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A TABLE OF DERIVATIVES ON ARBITRARY TIME SCALES

DELFIM F. M. TORRES

Abstract. This paper presents a collection of useful formulas of dynamic derivatives on time scales, systematically collected for reference purposes. As an application, we define trigonometric and hyperbolic functions on time scales in such a way the most essential qualitative properties of the corresponding continuous functions are generalized in a proper way.

1. Introduction

The calculus on time scales is a unification of the theory of difference equations with that of differential equations, unifying integral and differential calculus with the calculus of finite differences and offering a formalism for studying hybrid discrete-continuous dynamical systems [4, 9]. It has applications in any field that requires simultaneous modeling of discrete and continuous data [1, 6]. Roughly speaking, the theory is based on a new definition of derivative, such that if one differentiates a function which acts on the real numbers, then the definition is equivalent to standard differentiation, but if one uses a function acting on the integers, then it is equivalent to the forward difference operator. The time-scale derivative applies, however, not only to the set of real numbers or set of integers, but to any time scale, that is, any closed subset $\mathbb{T}$ of the reals, such as the Cantor set. It turns out that closed form formulas of derivatives of concrete functions, valid on arbitrary time scales, are difficult to find in the literature, with some examples scattered over the literature. This paper intends to fill the gap by gathering systematically some of the most useful formulas of derivatives on time scales for reference purposes. Moreover, we present a simple method on how to prove all such formulas, based on the graininess function of the time scale and on a suitable integral representation of the time-scale derivative (Theorem 4.1).

The text is organized as follows. Section 2 presents the notations used along the manuscript and recalls the definition of delta derivative on time scales and one of its well-know important properties (Theorem 2.5). Section 3 provides a "table" of delta derivatives for twenty functions, organized in different subsections: derivatives of basic functions (Section 3.1); roots (Section 3.2); logarithms (Section 3.3); exponentials (Section 3.4); trigonometric functions (Section 3.5); products of trigonometric functions and monomials (Section 3.6) and trigonometric functions and exponentials (Section 3.7); and hyperbolic functions (Section 3.8).

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In Section 4 we prove a simple but powerful integral representation of the time-scale derivative (Theorem 4.1) that allows to obtain all given relations at Section 3 and others the reader may wish to prove. We end with Section 5 where, as an application of the given table of derivatives, we propose new definitions of trigonometric and hyperbolic functions on time scales that extend, in a proper way, the most essential qualitative properties of the corresponding continuous functions.

Two remarks are in order. Similarly to the classical calculus, integration on time scales is the inverse operation of differentiation, that is, the delta integral is defined as the antiderivative with respect to the delta derivative. This means that our formulas provide also closed formulas for integration on time scales. Our second remark concerns the dual concept of nabla derivative [3, 7]: all formulas given here are also true for the nabla derivative if one substitutes the forward graininess function by the symmetric of the backward graininess function, that is, $\mu(t)$ by $-\nu(t)$.

2. The delta derivative

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. Besides standard cases of $\mathbb{R}$ (continuous time) and $\mathbb{Z}$ (discrete time), many different models of time may be used, e.g., the h-numbers, $\mathbb{T} = h\mathbb{Z} := \{hz \mid z \in \mathbb{Z}\}$, where $h > 0$ is a fixed real number, and the q-numbers, $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k \mid k \in \mathbb{N}_0\}$, where $q > 1$ is a fixed real number. A time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. For each time scale $\mathbb{T}$, the following operators are used:

- the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$, defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ if $\sup \mathbb{T} < +\infty$;
- the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$, defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ for $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$;
- the *forward graininess function* $\mu : \mathbb{T} \to [0, \infty)$, defined by $\mu(t) := \sigma(t) - t$;
- the *backward graininess function* $\nu : \mathbb{T} \to [0, \infty)$, defined by $\nu(t) := t - \rho(t)$.

Example 2.1. If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{T}$ one has $\sigma(t) = \rho(t) = t$ and $\mu(t) = \nu(t) \equiv 0$. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then for every $t \in \mathbb{T}$ one has $\sigma(t) = t + h$, $\rho(t) = t - h$, and $\mu(t) = \nu(t) \equiv h$. On the other hand, if $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then we have $\sigma(t) = qt$, $\rho(t) = q^{-1}t$, $\mu(t) = (q-1)t$, and $\nu(t) = (1 - q^{-1})t$.

A point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* or *left-scattered*, if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ or $\rho(t) < t$, respectively. We say that t is *isolated* if $\rho(t) < t < \sigma(t)$, that t is *dense* if $\rho(t) = t = \sigma(t)$.

If $\sup \mathbb{T}$ is finite and left-scattered, we define $\mathbb{T}^\kappa := \mathbb{T} \setminus \{\sup \mathbb{T}\}$, otherwise $\mathbb{T}^\kappa := \mathbb{T}$.

Definition 2.2. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* at $t \in \mathbb{T}^\kappa$ if there exists a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \text{ for all } s \in U.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t and f is said *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Remark 2.3. If $t \in \mathbb{T} \setminus \mathbb{T}^\kappa$, then $f^\Delta(t)$ is not uniquely defined, since for such a point t, small neighborhoods U of t consist only of t and, besides, we have $\sigma(t) = t$. For this reason, maximal left-scattered points are omitted in Definition 2.2.

Example 2.4. If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if f is differentiable in the ordinary sense at t. Then, $f^\Delta(t) = f'(t)$. If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $f : \mathbb{Z} \to \mathbb{R}$ is always delta differentiable at every $t \in \mathbb{Z}$ with $f^\Delta(t) = \frac{f(t+h)-f(t)}{h}$. If $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, then $f^\Delta(t) = \frac{f(qt)-f(t)}{(q-1)t}$, i.e., we get the usual q-derivative of quantum calculus [10].

Theorem 2.5 (See Theorem 1.16 of [6])**.** *Let $\mathbb{T}$ be an arbitrarily given time scale. If f is delta differentiable at $t \in \mathbb{T}^\kappa$, then $f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t)$.*

3. Table of derivatives on time scales

In what follows, $\mathbb{T}$ is an arbitrary time scale with forward graininess function $\mu(t)$; k and c denote arbitrary constants; while n is a positive integer number: $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. The following formulas hold for $t \in \mathbb{T}^\kappa$ in the domain of the function under consideration. For example, expression (3.5) for the root function $f(t) = \sqrt{t}$ holds for any $t \in \mathbb{T}^\kappa$ such that $t > 0$. Note that in all the formulas given here, if substituting a value into an expression gives 0/0, the expression has an actual finite value and one should use limits to determine it. For example, if $\mu(t) = 0$, then formula (3.3) gives $\ln(k)\, k^t$ and formula (3.5) gives $\frac{1}{2\sqrt{t}}$.

3.1. Derivatives of basic functions.

$$(k)^\Delta = 0 \tag{3.1}$$

$$(t^n)^\Delta = \sum_{i=1}^{n} \binom{n}{i-1} \mu(t)^{n-i}\, t^{i-1} \tag{3.2}$$

$$\text{For } k > 0, \quad \left(k^t\right)^\Delta = \frac{k^{\mu(t)} - 1}{\mu(t)}\, k^t \tag{3.3}$$

$$((t+k)^n)^\Delta = \sum_{i=1}^{n} \binom{n}{i-1} \mu(t)^{n-i} (t+k)^{i-1} \tag{3.4}$$

3.2. Derivatives of roots.

$$(\sqrt{t})^\Delta = \frac{\sqrt{t+\mu(t)} - \sqrt{t}}{\mu(t)} \tag{3.5}$$

$$(\sqrt{k+t^n})^\Delta = \frac{\sqrt{k+(t+\mu(t))^n} - \sqrt{k+t^n}}{\mu(t)} \tag{3.6}$$

$$\left(t^n \sqrt{k+ct}\right)^\Delta = \frac{\sqrt{k+c(t+\mu(t))}\left(\sum_{i=0}^{n} \binom{n}{i} \mu(t)^{n-i} t^i\right) - \sqrt{k+ct}\, t^n}{\mu(t)} \tag{3.7}$$

3.3. Derivatives of logarithms.

$$(\ln(t^n))^\Delta = \frac{n \ln\left(1 + \frac{\mu(t)}{t}\right)}{\mu(t)} \tag{3.8}$$

$$(\ln(kt + c))^\Delta = \frac{\ln\left(1 + \frac{k\mu(t)}{kt+c}\right)}{\mu(t)} \tag{3.9}$$

3.4. Derivatives of exponentials.

$$\left(e^{kt}\right)^\Delta = \frac{e^{k\mu(t)} - 1}{\mu(t)}\, e^{kt} \tag{3.10}$$

$$\left(t^n\, e^{kt}\right)^\Delta = \frac{\left(\sum_{i=0}^{n} \binom{n}{i} \mu(t)^{n-i}\, t^i\right) e^{k\mu(t)} - t^n}{\mu(t)}\, e^{kt} \tag{3.11}$$

3.5. Derivatives of trigonometric functions.

$$(\sin t)^\Delta = \frac{\sin t\,(\cos \mu(t) - 1) + \cos t\, \sin \mu(t)}{\mu(t)} \tag{3.12}$$

$$(\cos t)^\Delta = \frac{\cos t\,(\cos \mu(t) - 1) - \sin t\, \sin \mu(t)}{\mu(t)} \tag{3.13}$$

3.6. Derivatives of products of trigonometric functions and monomials.

$$(t \sin(kt))^\Delta = \frac{(t + \mu(t))\,(\sin(kt) \cos(k\mu(t)) + \cos(kt) \sin(k\mu(t))) - t \sin(kt)}{\mu(t)} \tag{3.14}$$

$$(t \cos(kt))^\Delta = \frac{(t + \mu(t))\,(\cos(kt) \cos(k\mu(t)) - \sin(kt) \sin(k\mu(t))) - t \cos(kt)}{\mu(t)} \tag{3.15}$$

3.7. Products of trigonometric functions and exponentials.

$$\left(e^{kt} \sin(ct)\right)^\Delta = \frac{(\cos(ct) \sin(c\mu(t)) + \sin(ct) \cos(c\mu(t)))\, e^{k\mu(t)} - \sin(ct)}{\mu(t)}\, e^{kt} \tag{3.16}$$

$$\left(e^{kt} \cos(ct)\right)^\Delta = \frac{(\cos(ct) \cos(c\mu(t)) - \sin(ct) \sin(c\mu(t)))\, e^{k\mu(t)} - \cos(ct)}{\mu(t)}\, e^{kt} \tag{3.17}$$

3.8. Derivatives of hyperbolic functions.

$$(\sinh(kt))^{\Delta} = \frac{\left(e^{k(t+\mu(t))} - e^{kt}\right) - \left(e^{-k(t+\mu(t))} - e^{-kt}\right)}{2\mu(t)} \tag{3.18}$$

$$(\cosh(kt))^{\Delta} = \frac{\left(e^{k(t+\mu(t))} - e^{kt}\right) + \left(e^{-k(t+\mu(t))} - e^{-kt}\right)}{2\mu(t)} \tag{3.19}$$

$$(\sinh(kt)\cosh(kt))^{\Delta} = \frac{\left(e^{2k(t+\mu(t))} - e^{2kt}\right) - \left(e^{-2k(t+\mu(t))} - e^{-2kt}\right)}{4\mu(t)} \tag{3.20}$$

4. On the proof of equalities (3.1)–(3.20)

All the equalities (3.1)–(3.20) can be proved from the following result.

Theorem 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $\mathbb{T}$ be a given time scale with graininess function $\mu(t)$. If f is delta differentiable at $t \in \mathbb{T}^{\kappa}$, then*

$$f^{\Delta}(t) = \int_0^1 f'(t + \tau\mu(t))d\tau. \tag{4.1}$$

Proof. Let $\phi(\tau) = f(t + \tau\mu(t))$. Then $\phi(0) = f(t)$, $\phi(1) = f(\sigma(t))$, and $\phi'(\tau) = f'(t + \tau\mu(t))\mu(t)$. From the fundamental theorem of calculus, one has

$$\begin{aligned} f(\sigma(t)) - f(t) = \phi(1) - \phi(0) &= \int_0^1 \phi'(\tau)d\tau \\ &= \mu(t)\int_0^1 f'(t + \tau\mu(t))d\tau. \end{aligned}$$

Equality (4.1) follows from Theorem 2.5. □

5. Trigonometric and hyperbolic functions on time scales

The question of finding appropriate special functions on arbitrary time scales is an interesting and nontrivial subject under strong investigation: see, e.g., [5, 11]. For a survey of existing definitions and some new proposals of trigonometric and hyperbolic functions on time scales see [2, 8] and references therein. Here, based on our formulas of Section 3, we propose new versions of the sine, cosine, hyperbolic sine and hyperbolic cosine functions that extend, in a proper way, the most essential qualitative properties of the corresponding continuous functions.

5.1. New definitions of trigonometric functions on time scales. Having in mind (3.12) and (3.13) and the properties in $\mathbb{R}$

$$(\sin t)' = \cos t \quad \text{and} \quad (\cos t)' = -\sin t,$$

we define the sine and cosine functions on time scales, denoted respectively by $\sin_{\mathbb{T}}(t)$ and $\cos_{\mathbb{T}}(t)$, as follows.

Definition 5.1 (Trigonometric functions)**.** Let $\mathbb{T}$ be an arbitrarily given time scale with forward graininess function $\mu(t)$. The sine function on the time scale $\mathbb{T}$, $\sin_{\mathbb{T}} : \mathbb{T} \to \mathbb{R}$, is defined by

$$\sin_{\mathbb{T}}(t) = \frac{\sin t \, \sin \mu(t) - \cos t \, (\cos \mu(t) - 1)}{\mu(t)} \tag{5.1}$$

while the cosine function on the time scale $\mathbb{T}$, $\cos_{\mathbb{T}} : \mathbb{T} \to \mathbb{R}$, is defined by

$$\cos_{\mathbb{T}}(t) = \frac{\sin t \, (\cos \mu(t) - 1) + \cos t \, \sin \mu(t)}{\mu(t)}. \tag{5.2}$$

Remark 5.2. For $\mathbb{T} = \mathbb{R}$ one has $\mu(t) \equiv 0$ and we obtain from (5.1) and (5.2) that

$$\sin_{\mathbb{R}}(t) = \sin(t) \quad \text{and} \quad \cos_{\mathbb{R}}(t) = \cos(t).$$

Theorem 5.3. *Let $\mathbb{T}$ be an arbitrarily given time scale. The following relation holds:*

$$\sin^2_{\mathbb{T}}(t) + \cos^2_{\mathbb{T}}(t) = \frac{2(1 - \cos \mu(t))}{\mu^2(t)}. \tag{5.3}$$

Proof. Follows from Definition 5.1 by a direct computation. □

Remark 5.4. For $\mathbb{T} = \mathbb{R}$ one obtains from Remark 5.2 and (5.3) the fundamental Pythagorean trigonometric identity of the continuous case: $\sin^2(t) + \cos^2(t) = 1$.

5.2. New definitions of hyperbolic functions on time scales. Based on (3.18) and (3.19), and having in mind the properties in $\mathbb{R}$

$$(\sinh(t))' = \cosh(t) \quad \text{and} \quad (\cosh(t))' = \sinh(t),$$

we now define suitable hyperbolic functions sine and cosine on time scales, denoted respectively by $\sinh_{\mathbb{T}}(t)$ and $\cosh_{\mathbb{T}}(t)$.

Definition 5.5 (Hyperbolic functions)**.** Let $\mathbb{T}$ be an arbitrarily given time scale with graininess function $\mu(t)$. The hyperbolic sine on the time scale $\mathbb{T}$ is the function $\sinh_{\mathbb{T}} : \mathbb{T} \to \mathbb{R}$ defined by

$$\sinh_{\mathbb{T}}(t) = \frac{\left(e^{t+\mu(t)} - e^{t}\right) + \left(e^{-(t+\mu(t))} - e^{-t}\right)}{2\mu(t)}. \tag{5.4}$$

The hyperbolic cosine on the time scale $\mathbb{T}$ is the function $\cosh_{\mathbb{T}} : \mathbb{T} \to \mathbb{R}$ given by

$$\cosh_{\mathbb{T}}(t) = \frac{\left(e^{t+\mu(t)} - e^{t}\right) - \left(e^{-(t+\mu(t))} - e^{-t}\right)}{2\mu(t)}. \tag{5.5}$$

Remark 5.6. For $\mathbb{T} = \mathbb{R}$ one has $\mu(t) \equiv 0$ and we obtain from (5.4) and (5.5) the classical definitions:

$$\sinh_{\mathbb{R}}(t) = \frac{e^{2t} - 1}{2e^{t}} = \sinh(t), \quad \cosh_{\mathbb{R}}(t) = \frac{e^{2t} + 1}{2e^{t}} = \cosh(t).$$

Theorem 5.7. *Let $\mathbb{T}$ be an arbitrarily given time scale. The following relation holds:*

$$\cosh^2_{\mathbb{T}}(t) - \sinh^2_{\mathbb{T}}(t) = \frac{e^{\mu(t)} + e^{-\mu(t)} - 2}{\mu^2(t)}. \tag{5.6}$$

Proof. Follows from Definition 5.5 by a direct computation.

Remark 5.8. For $\mathbb{T} = \mathbb{R}$ one obtains from Remark 5.6 and (5.6) the most important qualitative property of the hyperbolic functions in the continuous case: $\cosh^2(t) - \sinh^2(t) = 1$.

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Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, 3810–193 Aveiro, Portugal

E-mail address: delfim@ua.pt

URL: http://orcid.org/0000-0001-8641-2505