i|\right)^{\dfrac{2}{n}}\\
&= P^{\frac{2}{n}},
\end{aligned}$$

where $P=|M_{R_p}(G\oplus S)|$.

$$\sum_{i\neq j}|\lambda_i||\lambda_j| \geq n(n-1)P^{\frac{2}{n}}.$$

Now,

$$(EM_R(G\oplus S))^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)^2$$

$$(EM_R(G\oplus S))^2 = \sum_{i=1}^{n}|\lambda_i|^2+\sum_{i\neq j}|\lambda_i||\lambda_j|$$

$$EM_R(G\oplus S) \geq \sqrt{|S|+\beta+n(n-1)P^{\frac{2}{n}}}.$$

Equality holds, when

(1) $G = K_n$ with $|S| = n$.
(2) $G = \overline{K_n}$ with $|S| = n$.

$\square$

Theorem 3.4. *Let $G \oplus S = (V, E_S)$ be a connected subgraph complement of a graph $G = (V, E)$ and $\rho(G \oplus S) = \max_{1 \le i \le n} |\lambda_i|$ be the maximum reverse degree spectral radius of $G \oplus S$. Then,* $\sqrt{\frac{|S| + \beta}{n}} \le \rho(G \oplus S) \le \sqrt{|S| + \beta}$.

Proof. Consider

$$\begin{aligned}\rho^2(G \oplus S) &= \max_{1 \le i \le n} \{|\lambda_i|\} \\ &\le \sum_{j=1}^{n} \lambda_j^2 \\ &= |S| + 2\sum_{i=1}^{n}(x_i + y_i)c_{v_i}^2 \\ \rho(G \oplus S) &\le \sqrt{|S| + \beta},\end{aligned}$$

where $\beta = 2\sum_{i=1}^{n}(x_i + y_i)c_{v_i}^2$.
Equality holds if $G = \overline{K_n}$ with $|S| = n$.
Next,

$$\begin{aligned}n\rho^2(G \oplus S) &\ge \max_{1 \le i \le n} \{|\lambda_i|\} \\ &\ge |S| + \beta \\ \rho(G \oplus S) &\ge \sqrt{\frac{|S| + \beta}{n}}\end{aligned}$$

$$\sqrt{\frac{|S| + \beta}{n}} \le \rho(G \oplus S) \le \sqrt{|S| + 2\beta}.$$

Equality holds if $G = K_n$ with $|S| = n$. $\square$

4. Maximum reverse degree energy of subgraph complement of some families of graphs

Theorem 4.1. *Let K_n be complete graph, where $|S| = k, k < n$. Then*

$$EM_R(K_n \oplus S) = (n - 2) + \sqrt{(k - n)^2 - 4\{(k - n)(k^3 - 1) - 1\}}.$$

Proof. Let $M_R(K_n \oplus S) = \begin{bmatrix} (I)_k & kJ_{k \times (n-k)} \\ kJ_{(n-k) \times k} & (J - I)_{(n-k)} \end{bmatrix}_n$ be the maximum reverse degree subgraph complement matrix of $K_n \oplus S$, where J is matrix of all 1's. The result is proved by showing $M_R(K_n \oplus S)Z = \lambda Z$ for certain vector Z and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as $M_R(K_n \oplus S)$ is real and symmetric.

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $M_R(K_n \oplus S)$. Consider,

$$[M_R(K_n \oplus S) - \lambda I] \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} [(1-\lambda)I]X + (kJ)Y \\ (kJ)X + [J - (1+\lambda)I]Y \end{bmatrix}_n . \tag{4.1}$$

Case 1. Let $X = X_j$, $j = 2, 3, \dots, k$ and $Y = 0_{n-k}$. Using equation (4.1), $[(1-\lambda)I]X_j + (kJ)0_{n-k} = (1-\lambda)X_j$, then $\lambda = 1$ is the eigenvalue with multiplicity of at least $k-1$ since there are $k-1$ independent vectors of the form X_j.

Case 2. Let $X = 0_k$ and $Y = Y_j$, $j = 2, 3, \dots, n-k$. From equation (4.1), $(kJ)0_k + [J - (1+\lambda)I]Y_j = -(1+\lambda)Y_j$, then $\lambda = -1$ is the eigenvalue with multiplicity of at least $n-k-1$ since there are $n-k-1$ independent vectors of the form Y_j.

Case 3. Let $X = 1_k$ and $Y = \left(\frac{1-\lambda}{k^2 - kn}\right) 1_{n-k}$. Here, λ denotes root of the equation

$$\lambda^2 + \lambda(k-n) + k^4 - k^3 n + n - k - 1 = 0.$$

From equation (4.1),

$$\begin{aligned}
(kJ)_{(n-k)\times k} 1_k + [J - (1+\lambda)I]_{(n-k)\times(n-k)} \left(\frac{1-\lambda}{k^2-kn}\right) 1_{n-k} \\
= k^2 1_{n-k} + (n-k-\lambda-1)\left(\frac{1-\lambda}{k^2-kn}\right) 1_{n-k} \\
= \left\{ k^2 + (n-k-\lambda-1)\left(\frac{1-\lambda}{k^2-kn}\right) \right\} 1_{n-k} \\
= \frac{\lambda^2 + \lambda(k-n) + k^4 - k^3 n + n - k - 1}{k^2 - kn} 1_{n-k}.
\end{aligned}$$

So,

$$\lambda = \frac{-(k-n) + \sqrt{(k-n)^2 - 4\{(k-n)(k^3-1) - 1\}}}{2}$$

and

$$\lambda = \frac{-(k-n) - \sqrt{(k-n)^2 - 4\{(k-n)(k^3-1) - 1\}}}{2}$$

are the eigenvalues with multiplicity at least one.

Thus, spectrum of $K_n \oplus S$ is $\begin{pmatrix} 1 & -1 & \lambda_1 & \lambda_2 \\ k-1 & n-k-1 & 1 & 1 \end{pmatrix}$,

where $\lambda_1 = \dfrac{-(k-n) + \sqrt{(k-n)^2 - 4\{(k-n)(k^3-1) - 1\}}}{2}$,

$\lambda_2 = \dfrac{-(k-n) - \sqrt{(k-n)^2 - 4\{(k-n)(k^3-1) - 1\}}}{2}$.

Therefore,

$$EM_R(K_n \oplus S) = (n-2) + \sqrt{(k-n)^2 - 4\{(k-n)(k^3-1) - 1\}}.$$

□

Theorem 4.2. *Let $K_{m,n}$ be complete bipartite graph, where $|S| = m$. Then* $EM_R(K_{m,n} \oplus S) = \sqrt{m^2 + 4mn^3}$.

Proof. Let

$$M_R(K_{m,n} \oplus S) = \begin{bmatrix} (J)_m & (nJ)_{m\times n} \\ (nJ)_{n\times m} & 0_n \end{bmatrix}_{m+n}$$

be the maximum reverse degree subgraph complement matrix of $K_{m,n} \oplus S$. The result is proved by showing $M_R(K_{m,n} \oplus S)Z = \lambda Z$ for certain vector Z and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as $M_R(K_{m,n} \oplus S)$ is real and symmetric.

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $M_R(K_{m,n} \oplus S)$. Consider,

$$\big[M_R(K_{m,n} \oplus S) - \lambda I\big] \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} (J-\lambda I)X + (nJ)Y \\ (nJ)X - (\lambda I)Y \end{bmatrix}_{m+n}. \tag{4.2}$$

Case 1. Let $X = X_j$, $j = 2, 3, \ldots, m$ and $Y = 0_n$. Using equation (4.2), $(J - \lambda I)X_j + (nJ)0_n = -\lambda X_j$, then $\lambda = 0$ is the eigenvalue with multiplicity of at least $m-1$ since there are $m-1$ independent vectors of the form X_j.

Case 2. Let $X = 0_m$ and $Y = Y_j$, $j = 2, 3, \ldots, n$. From equation (4.2), $(nJ)0_m - \lambda I Y_j = \lambda Y_j$, then $\lambda = 0$ is the eigenvalue with multiplicity of at least $n-1$ since there are $n-1$ independent vectors of the form Y_j.

Case 3. Let $X = 1_m$ and $Y = \frac{mn}{\lambda}1_n$. Here, λ denotes root of the equation,

$$\lambda^2 - m\lambda - mn^3 = 0.$$

From equation (4.1),

$$\begin{aligned} (J-\lambda I)1_m + nJ\left(\frac{mn}{\lambda}\right)1_n &= \left(m - \lambda + n^2\left(\frac{mn}{\lambda}\right)\right)1_m \\ &= \left(m - \lambda + mn\left(\frac{n^2}{\lambda}\right)\right)1_m \\ &= \left(\frac{\lambda^2 - m\lambda - mn^3}{\lambda}\right)1_m. \end{aligned}$$

So, $\lambda = \dfrac{m + \sqrt{m^2 + 4mn^3}}{2}$ and $\lambda_2 = \dfrac{m - \sqrt{m^2 + 4mn^3}}{2}$ are the eigenvalues with multiplicity of at least one.

Thus, the spectrum of $K_{m,n} \oplus S$ is $\begin{pmatrix} 0 & 0 & \lambda_1 & \lambda_2 \\ m-1 & n-1 & 1 & 1 \end{pmatrix}$, where $\lambda_1 = \dfrac{m + \sqrt{m^2 + 4mn^3}}{2}$, $\lambda_2 = \dfrac{m - \sqrt{m^2 + 4mn^3}}{2}$.

Therefore,

$$EM_R(K_{m,n} \oplus S) = \sqrt{m^2 + 4mn^3}.$$

□

Corollary 4.3. *For the star graph $K_{1,n-1}$, let S be the set of k non-pendant vertices. Then $EM_R(K_{1,n-1} \oplus S) = \sqrt{1+4(n-1)^3}$.*

Proof. On substituting $m = 1$ and $n = n-1$ in theorem (4.2), we get

$$EM_R(K_{1,n-1} \oplus S) = \sqrt{1+4(n-1)^3}.$$

□

Theorem 4.4. *Let $K_{n\times 2}$ be cocktail party graph, where $|S| = n$. Then $EM_R(K_{n\times 2} \oplus S) = \sqrt{n^2 - 4\{n - n^2(n-1)^2 - 1\}}$.*

Proof. Let $M_R(K_{n\times 2}\oplus S) = \begin{bmatrix} (I)_n & n(J-I)_n \\ (n(J-I))_n & (J-I)_n \end{bmatrix}_{2n}$ be the maximum reverse degree subgraph complement matrix of $K_{n\times 2}\oplus S$. The result is proved by showing $M_R(K_{n\times 2} \oplus S)Z = \lambda Z$ for certain vector Z and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as $M_R(K_{n\times 2} \oplus S)$ is real and symmetric.

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $M_R(K_{n\times 2} \oplus S)$. Consider,

$$\left[M_R(K_{n\times 2} \oplus S) - \lambda I\right] \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} (1-\lambda)IX + n(J-I)Y \\ n(J-I)X + (J - I(1+\lambda))Y \end{bmatrix}_{2n}. \tag{4.3}$$

Case 1. Let $X = X_j$, $j = 2, 3, \ldots, n$ and $Y = -\frac{\lambda - 1}{n} X_j$.

Now, equation (4.3) implies

$$\begin{aligned} n(J-I)X_j + (J - I(1+\lambda)) &= n(-X_j) - (J - (1+\lambda)I)\frac{\lambda-1}{n}X_j \\ &= -n + \frac{1}{n}(\lambda^2 - 1)X_j \\ &= \frac{(\lambda^2-1) - n^2}{n}X_j. \end{aligned}$$

So, $\lambda = \pm\sqrt{1+n^2}$ are the eigenvalues with multiplicity at least $n-1$ since there are $n-1$ independent vectors of the form X_j.

Case 2. Let $X = 1_n$ and $Y = \frac{\lambda - 1}{n(n-1)^2}1_n$. Here, λ denotes root of the equation

$$\lambda^2 - \lambda n + (n-1) - n^2(n-1)^2 = 0.$$

Equation (4.3) implies

$$\begin{aligned} n(J-I)1_n + (J - I(1+\lambda))\frac{\lambda-1}{n(n-1)^2}1_n &= n + (n-1-\lambda)\frac{\lambda-1}{n(n-1)^2}1_n \\ &= \frac{\lambda^2 - n\lambda + (n-1) - n^2(n-1)^2}{n(n-1)^2}1_n. \end{aligned}$$

So, $\lambda = \dfrac{n + \sqrt{n^2 - 4\{n - n^2(n-1)^2 - 1\}}}{2}$ and

$\lambda = \dfrac{n - \sqrt{n^2 - 4\{n - n^2(n-1)^2 - 1\}}}{2}$ are the eigenvalues with multiplicity of at least one.

Thus, the spectrum of $K_{n\times 2} \oplus S$ is $\begin{pmatrix} \sqrt{n^2+1} & -\sqrt{n^2+1} & \lambda_1 & \lambda_2 \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$,

where $\lambda_1 = \dfrac{n + \sqrt{n^2 - 4\{n - n^2(n-1)^2 - 1\}}}{2}$,

$\lambda_2 = \dfrac{n - \sqrt{n^2 - 4\{n - n^2(n-1)^2 - 1\}}}{2}$.

Therefore,

$$EM_R(K_{n\times 2} \oplus S) = \sqrt{n^2 - 4\{n - n^2(n-1)^2 - 1\}}.$$

$\square$

Theorem 4.5. *Let S_n^0 be crown graph, where $|S| = n$, then* $EM_R(S_n^0 \oplus S) = 2n(n-1) + \sqrt{n^2 + 4n^2(n-1)^2}$.

Proof. Let $M_R(S_n^0 \oplus S) = \begin{bmatrix} (J)_n & n(J-I)_n \\ (n(J-I))_n & (0)_n \end{bmatrix}_{2n}$ be the maximum reverse degree subgraph complement matrix of $S_n^0 \oplus S$. The result is proved by showing $M_R(S_n^0 \oplus S)Z = \lambda Z$ for certain vector Z and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as $M_R(S_n^0 \oplus S)$ is real and symmetric.

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $M_R(S_n^0 \oplus S)$. Consider

$$|M_R(S_n^0 \oplus S) - \lambda I| \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} (J-\lambda I)X + [n(J-I)]Y \\ [n(J-I)]X - \lambda IY \end{bmatrix}_{2n}. \tag{4.4}$$

Case 1. Let $X = X_j$, $j = 2, 3, \dots, n$ and $Y = \dfrac{n(J-I)}{\lambda I} X_j$.

Now, (4.4) implies

$$\begin{aligned} (J-\lambda I)X_j + n(J-I)\frac{n(J-I)}{\lambda I}X_j &= -\lambda X_j + \frac{n^2}{\lambda}(J-I)(-X_j) \\ &= -\lambda X_j + \frac{n^2}{\lambda}(X_j) \\ &= \frac{n^2 - \lambda^2}{\lambda} X_j. \end{aligned}$$

So, $\lambda = \pm n$ are the eigenvalues with multiplicity of at least $n-1$ since there are $n-1$ independent vectors of the form X_j.

Case 2. Let $X = 1_n$ and $Y = \dfrac{\lambda - n}{n(n-1)} 1_n$. Here, λ denotes root of the equation

$$\lambda^2 - n\lambda - n^2(n-1)^2 = 0.$$

From equation (4.4), we have

$$\begin{aligned} n(J-I)1_n - (\lambda I)\frac{\lambda - n}{n(n-1)}1_n &= \left(n(n-1) - \lambda\frac{\lambda-n}{n(n-1)}\right)1_n \\ &= \frac{\lambda^2 - \lambda n - n^2(n-1)^2}{n(n-1)}1_n. \end{aligned}$$

So, $\lambda = \dfrac{n+\sqrt{n^2+4n^2(n-1)^2}}{2}$ and $\lambda = \dfrac{n-\sqrt{n^2+4n^2(n-1)^2}}{2}$ are the eigenvalues with multiplicity of at least one.

Thus, the spectrum of $S_n^0 \oplus S$ is given by $\begin{pmatrix} n & -n & \lambda_1 & \lambda_2 \\ n-1 & n-1 & 1 & 1 \end{pmatrix}$,

where $\lambda_1 = \dfrac{n+\sqrt{n^2+4n^2(n-1)^2}}{2}$, $\lambda_2 = \dfrac{n-\sqrt{n^2+4n^2(n-1)^2}}{2}$.

Therefore,

$$EM_R(S_n^0 \oplus S) = 2n(n-1) + \sqrt{n^2+4n^2(n-1)^2}.$$

□

Theorem 4.6. *Let $S(l,m)$ be the double star and $S = \{u_1, u_2, \ldots, u_l\}$. Then, characteristic polynomial of $S(l,m) \oplus S$ is given by,*

$(-\lambda)^{m-1}(1-\lambda-r)^{l-1}\{\lambda^4 + \lambda^3(r-1-lr) - \lambda^2(k^2+mp^2+lr^2) + \lambda(k^2+mp^2-k^2r+k^2lr-mp^2r+lmp^2r) + lmp^2r^2\}$.

Proof. Let

$$M_{R_p}(S(l,m)\oplus S) = \begin{bmatrix} 0 & k & q & q & \cdots & q & 0 & 0 & 0 & \cdots & 0 & 0 \\ k & 0 & 0 & 0 & \cdots & 0 & n & n & n & \cdots & n & n \\ q & 0 & 1 & r & \cdots & r & 0 & 0 & 0 & \cdots & 0 & 0 \\ q & 0 & r & 1 & \cdots & r & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q & 0 & r & r & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{l+m+2}$$

be the maximum reverse degree subgraph complement matrix of $S(l,m)\oplus S$, where

$k = max\{c_{u_o}, c_{v_o}\}$,
$q = max\{c_{u_o}, c_{u_i}\}, 1 \le i \le l$
$n = max\{c_{v_o}, c_{v_j}\}, 1 \le j \le m$
$r = max\{c'_{u_i}s\}$, $1 \le i \le l$
$p = max\{c'_{v_j}s\}$, $1 \le j \le m$.

On applying row operation $R_i \longrightarrow R_i - R_{i+1}$, $1 \le i \le l-1$, $1 \le j \le m-1$ and column operations $C_i \longrightarrow C_i + C_{i-1} + \ldots + C_1$, $1 \le i \le l$, $1 \le j \le m$ in $|M_{R_p}(S(l,m)\oplus S) - \lambda I|$, we get

$$(-\lambda)^{m-1}(1-\lambda-r)^{l-1}\begin{vmatrix} -\lambda & k & lr & 0 \\ k & -\lambda & 0 & mp \\ r & 0 & ((l-1)r+1)-\lambda & 0 \\ 0 & p & 0 & -\lambda \end{vmatrix}.$$

On further simplifying, we get

$\phi_p\{M_R(S_n^0 \oplus S)\} = (-\lambda)^{m-1}(1-\lambda-r)^{l-1}\{\lambda^4+\lambda^3(r-1-lr)-\lambda^2(k^2+mp^2+lr^2)+\lambda(k^2+mp^2-k^2r+k^2lr-mp^2r+lmp^2r)+lmp^2r^2\}$. □

Theorem 4.7. *Let B_n^3 be triangular book graph and $S=\{v_1,v_2\}$, where v_1v_2 is the base of n triangles, then $EM_R(B_n^3 \oplus S) = 1+\sqrt{1+8n(1-n)^2}$.*

Proof. Let $M_R(B_n^3 \oplus S) = \begin{bmatrix} (I)_2 & (n-1)J_{2\times n} \\ (n-1)J_{n\times 2} & (0)_n \end{bmatrix}_{n+2}$ be the maximum reverse degree subgraph complement matrix of $B_n^3 \oplus S$. The result is proved by showing $M_R(B_n^3 \oplus S)Z = \lambda Z$ for certain vector Z and by making use of fact that the geometric multiplicity and algebraic multiplicity of each eigenvalue λ is same, as $M_R(B_n^3 \oplus S)$ is real and symmetric.

Let $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ be an eigenvector of order $2n$ partitioned conformally with $M_R(B_n^3 \oplus S)$. Consider,

$$\left[\lambda I - M_R(B_n^3 \oplus S)\right]\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} (\lambda-1)IX+(1-n)JY \\ (1-n)JX+\lambda IY \end{bmatrix}_{n+2}. \quad (4.5)$$

Case 1. Let $X = 1_2$ and $Y = 0_n$. Using equation (4.5), $[(\lambda-1)I]1_2+[(1-n)J]0_n$, then $\lambda=1$ is an eigenvalue with multiplicity of at least 1.

Case 2. Let $X = 0_2$ and $Y = Y_j$, $j=2,3,\ldots,n$. From equation (4.5), $(1-n)J0_2+\lambda IY_j = \lambda Y_j$, then $\lambda = 0$ is the eigenvalue with multiplicity of at least $n-1$ since there are $n-1$ independent vectors of the form Y_j.

Case 3. Let $X = 1_2$ and $Y = \left(\frac{-2(1-n)}{\lambda}\right)1_n$. Here, λ denotes root of the equation

$$\lambda^2-\lambda-2n(1-n)^2 = 0.$$

From equation (4.5),

$$\begin{aligned}[(\lambda-1)I]1_2+(1-n)J\left(\frac{-2(1-n)}{\lambda}\right)1_n &= \left\{(\lambda-1)-(1-n)n\left(\frac{2(1-n)}{\lambda}\right)\right\}1_2 \\ &= \left\{\frac{\lambda^2-\lambda-2n(1-n)^2}{\lambda}\right\}1_2.\end{aligned}$$

So, $\lambda = \frac{1+\sqrt{1+8n(1-n)^2}}{2}$ and $\lambda = \frac{1-\sqrt{1+8n(1-n)^2}}{2}$ are the eigenvalues with multiplicity of at least one.

Thus, spectrum of $B_n^3 \oplus S$ is $\begin{pmatrix} 0 & 1 & \lambda_1 & \lambda_2 \\ n-1 & 1 & 1 & 1 \end{pmatrix}$,

where $\lambda_1 = \frac{1+\sqrt{1+8n(1-n)^2}}{2}$, $\lambda_2 = \frac{1-\sqrt{1+8n(1-n)^2}}{2}$.

Therefore,

$$EM_R(B_n^3 \oplus S) = 1 + \sqrt{1 + 8n(1-n)^2}.$$

□

Theorem 4.8. *Let $Amal(k, K_n)$ be the k times amalgamation of complete graph K_n. If $S = v_0$ which is the central vertex, then $EM_R(Amal(k, K_n) \oplus S) = C(n-2)(2k-1) + \sqrt{(1 + C(n-1))^2 - 4\{C(n-1) - C^2k(n-1)\}}$.*

Proof. Let

$$M_R(Amal(k, K_n) \oplus S) = \begin{bmatrix} J_1 & CJ_{1\times n-1} & CJ_{1\times n-1} & \cdots & CJ_{1\times n-1} \\ CJ_{1\times n-1} & CB_{n-1} & 0_{n-1} & \cdots & 0_{n-1} \\ CJ_{1\times n-1} & 0_{n-1} & CB_{n-1} & \cdots & 0_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CJ_{1\times n-1} & 0_{n-1} & 0_{n-1} & \cdots & CB_{n-1} \end{bmatrix}_{k+1}$$

be the maximum reverse degree subgraph complement matrix of $Amal(k, K_n)$, where $C = (n-1)(k-1) + 1$ and B is the adjacency matrix of subgraph of the complete graph.

$$\begin{vmatrix} |\lambda I - M_R(Amal(k, K_n) \oplus S)| = \lambda - 1_1 & -CJ_{1\times n-1} & -CJ_{1\times n-1} & \cdots & -CJ_{1\times n-1} \\ -CJ_{1\times n-1} & \lambda I - CB_{n-1} & 0_{n-1} & \cdots & 0_{n-1} \\ -CJ_{1\times n-1} & 0_{n-1} & \lambda I - CB_{n-1} & \cdots & 0_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -CJ_{1\times n-1} & 0_{n-1} & 0_{n-1} & \cdots & \lambda I - CB_{n-1} \end{vmatrix}_{k+1}. \tag{4.6}$$

On replacing R_i by $R_i - R_{i+1}$ for $i = 2, \dots, k+1$ and replacing C_i by $C_i + C_{i-1} + \cdots + C_2$ for $i = k+1, k, \dots, 3, 2$ in (4.6), a new determinant say $det(D)$ is obtained.

$$det(D) = |(\lambda I - CB)_{n-1)}|^{(k-1)} \begin{vmatrix} \lambda - 1 & -CkJ \\ -CJ & \lambda I - CB \end{vmatrix}_n. \tag{4.7}$$

Consider,

$$|(\lambda I - CB)_{n-1)}| = \begin{vmatrix} \lambda & -C & -C & \cdots & -C \\ -C & \lambda & -C & \cdots & -C \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -C & -C & -C & \cdots & \lambda \end{vmatrix}_{n-1}. \tag{4.8}$$

On replacing R_i by $R_i - R_{i+1}$ for $i = 1, \dots, n-2$ and replacing C_i by $C_i + C_{i-1} + \cdots + C_2 + C_1$ for $i = 1, 2, \dots, n-1$ in (4.8), we have

$$\{(\lambda + C)^{n-2}(\lambda - C(n-2))\}^{k-1}. \tag{4.9}$$

Next,

$$\begin{vmatrix} \lambda - 1 & -CkJ \\ -CJ & \lambda I - CB \end{vmatrix}_n = \begin{vmatrix} \lambda - 1 & -Ck & -Ck & \cdots & -Ck \\ -C & \lambda & -C & \cdots & -C \\ -C & -C & \lambda & \cdots & -C \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -C & -C & -C & \cdots & \lambda \end{vmatrix}_n . \tag{4.10}$$

On replacing R_i by $R_i - R_{i+1}$ for $i = 2, \ldots, n-1$ and replacing C_i by $C_i + C_{i-1} + \cdots + C_2$, $i = 2, \ldots, n$ in (4.10), we have

$$(\lambda + C)^{n-2}\{\lambda^2 - \lambda(1 + C(n-1)) + C(n-1) - C^2k(n-2)\}. \tag{4.11}$$

Substituting (4.9) and (4.11) in (4.7), we obtain
$\phi_p\{M_R(Amal(k, K_n) \oplus S)\} = (\lambda + C)^{k(n-2)}\{\lambda - C(n-2)\}^{k-1}\{\lambda^2 - \lambda(1 + C(n-1)) + C(n-1) - C^2k(n-2)\}$.

Thus, spectrum of $Amal(k, K_n) \oplus S$ is $\begin{pmatrix} -C & C(n-2) & \lambda_1 & \lambda_2 \\ (n-2)k & k-1 & 1 & 1 \end{pmatrix}$,

where $\lambda_1 = \dfrac{1 + C(n-1) + \sqrt{(1 + C(n-1))^2 - 4\{C(n-1) - C^2k(n-1)\}}}{2}$

and $\lambda_2 = \dfrac{1 + C(n-1) - \sqrt{(1 + C(n-1))^2 - 4\{C(n-1) - C^2k(n-1)\}}}{2}$.

Therefore, $EM_R(Amal(k, K_n) \oplus S) = C(n-2)(2k-1)$
$+ \sqrt{(1 + C(n-1))^2 - 4\{C(n-1) - C^2k(n-1)\}}$. □

5. Conclusion

Graph energy has so many applications in the field of chemistry, physics and mathematics. The maximum reverse degree subgraph complement energy depends on the chosen induced set of graph such that resultant subgraph complement is connected. In this paper, we have obtained some bounds for maximum reverse degree energy of subgraph complement of graphs. Also, a generalized expression for maximum reverse degree energy of subgraph complement of complete graph, cocktail party graph, crown graph, complete bipartite graph, double star graph, triangular book graph and amalgamation of K_n are also computed.

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