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GLOBAL INVARIANT MANIFOLDS AND DECOMPOSITION OF ROBOTIC TYPE DYNAMICAL MODELS WITH SINGULAR PERTURBATIONS

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Abstract. The decomposition problem for the control systems with singular perturbations is considered in the paper. A global invariant manifolds of slow motions are applied to show that the system under considferation can be reduced to two subsystems of the lower dimension. A two-link horisontal manipulator model is considered to illustrate the mathematical results. In contradictory to the works of other authors, asymptotic expansions are used for the approximate construction of splitting transformations, and not for solving initial or boundary value problems.

1. Introduction

Consider the following differential system:

$$\dot{x} = f(x, y, t, \varepsilon), \varepsilon\dot{y} = g(x, y, t, \varepsilon), \tag{1.1}$$

where $t \in R$, $x \in R^m$, $y \in R^n$, and ε is a small positive parameter. The second equation of the system contains ε at the derivative. That makes the system singularly perturbed. Systems with singular perturbations are a typical object of study in control theory (see, for example, [1, 2, 3, 4] and references therein). Such systems are typical for some classes of robotic systems. The goals of the paper are to construct a transformation reducing (1.1) to the system

$$\dot{v} = \varphi(v, t, \varepsilon),\ \varepsilon\dot{z} = \eta(v, z, t, \varepsilon).$$

For example, the differential system $\dot{x} = x,\ \varepsilon\dot{y} = -y - x^2$ can be reduced to the form $\dot{v} = v,\ \varepsilon z = -z$ using transformation $x = v,\ y = z - v^2/(1 + 2\varepsilon)$, and the differential system $\dot{x} = y,\ \varepsilon\dot{y} = -y - y^2$ can be reduced to the form $\dot{v} = 0,\ \varepsilon z = -z - z^2$ using the transformation $x = v - \varepsilon \ln(1 + z),\ y = z$.

In the case of linear stationary systems, this is a well-known transformation that brings a linear homogeneous system to a block-diagonal form. A detailed description and history of the issue are reflected in the book [2]. In the case of non-stationary systems, this transformation was extended in the works of [5, 6].

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It should be noted that in the paper[6] the case of weak extinction of transient processes was also considered. This approach was suggested in [5] and it has been successfully used to solve a number of problems in control theory [7, 8, 9, 10, 11]. However, it is not a straightforward procedure to construct the transformation in the general case. One of the crucial steps is to find functions that describe slow and fast integral manifolds. The pioneered work [12] described the construction of a slow integral manifold as an asymptotic expansion in powers of a small parameter. This method was later used and well-studied by many authors (see, for example, the book [13] and references therein). Unfortunately, the construction of a fast integral manifold function is a much more complicated problem. There are no methods that work in the general case, and it is more art than science. Nevertheless, we describe some classes of systems for which these functions can be effectively constructed.

2. Method of Decomposition

We will use the method of decomposition of the system into two independent subsystems using the splitting transformation. Consider the differential system

$$\dot{x} = y, \tag{2.1}$$

$$\varepsilon\Psi(x)\dot{y} = \xi_0(x) + \varepsilon\xi_1(x) + [\Xi_0(x) + \varepsilon\Xi_1(x)]y + \varepsilon\Upsilon(x, y), \tag{2.2}$$

where $x \in R^n$, $y \in R^n$, $t > 0$, elements of vector-function $\Upsilon(x, y)$ are quadratic forms with respect to coordinates of vector y.

2.1. Main Assumptions. We will consider two types of systems. The first type is systems with a boundary layer, which main characteristic is a very rapid extinction of transient processes. The second type is systems with so-called weak energy dissipation. Transient processes in such systems fade away relatively slowly, but they make high-frequency oscillations.

2.1.1. *Systems with Boundary Layer.* The most common version of the basic assumption is as follows. We assume that the roots $\lambda_i(x)$ of the equation $\det|\lambda\Psi(x) - \Xi_0(x)| = 0$ have the property Re $\lambda_i(x) \le -2\gamma < 0$, where $x \in R^n$. Moreover, the matrix- and vector-functions Ψ, ξ_0, ξ_1, Ξ_0, Ξ_1, the coefficients of all quadratic forms of $\Upsilon(x, y)$, and their partial derivatives with respect to the arguments $x \in R^n$ are continuous and bounded. Then, any function of the boundary layer type can be considered as a function describing the transient process, for example, $\exp(-\gamma t/\varepsilon)$ or $\exp(-\gamma t/\varepsilon)\cos(t/\varepsilon)$.

2.1.2. *Systems with Weak Dissipation.* In this case, any function of the form $\exp(-t)\cos(t/\varepsilon)$ can be considered as a typical function describing the transient process. If matrices Ψ, Ξ_0, and Ξ_1 are constant, we assume that Ψ and $-\Xi_1$ are symmetric and positive definite, and Ξ_0 is nonsingular and skew-symmetric. If these matrices depend on x, we assume the fulfillment of similar conditions. For gyroscopic systems and manipulators, these matrices usually depend on x in a periodic manner, and it can be assumed that the above conditions are satisfied for all x.

2.2. Slow Invariant Manifold. When the assumptions for the corresponding type of system hold, (2.1), (2.2) has an attractive global slow invariant manifold

$$y = h(x,\varepsilon) = h_0(x) + \varepsilon h_1(x) + \dots .$$

The functions h_i can be derived from the invariance equation

$$\varepsilon \Psi \frac{\partial h}{\partial x} h = \xi + \Xi h + \varepsilon \Upsilon(x,h),$$

where $\Xi(x,\varepsilon) = \Xi_0(x) + \varepsilon \Xi_1(x)$, $\xi(x,\varepsilon) = \xi_0(x) + \varepsilon \xi_1(x)$. The formulae for the coefficients of the asymptotic expansions of slow invariant manifold $h = h(x,\varepsilon)$ take the form $h_0 = -\Xi_0^{-1}\xi_0$, $h_1 = \Xi_0^{-1}\left[\Psi \frac{\partial h_0}{\partial x} h_0 - \xi_1 - \Xi_1 h_0 - \Upsilon(x,h_0)\right]$.

2.3. Fast Invariant Manifold. In this case the invariance equation for the fast invariant manifold $H = H(v,z,\varepsilon)$ takes the form [5, 7]

$$\varepsilon \frac{\partial H}{\partial v} h(v,\varepsilon) + \frac{\partial H}{\partial z} \Psi^{-1}(v+\varepsilon H)[\Xi(v+\varepsilon H,\varepsilon)$$

$$-\varepsilon \frac{\partial h}{\partial x}(v+\varepsilon H,\varepsilon)]z = z + h(v+\varepsilon H,\varepsilon) - h(v,\varepsilon).$$

Setting $\varepsilon = 0$, we obtain $\dfrac{\partial H_0}{\partial z}\Psi^{-1}(v)\Xi_0(v)z = z$. It is possible to represent $H(v,z,\varepsilon)$ in the form $H(v,z,\varepsilon) = D(v,z,\varepsilon)z$ [5]. This implies $H_0(v,z) = D_0(v)z$, where the matrix $D_0(v)$ satisfies the equation $D_0(v)\Psi^{-1}(v)\Xi_0(v) = I$, and, therefore, $H_0(v,z) = \Xi_0^{-1}(v)\Psi(v)z$.

2.4. Representation of Solutions. Let $(x(t),\ y(t))$ be a solution to (2.1), (2.2) with an initial condition $x(t_0) = x_0$, $y(t_0) = y_0$. There exists a solution $(v(t),\ z(t))$ of

$$\dot v = h(v,\varepsilon),\ \varepsilon \Psi\left(v + \varepsilon H(v,z,\varepsilon)\right)\dot z = Z(v,z,\varepsilon),$$

with the initial condition $v(t_0) = v_0$, $z(t_0) = z_0$, such that [5]

$$x(t) = v(t) + \varepsilon H(v(t),z(t),\varepsilon),\ y(t) = z(t) + h(x(t),t,\varepsilon). \tag{2.3}$$

This means that the solution $x = x(t,\varepsilon)$, $y = y(t,\varepsilon)$ of the original system (2.1), (2.2) that satisfied the initial condition $x(0,\varepsilon) = x_0$, $y(t_0,\varepsilon) = y_0$ can be represented as

$$\begin{array}{rcl} x(t,\varepsilon) & = & v(t,\varepsilon) + \varepsilon\varphi_1(t,\varepsilon), \\ y(t,\varepsilon) & = & \bar y(t,\varepsilon) + \varphi_2(t,\varepsilon). \end{array} \tag{2.4}$$

From the main assumption it follows the existence of number $M > 1$ such that $\|z(t,\varepsilon)\| \le M\exp(-\gamma t/\varepsilon)\|z_0\|,\ t \ge 0$, i.e.,

$$\|\varphi_i\| \le M_i \exp(-\gamma t/\varepsilon)\|z_0\|,\ t \ge 0,\ i - 1, 2, \tag{2.5}$$

for the systems with boundary layers and $\|z(t,\varepsilon)\| \le M\exp(-\gamma t)\|z_0\|,\ t \ge 0$, i.e.,

$$\|\varphi_i\| \le M_i \exp(-\gamma t)\|z_0\|,\ t \ge 0,\ i - 1, 2, \tag{2.6}$$

for the systems with weak dissipation. Thus, this solution is represented as a sum of solution which lies on the slow invariant manifold, i.e.

$$x = x(t,\varepsilon) = v(t,\varepsilon),\ \bar y(t,\varepsilon) = h(v(t,\varepsilon),\varepsilon),$$

and exponentially decreasing functions

$$\varepsilon\varphi_1(t,\varepsilon) = \varepsilon H(v(t,\varepsilon), z(t,\varepsilon),\varepsilon),$$
$$\varphi_2(t,\varepsilon) = z(t,\varepsilon) + h(v(t,\varepsilon)$$
$$+\varepsilon H\left(v(t,\varepsilon), z(t,\varepsilon),\varepsilon\right) - h\left(v(t,\varepsilon),\varepsilon\right).$$

Neglecting terms of order $o(\varepsilon)$, we use the transformation

$$x = v + \varepsilon H_0(v,z), \quad y = z + h_0(x) + \varepsilon h_1(x) \tag{2.7}$$

to reduce (2.1) to a nonlinear block-triangular form:

$$\dot v = h_0(v) + \varepsilon h_1(v) + O(\varepsilon^2),$$
$$\varepsilon\Psi(v)\dot z = \Big[\Xi_0(v) + \varepsilon\Big(\Xi_1(v) + \frac{\partial \Xi_0}{\partial x}(v)\Xi_0^{-1}(v)\Psi(v)z$$
$$-\Psi(v)\frac{\partial h_0}{\partial x}(v)\Big) + O(\varepsilon^2)\Big]\, z.$$

2.5. Lyapunov Reduction Principle. The representation (2.4) and inequalities (2.5), (2.6) immediately imply the validity of the Lyapunov Reduction Principle. It means that any solution which lies on the slow invariant manifold is stable (asymptotically stable, unstable), if and only if the corresponding solution of

$$\dot v = h(v,\varepsilon),$$

is stable (asymptotically stable, unstable). The Lyapunov Reduction Principle can be extended to systems with a manifold of equilibrium states.

3. Two-Link Robotic Manipulator Model with Symmetric Manifolds of Steady States

Let us consider the model of horizontal robotic manipulator [15]

$$\varepsilon(\alpha+\beta+2\cos q_2)\ddot q_1 + \varepsilon(\beta+\cos q_2)\ddot q_2$$
$$-\varepsilon(2\dot q_1\dot q_2 + \dot q_2^2)\sin q_2 = u_1,$$
$$\varepsilon(\beta+\cos q_2)\ddot q_1 + \varepsilon\beta\ddot q_2 + \varepsilon\dot q_1^2\sin q_2 = u_2.$$

It should be noted that in the absence of control, the considered differential system has a symmetric manifold of steady states $q_1 = const,\ q_2 = const,\ \dot q_1 = 0,\ \dot q_2 = 0$. Therefore, it seems natural to consider the problem of reaching one of the states on this manifold.

Let the goal of control is to attach some given position $q_1 = \tilde q_1,\ q_2 = \tilde q_2,\ \dot q_1 = 0,\ \dot q_2 = 0$. Let

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} q_1 - \tilde q_1 \\ q_2 - \tilde q_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \dot q_1 \\ \dot q_2 \end{pmatrix}, u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

then

$$\Psi(x) = \begin{pmatrix} \alpha+\beta+2\cos(x_2+\tilde q_2) & \beta+\cos(x_2+\tilde q_2) \\ \beta+\cos(x_2+\tilde q_2) & \beta \end{pmatrix},$$
$$\Upsilon(x,y) = \begin{pmatrix} -(2y_1y_2 + y_2^2)\sin(x_2+\tilde q_2) \\ y_1^2\sin(x_2+\tilde q_2) \end{pmatrix}.$$

We will consider two cases.

3.1. Rapid Control. Setting the control of form $u_1 = -\gamma_1 x_1 - \kappa_1 y_1$, $u_2 = -\gamma_2 x_2 - \kappa_2 y_2$, with positive γ_1, γ_2, κ_1, κ_2, we obtain the following representations

$$\xi_0 = \begin{pmatrix} -\gamma_1 x_1 \\ -\gamma_2 x_2 \end{pmatrix} \ \Xi_0 = \begin{pmatrix} -\kappa_1 & 0 \\ 0 & -\kappa_2 \end{pmatrix},$$

$\xi_1 = 0, \ \Xi_1 = 0.$

3.2. Splitting Transformation. In the case under consideration the characteristic equation takes the following form

$$\left[\alpha\beta - \cos^2(x_2 + \tilde{q}_2)\right] \lambda^2 + \{\kappa_1\beta + \kappa_2 \left[\alpha + \beta + 2\cos(x_2 + \tilde{q}_2)\right]\} \lambda + \kappa_1\kappa_2 = 0.$$

Taking into account that $\alpha\beta > 1$ [15], it is easy to see that all the coefficients of this equation are positive, which immediately entails the fulfillment of the condition on the roots of this equation.

The zero-order approximation of slow invariant manifold takes the form

$$h_0(x_1, x_2) = \begin{pmatrix} -\frac{\gamma_1}{\kappa_1} x_1 \\ -\frac{\gamma_2}{\kappa_2} x_2 \end{pmatrix}.$$

The main term of the fast manifold takes the form

$$H_0(v, z) = \Xi_0^{-1}(v)\Psi(v)z$$
$$= \begin{pmatrix} -\frac{1}{\kappa_1}(\alpha + \beta + 2\cos v_2) & -\frac{1}{\kappa_1}(\beta + \cos v_2) \\ -\frac{1}{\kappa_2}(\beta + \cos v_2) & -\frac{1}{\kappa_2}\beta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Using the splitting transformation (2.7) we obtain the independent slow subsystem

$$\dot{v} = h_0(v) + \varepsilon h_1(v) + O(\varepsilon^2),$$

and fast subsystem

$$\varepsilon\Psi(v)\dot{z} = [\Xi_0(v) - \varepsilon\Psi(v)\frac{\partial h_0}{\partial x})(v) + O(\varepsilon^2)]z,$$

since Ξ_0 is a constant matrix, with

$$\Xi_0(v) - \varepsilon\Psi(v)\frac{\partial h_0}{\partial x}(v) = \begin{pmatrix} -\kappa_1 & 0 \\ 0 & -\kappa_2 \end{pmatrix}$$
$$+\varepsilon \begin{pmatrix} \frac{\gamma_1}{\kappa_1}(\alpha + \beta + 2\cos v_2) & \frac{\gamma_2}{\kappa_2}(\beta + \cos v_2) \\ \frac{\gamma_1}{\kappa_1}(\beta + \cos v_2) & \frac{\gamma_2}{\kappa_2}\beta \end{pmatrix}.$$

Strictly speaking, in the case of quick control, there is no need to calculate the first-order terms. The motion on a slow invariant manifold is described by a differential system

$$\dot{v} = \left[\begin{pmatrix} -\frac{\gamma_1}{\kappa_1} & 0 \\ 0 & -\frac{\gamma_2}{\kappa_2} \end{pmatrix} + O(\varepsilon)\right] v,$$

and transient processes are described by a differential system

$$\varepsilon\Psi(v)\dot{z} = \left[\begin{pmatrix} -\kappa_1 & 0 \\ 0 & -\kappa_2 \end{pmatrix} + O(\varepsilon)\right] z.$$

It is clear that the zero solution is exponentially stable, and the transients decay almost instantly. In conclusion, we present the reduced equations for a system with weak decay of transient processes.

3.3. Soft Control. Setting the control of form $u_1 = -\gamma_1 x_1 - \varepsilon\kappa_1 y_1 + \kappa y_2,\ u_2 = -\gamma_2 x_2 - \varepsilon\kappa_2 y_2 - \kappa y_1$, with positive $\gamma_1, \gamma_2, \kappa, \kappa_1, \kappa_2$, we obtain the following representations

$$\xi_0 = \begin{pmatrix} -\gamma_1 x_1 \\ -\gamma_2 x_2 \end{pmatrix},\ \Xi_0 = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix},$$

$$\Xi_1 = \begin{pmatrix} -\kappa_1 & 0 \\ 0 & -\kappa_2 \end{pmatrix},\ \xi_1 = 0.$$

The zero-order approximation of slow invariant manifold takes the form

$$h_0(x_1, x_2) = \begin{pmatrix} -\frac{\gamma_2}{\kappa} x_2 \\ \frac{\gamma_1}{\kappa} x_1 \end{pmatrix}.$$

To obtain the first order approximation it is necessary to use the representation

$$h_1 = \Xi_0^{-1} \left[\Psi \frac{\partial h_0}{\partial x} h_0 - \Xi_1 h_0 - \Upsilon(x, h_0) \right].$$

The flow on the slow invariant manifold is described by the differential system

$$\dot{v} = \left[\begin{pmatrix} 0 & -\frac{\gamma_2}{\kappa} \\ \frac{\gamma_1}{\kappa} & 0 \end{pmatrix} \right.$$

$$+ \frac{\varepsilon}{\kappa^2} \begin{pmatrix} -\gamma_1\kappa_2 + \gamma_1\gamma_2\psi_2/\kappa & \gamma_1\gamma_2\psi_2/\kappa \\ -\gamma_1\gamma_2\psi_1/\kappa & -\gamma_2\kappa_1 - \gamma_1\gamma_2\psi_2/\kappa \end{pmatrix}$$

$$\left. + O(\varepsilon^2) + \varepsilon O(\|v\|) \right] v.$$

Here $\psi_i,\ i = 1, 2, 3$ are elements of $\Psi(0)$, i.e.,

$$\psi_1 = \alpha + \beta + 2\cos(\tilde{q}_2),\ \psi_2 = \beta + \cos(\tilde{q}_2),\ \psi_3 = \beta.$$

By virtue of the Lyapunov's Indirect Method, this subsystem is asymptotically stable. Thus, we can conclude that the control goal has been achieved taking into account that the solutions of the subsystem which describes the transient processes, exponentially fade away. To verify that the transient processes decay exponentially we can use a well-known Ważevski inequality. Recall that any solution of the linear homogeneous system

$$\frac{dx}{dt} = A(t)x$$

with continuous on $(0, +\infty)$ matrix A satisfies the Ważevski inequality

$$\|x(0)\| e^{\left(\int_0^t \lambda(\tau) d\tau\right)} \le \|x(t)\| \le \|x(0)\| e^{\left(\int_0^t \Lambda(\tau) d\tau\right)}.$$

Here $\|\cdot\|$ is the Euclidean norm, $\lambda(t)$ and $\Lambda(t)$ are smallest and greatest eigenvalues of matrix

$$\frac{1}{2}\left(A(t) + A^T(t)\right),$$

where $A^T(t)$ is a transpose matrix, see [16].

Transient processes are described by a differential system

$$\varepsilon\Psi(v)\dot{z} = \left[\Xi_0 - \varepsilon\Psi(v) \frac{\partial h_0}{\partial x} + \varepsilon\Xi_1 + O(\varepsilon^2) \right] z.$$

Here
$$\Psi(v)\frac{\partial h_0}{\partial x}(v) = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_2 & \psi_3 \end{pmatrix} \begin{pmatrix} 0 & -\frac{\gamma_2}{\kappa} \\ \frac{\gamma_1}{\kappa} & 0 \end{pmatrix},$$
where $\psi_i(v),\ (i = 1, 2, 3)$ are elements of $\Psi(v)$, i.e., $\psi_1(v) = \alpha + \beta + 2\cos(v_2 + \tilde{q}_2),\ \psi_2(v) = \beta + \cos(v_2 + \tilde{q}_2),\ \psi_3 = \beta$. Thus, setting $\gamma_1 = \gamma_2 = \gamma$ and $\kappa_2 = \kappa_1$, we obtain the following equation for z
$$\varepsilon\Psi(v)\dot{z} = \Bigg[\begin{pmatrix} 0 & -\frac{\gamma}{\kappa} \\ \frac{\gamma}{\kappa} & 0 \end{pmatrix}$$
$$+\varepsilon \begin{pmatrix} -\kappa_1 - \gamma\psi_2(v)/\kappa & \gamma\psi_1(v)/\kappa \\ -\gamma\psi_3/\kappa & -\kappa_1 + \gamma\psi_2(v)/\kappa \end{pmatrix}$$
$$+O(\varepsilon^2) + \varepsilon O(\|z\|)\Bigg] z.$$
To analyse the last system it is convenient to introduce a new vector variable z_1 by the formula
$$z_1 = \Phi(v)z,$$
where $\Phi(v)$ is a unique positive definite square root of $\Psi(v)$, i.e., $\Phi^2 = \Psi$. Then,
$$\varepsilon\dot{z}_1 = [A_0(v) + \varepsilon A_1(v) + O(\varepsilon^2) + \varepsilon O(\|z_1\|)]z_1.$$
Here
$$A_0(v) = \Phi(v)^{-1} \begin{pmatrix} 0 & -\frac{\gamma}{\kappa} \\ \frac{\gamma}{\kappa} & 0 \end{pmatrix} \Phi(v)^{-1}.$$
Note that $A_0(v)$ is a skew-symmetric matrix and, therefore, $A_0(v) + A_0^T(v) = 0$.

The matrix $A_1(v)$ may be represented in the form
$$A_1(v) = -\kappa_1\Psi^{-1}(v) + \frac{\gamma}{\kappa}\begin{pmatrix} -\psi_2(v) & \psi_1(v) \\ -\psi_3 & \psi_2(v) \end{pmatrix}$$
$$+\frac{\gamma}{2\kappa}\frac{\partial\Psi(v)}{\partial v}\Psi^{-1}(v)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} v.$$

Since $\Psi^{-1}(v)$ is a symmetric positive definite matrix , then with a suitable choice of parameters κ_1 and κ (or γ), the largest eigenvalue of the matrix
$$\frac{1}{2}\left(A_1(v) + A_1^T(v)\right)$$
will be negative. By virtue of the Ważevski equality, this means that transient processes exponentially fade away and the control goal can be considered achieved.

4. Conclusion

In this paper, we investigated the decomposition problem for a special class of systems of differential equations with singular perturbations. Unlike most of the available literature on the subject, we considered not only systems with a boundary layer, but also systems with weak energy dissipation, a characteristic feature of which is a relatively slow decay of transient processes. The proposed approach tested here shows that the use of the decomposition method greatly simplifies the analysis of models of manipulation robots.

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