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NOTE ON DEGREE EQUIVALENCE SIGNED GRAPHS

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Abstract. The new concept of a degree equivalence signed graph of a signed graph was presented in this research, and its characteristics were examined. The structural characterization of this novel concept was also acquired, and some switching equivalent characterizations were offered.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ are denoted by n and m respectively. For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [1]. The non-standard will be given in this paper as and when required.

Let R be a binary relation defined on the set $\mathcal{S}$. If R satisfies reflexive, symmetric, and transitive properties, then R is said to be an equivalence relation.

Let $\mathcal{S} \neq \phi$ and R be an equivalence relation on $\mathcal{S}$ with respect to the relation R, we can draw an undirected graph G_R as: for any $u, v \in \mathcal{S}$, with $u \neq v$,

$$u \text{ and } v \text{ are adjacent in } G_R \Leftrightarrow uRv.$$

The graph G_R is called *equivalence graph* on S with respect to the relation R. We have the following observations:

(1) If there are two or more equivalence classes in the partition of S with respect to the relation R, then G_R is disconnected and the number of components is the number of distinct equivalence classes. Each component is a complete graph. If there is only one equivalence class, then G_R is the complete graph with $|S|$ vertices.
(2) For given $G = (V, E)$, we can construct new graphs with a vertex set V by defining equivalence relations on V with respect to some property of elements of V in G.

Consider a graph $G = (V, E)$ with n vertices. We now define a equivalence relation $\sim$ on the vertex set V as follows: for $u, v \in V$,

$$u \sim v \Leftrightarrow deg(u) = deg(v).$$

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We can clearly notice that $\sim$ is an equivalence relation on V. Let $V_1, V_2, \ldots, V_k$ be the partition of V into disjoint classes by the relation $\sim$. Let $|V_i| = n_i, 1 \leq i \leq k$ so that $n_1 + n_2 + \ldots + n_k = n$. The equivalence class graph on V defined by $\sim$ is called *degree equivalence graph* of G and is symbolized by $D(G)$. We observe that $D(G)$ is a simple graph. Consequently, we have the subsequent proposition from the elucidation $D(G)$ (See [4]).

In [4], proved the following results:

Theorem 1.1. *For any graph, $D(G) = D(\overline{G})$, where $\overline{G}$ is the complement of G.*

Theorem 1.2. *Let G be a graph and $L(G)$ be the line graph of G. Then $D(L(G)) = D(\overline{L(G)})$*

To model individuals' preferences towards each other in a group, Harary [2] introduced the concept of signed graphs in 1953. A signed graph $S = (G, \sigma)$ is a graph $G = (V, E)$ whose edges are labeled with positive and negative signs (i.e., $\sigma : E(G) \to \{+, -\}$). The vertexes of a graph represent people and an edge connecting two nodes signifies a relationship between individuals. The signed graph captures the attitudes between people, where a positive (negative edge) represents liking (disliking). An unsigned graph is a signed graph with the signs removed. Similar to an unsigned graph, there are many active areas of research for signed graphs.

The sign of a cycle (this is the edge set of a simple cycle) is defined to be the product of the signs of its edges; in other words, a cycle is positive if it contains an even number of negative edges and negative if it contains an odd number of negative edges. A signed graph S is said to be balanced if every cycle in it is positive. A signed graph S is called totally unbalanced if every cycle in S is negative. A chord is an edge joining two non adjacent vertices in a cycle.

A *marking* of S is a function $\zeta : V(G) \to \{+, -\}$. Given a signed graph S one can easily define a marking ζ of S as follows: For any vertex $v \in V(S)$,

$$\zeta(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking ζ of S is called *canonical marking* of S.

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.3. *A signed graph S is balanced if and only if either of the following equivalent conditions is satisfied:*

- **(i):** *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary [2]).*

(ii): *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. (Sampathkumar [5]).*

Switching S with respect to a marking ζ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The resulting signed graph $S_\zeta(S)$ is said switched signed graph. A signed graph S is called to switch to another signed graph S'written $S \sim S'$, whenever their exists a marking ζ such that $S_\zeta(S) \cong S'$, where $\cong$ denotes the usual equivalence relation of isomorphism in the class of signed graphs. Hence, if $S \sim S'$, we shall say that S and S' are switching equivalent. Two signed graphs S_1 and S_2 are signed isomorphic (written $S_1 \cong S_2$) if there is a one-to-one correspondence between their vertex sets which preserve adjacency as well as sign.

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be *weakly isomorphic* (see [6]) or *cycle isomorphic* (see [7]) if there exists an isomorphism $\phi : G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (see [7]):

Theorem 1.4. (**T. Zaslavsky** [7]) *Given a graph G, any two signed graphs in $\psi(G)$, where $\psi(G)$ denotes the set of all the signed graphs possible for a graph G, are switching equivalent if and only if they are cycle isomorphic.*

2. Degree Equivalence Signed Graph of a Signed Graph

By the motivation of degree equivalence graph of a graph defined in the above, in this section we defined the new notion called degree equivalence signed graph of S as: the degree equivalence signed graph $D(S) = (D(G), \sigma')$ of $S = (G, \sigma)$ is a signed graph, the sign of any edge $pq \in E(D(S))$ is the product of canonical marking of the vertices p and q. If any S is isomorphic to degree equivalence signed graph of some signed S' (i.e., $D(S) \cong S'$), then S is termed a degree equivalence signed graph. The following result restricts the class of token graphs.

Theorem 2.1. *For any signed graph $S = (G, \sigma)$, its token signed graph $D(S)$ is balanced.*

Proof. Since sign of any edge $e = uv$ in $D(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S, by Theorem 1.3, $D(S)$ is balanced. □

Consider the $\mathbb{Z}^+$ and $k \in \mathbb{Z}^+$, the k^{th} iterated degree equivalence signed graph $D(S)$ of S is explicate as:

$$D^0(S) = S,\ D^k(S) = D(D^{k-1}(S)).$$

Corollary 2.2. *The k^{th} iterated degree equivalence signed graph $D^k(S)$ is always balanced, for any $S = (G, \sigma)$.*

Theorem 2.3. *The degree equivalence signed graphs of $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$ are switching equivalent (i.e., $D(S_1) \sim D(S_2)$), if G_1 and G_2 are isomorphic.*

Proof. Consider two signed graphs S_1 and S_2 with $G_1 \cong G_2$. Thereupon, the corresponding the degree equivalence signed graphs $D(S_1)$ and $D(S_2)$ are positive. From Theorem 1.4, $D(S_1)$ and $D(S_2)$ are switching equivalent. □

The following result characterizes S such that S and degree equivalence signed graphs are cycle isomorphic.

Theorem 2.4. *For any signed graph $S = (G, \sigma)$, the signed graph and its degree equivalence signed graph are cycle isomorphic iff S is positive and for each $v \in V(G)$ is of same degree.*

Proof. Consider S is positive and for each $v \in V(G)$ is of same degree. Then, G and $D(G)$ are the same. Now the degree equivalence signed graph $D(S)$ of S with underlying graph is regular, is positive. From the hypothesis, S is positive and just now we have seen that $D(S)$ is also positive and hence S and $D(S)$ are cycle isomorphic, from the Theorem 1.4.

Conversely, suppose that signed graph and its degree equivalence signed graph are cycle isomorphic. Then $G \cong D(G)$. Therefore G is a regular graph. Since $D(S)$ and S are cycle isomorphic. This satisfies only when S is positive. □

From Theorems 1.1 and 1.2, we are able to present the following results for degree equivalence signed graphs:

Theorem 2.5. *For any $S = (G, \sigma)$, $D(S) \sim D(\overline{S})$.*

Theorem 2.6. *Let S be a signed graph and $L(S)$ be the line signed graph of S. Then $D(L(S)) \sim D(\overline{L(S)})$.*

The concept negation of a signed graph introduced by Harary [3] as follows: Consider a signed graph $S = (G, \sigma)$, the negation of S is denoted by $\eta(S)$ and the underlying graph of S and $\eta(S)$ are isomorphic. Further, the marking of each line $e = pq$ in $\eta(S)$ is $+$ $(-)$, if the marking of the line $e = pq$ in S is $-$ $(+)$.

In view of the negation operator introduced by Harary [3], we have the following cycle isomorphic characterizations:

Corollary 2.7. *The negation of degree equivalence signed graphs of $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$ are cycle isomorphic (i.e., $\eta(D(S_1)) \sim \eta(D(S_2))$), if G_1 and G_2 are isomorphic.*

Corollary 2.8. *For any two signed graphs $S_1 = (G_1, \sigma)$ and $S_2 = (G_2, \sigma)$, $D(\eta(S_1))$ and $D(\eta(S_2))$ are cycle isomorphic, if G_1 and G_2 are isomorphic.*

Corollary 2.9. *For any $S = (G, \sigma)$, the signed graph S and degree equivalence signed graph of $\eta(S)$ are cycle isomorphic iff S is positive and for each $v \in V(G)$ is of same degree.*

We have observed that, the signed graph is either positive or negative but the degree equivalence signed graph of one such signed graph is always positive. Using the concept negation in signed graphs introduced by Harary [3], we have the following result to the degree equivalence signed graphs.

Theorem 2.10. *Suppose the degree equivalence signed graph $D(G)$ is bipartite. Then the negation of degree equivalence signed graph $\eta(D(S))$ is positive, where S is any signed graph.*

Proof. Since, by Theorem 2.1, degree equivalence signed graph $D(S)$ is positive. Then all the cycles in degree equivalence signed graph $D(S)$ are positive. By the hypothesis, the degree equivalence graph $D(G)$ is bipartite. Then each cycle C_n (where n is even) in $D(S)$ is positive. Therefore, the negation of degree equivalence signed graph $D(S)$ is positive. □

We now give the structural characterization of degree equivalence signed graphs.

Theorem 2.11. *Suppose $S = (G, \sigma)$ be any signed graph. Then S is positive and its underlying graph is degree equivalence graph if and only if S is a degree equivalence signed graph $D(S)$.*

Proof. Let us consider that S is a degree equivalence signed graph $D(S)$. Then the signed graph S and the degree equivalence signed graph of some signed graph S_1 (i.e., $D(S_1)$) are isomorphic. Since, the degree equivalence signed graph of any signed graph is positive and we have $S \cong D(S_1)$. Consequently, S is positive and its underlying graph is a degree equivalence signed graph.

Conversely, suppose that S is positive and its underlying graph is degree equivalence graph. Since, the signed graph S is positive, then establish the S_ζ. With the evidence of Sampathkumar's result (Theorem 1.3), every edge pq in S_ζ amuse $\sigma(pq) = \zeta(p)\zeta(q)$. Deliberate, the signed graph $S_1 = (G_1, \sigma_1)$ in which each edge $e = (pq)$ in G_1, $\sigma_1(e) = \zeta(p)\zeta(q)$. Therefore, the signed graph S and the degree equivalence signed graph of S_1 are isomorphic. Hence, S is a degree equivalence signed graph $D(S)$. □

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