Journal of Mathematical Control Science and Applications
Vol. 7 No. 1 (January-June 2021)

Received:18th January 2021 *Revised: 10th February 2021* *Accepted: 27th February 2021*

PROBLEM OF THE OPTIMAL AMOUNT OF RIGID THIN SEGMENTS FOR AN EQUILIBRIUM MODEL OF A TWO-DIMENSIONAL BODY WITH A CRACK

NYURGUN LAZAREV*, GALINA SEMENOVA, AND NATALYYA ROMANOVA

Abstract. Variational problems for composite two-dimensional cracked bodies with a system of joined rigid inclusions are considered. The deformation of the body's matrix is described using the classical elastic constitutive equations. Nonlinearity of the considered problem is caused by an inequality-type boundary condition that describes mutual nonpenetration of opposite crack faces. We investigate two types of equilibrium models that correspond to different types of rigid inclusions. For the first model, we suppose that the body has a volume rigid inclusion which is described by a corresponding domain, and the second one describes the body containing a set of fastened thin rigid inclusions, each of which corresponds to a curve. The crack is defined by the same curve in both models. An optimal control problem is formulated in the framework of the both model, such that a control is specified by the number of thin rigid thin rectilinear segments and by the limiting case which as it turned out, fits to the first model. A quality functional is defined by an arbitrary continuous functional in a suitable Sobolev space. The solvability of the optimal control problem is proved.

1. Introduction

Composite materials have emerged as the materials of choice for their attractive mechanical, thermal, environmental properties. Modern capabilities of engineering approaches allow us to simulate the possible behavior of composites based on appropriate mathematical models. Various problems concerned with deformations of composite bodies with inhomogeneities give rise to numerous new approaches and solutions in the field of applied mathematics. Difficulties in studying problems of this type can be associated with the non-smoothness of domains and the complexity of taking into account the conjugation of various materials.

It is well known that the presence of inclusions or cracks in loaded solids can cause significant stress concentrations. This, in turn, can lead to generation of delaminations and cracks near inclusions. Another cause of cracking may be the peculiarities of temperature regimes of an operating environment. Various models of composite solids with both rigid inclusions and cracks, are under active studying [1–9].

2000 *Mathematics Subject Classification.* Primary 49J55; Secondary 49J40.

Key words and phrases. optimal control problem, equilibrium problem, composite body, nonpenetration condition, variational inequality.

* This work has been supported by the Ministry of Education and Science of the Russian Federation within the framework of the base part of the state task FSRG-2020-0006.

In this work, we follow the well-known approach that uses inequality-type boundary conditions on the crack faces [10–17]. This circumstance determines the nonlinearity of boundary conditions and leads to variational formulations. As well as for the well-known Signorini problems, the problem statements exclude a priori knowledge of the contact zones of mechanical interaction of opposite crack faces. The generality of the variational calculus gives us the opportunity to provide successful formulations and investigations of various problems for composite solids with elastic or rigid inclusions, see, for example [18–32]. Optimal control problems concerning geometrical properties of cracks or rigid inclusions in elastic bodies are investigated in [5, 19, 24] and many other papers.

In fact, in this paper, we formulate a new optimal control problem, which, from a practical point of view, has a quite clear interpretation. In particular, it deals with the issues of strengthening the body with rigid elements. As well as in the paper [32], we consider two different types of two-dimensional models describing the equilibrium of an elastic body with a rigid inclusion. According to the paper [20], we will adopt the following characterizations for inclusions: the term "thin inclusion" is used when the dimension of inclusion's set is one less than the matrix's dimension, while the term "volume inclusion" is used when these dimensions coincide. For the first type of an equilibrium model, we suppose that the body has a volume inclusion with an initially-debonded patch on its interfacial surface. So, we have a initial delamination crack lying on a part of the volume inclusion boundary. The second type of model concerns a system of joined thin rigid inclusions, which specified by a union of a finite number of straight line segments and a curve joining these segments. It should be noted that both types of considered inclusions are connected in a clear geometrical sense and cracks are given by the same curve in both models. We formulate an optimal control problem, where a control is specified by the number of rectilinear segments that fit thin rigid inclusions and by the limiting case of infinite segments which corresponds to the first model. A quality functional is defined by an arbitrary continuous functional in a suitable Sobolev space. The solvability of the optimal control problem is proved.

2. Equilibrium problem for an elastic body with a volume rigid inclusion

Let us formulate an equilibrium problem for an elastic body containing a volume rigid inclusion. We consider the case of the partly delaminated inclusion. In this case we have an interfacial crack passing along the inclusion boundary. In addition, we suppose that the rest of the crack can be situated inside the elastic medium. Consider a bounded domain $\Omega \subset \mathbf{R}^2$ with the boundary $\Gamma \in C^{0,1}$. We consider a strictly inner subdomain ω of Ω ($\overline{\omega} \subset \Omega$) having the shape of a curvilinear rectangle of width a:

$$\omega = \{(x_1, x_2) \,|\, 0 < x_1 < 1, \quad g(x_1) < x_2 < g(x_1) + a\}, \quad a > 0,$$

where $g \in C^{0,1}(0, 2)$. The crack in the body is defined by the unclosed Lipschitz curve

$$\gamma = \{(x_1, x_2) \,|\, 0 < x_1 < 1 + \lambda, \quad x_2 = g(x_1) + a\}, \quad \bar{\gamma} \subset \Omega, \quad -1 < \lambda < 1,$$

which lies on the part of the boundary of ω. We assume that the domain Ω can be split into two subdomains Ω_1 and Ω_2 with Lipschitz boundaries such that $\gamma \subset \partial\Omega_1 \cap \partial\Omega_2$, $\mathrm{meas}(\partial\Omega_i \cap \Gamma) > 0$, $i = 1, 2$. This condition guarantees the validity of the Korn inequality in the non-Lipschitz domain $\Omega_\gamma = \Omega \backslash \bar{\gamma}$. Depending on the direction of the normal $\nu = (\nu_1, \nu_2)$ to γ we will speak about a positive face γ^+ or a negative face γ^- of the curve γ.

The domain ω fits a volume rigid inclusion, while the domain $\Omega_\gamma \backslash \overline{\omega}$ corresponds to an elastic part of the body. Denote by $W = (w_1, w_2)$ the displacement vector.

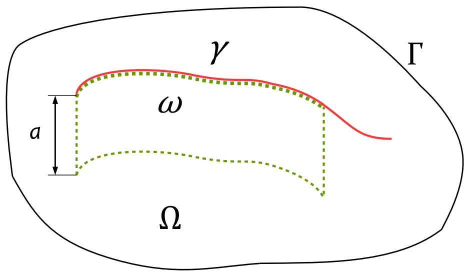

FIGURE 1. Geometry of the cracked body with a volume rigid inclusion.

Introduce the Sobolev spaces

$$H^{1,0}(\Omega_\gamma) = \{v \in H^1(\Omega_\gamma) \mid v = 0 \quad \text{on} \quad \Gamma\}.$$

Introduce the tensors describing the deformation of the body

$$\varepsilon_{ij}(W) = \frac{1}{2}(w_{i,j} + w_{j,i}),\ i, j = 1, 2, \quad (w_{i,j} = \frac{\partial w_i}{\partial x_j}),$$

$$\sigma_{ij}(W) = c_{ijkl}\varepsilon_{ij}(W),\ i, j = 1, 2,$$

where c_{ijkl} is the given elasticity tensor, assumed to be symmetric and positive definite:

$$c_{ijkl} = c_{klij} = c_{jikl}, \quad i, j, k, l = 1, 2, \quad c_{ijkl} = const.,$$

$$c_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2, \quad \forall\xi, \quad \xi_{ij} = \xi_{ji}, \quad i, j = 1, 2, \quad c_0 = const., \quad c_0 > 0.$$

Due to the presence of the rigid inclusion in the body, the displacement field satisfies a special kind of relations on the corresponding domain ω. The linear space of infinitesimal rigid displacements $R(\omega)$ is defined as follows [33]:

$$R(\omega) = \{\rho = (\rho_1, \rho_2) \mid \rho(x) = b(x_2, -x_1) + (c_1, c_2);\ b, c_1, c_2 \in \mathbf{R},\ x \in \omega\}.$$

The condition of mutual nonpenetration of opposite faces of the crack is given by [10, 33]

$$[W]\nu \geq 0 \quad \text{on} \quad \gamma,$$

where $[W] = W|_{\gamma^+} - W|_{\gamma^-}$ is the jump of W on γ with two opposite crack faces γ^+ and γ^-.

In order to provide a variational formulation describing the equilibrium state for the body with the rigid inclusion ω, and the crack γ, we introduce the energy functional

$$\Pi(W) = \frac{1}{2}\int\limits_{\Omega_\gamma} \sigma_{ij}(W)\varepsilon_{ij}(W) - \int\limits_{\Omega_\gamma} FW,$$

where the vector $F = (f_1, f_2) \in L^2(\Omega_\gamma)^2$ describes the external forces acting on the body, $FW = f_i w_i$. Consider the minimization problem:

$$\text{find} \quad U_\omega \in K \quad \text{such that} \quad \Pi(U_\omega) = \inf_{W\in K} \Pi(W), \tag{2.1}$$

where

$$K = \{W \in H^{1,0}(\Omega_\gamma)^2 \,|\, [W]\nu \geq 0 \text{ on } \gamma; \quad W|_\omega \in R(\omega)\}.$$

The problem (2.1) is known to have a unique solution $U_\omega \in K$, which satisfies the variational inequality [32, 33]

$$\int\limits_{\Omega_\gamma\setminus\overline{\omega}} \sigma_{ij}(U_\omega)\varepsilon_{ij}(W - U_\omega) \geq \int\limits_{\Omega_\gamma} F(W - U_\omega) \quad \forall W \in K. \tag{2.2}$$

We note that because of the structure of the displacement in the domain ω we have $\varepsilon_{ij}(W) = 0$, $i, j = 1, 2$, for all $W \in K$. Therefore, the inequality (2.2) can be rewritten as

$$U_\omega \in K, \quad \int\limits_{\Omega_\gamma} \sigma_{ij}(U_\omega)\varepsilon_{ij}(W - U_\omega) \geq \int\limits_{\Omega_\gamma} F(W - U_\omega) \quad \forall W \in K.$$

3. Family of equilibrium problems for an elastic bodies with a thin rigid inclusion

Along with the equilibrium problem (2.1), we will consider the following equilibrium problems for a special thin rigid inclusion [32]. We start with a description of the geometrical properties of the inclusions' shape. We suppose that Q_n is a union of line segments and a special Lipschitzian curve $\mathcal{L}$, so that

$$Q_n = (\bigcup_{k=1}^{k=2^n} l_k^n) \cup \mathcal{L}, \quad n = 1, 2, \ldots,$$

where 2^n is a quantity of the following similar line segments

$$l_k^n = \{(x_1, x_2) \,|\, x_1 = k/2^n,\ g(x_1) < x_2 < g(x_1) + a\}, \quad k = 1, 2, \ldots 2^n,$$

and $\mathcal{L}$ is the curve

$$\mathcal{L} = \{(x_1, x_2) \,|\, 0 \leq x_1 \leq 1, \quad x_2 = \psi(x_1)\}$$

defined by the function $\psi \in C^{0,1}[0, 1]$ satisfying

$$g(x_1) \leq \psi(x_1) < g(x_1) + a, \quad 0 \leq x_1 \leq 1.$$

Next we fix $n \in \mathbf{N}$ and assume that the set Q_n fits rigid inclusion, so that the corresponding space of infinitesimal rigid displacements has the form

$$R(Q_n) = \{\rho = (\rho_1, \rho_2) \,|\, \rho(x) = b(x_2, -x_1) + (c_1, c_2);\ b, c_1, c_2 \in \mathbf{R},\ x \in Q_n\}.$$

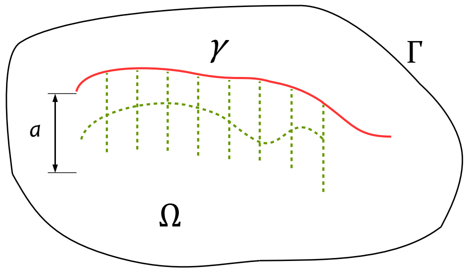

FIGURE 2. Geometry of the cracked body with a system of joined thin inclusions (example of Q_3).

A variational statement of the equilibrium problem for an elastic body with a system of joined thin rigid inclusions and a crack has the following form

$$\text{find} \quad U_n \in K_n \quad \text{such that} \quad \Pi(U_n, \Omega) = \inf_{W \in K_n} \Pi(W, \Omega), \tag{3.1}$$

$$K_n = \{W \in H^{1,0}(\Omega_\gamma)^2 \,|\, [W]\nu \geq 0 \text{ on } \gamma; \quad W|_{Q_n} \in R(Q_n)\}.$$

The existence and uniqueness of solution U_n of problem (3.1) can be proven as in the case for one delaminated inclusion, see [4]. The corresponding variational inequality takes the form

$$U_n \in K_n, \quad \int\limits_{\Omega_\gamma} \sigma_{ij}(U_n)\varepsilon_{ij}(W - U_n) \geq \int\limits_{\Omega_\gamma} F(W - U_n) \quad \forall W \in K_n.$$

4. Optimal control problem

It is known that the sequence of solutions $\{U_n\}$ converge to U_ω strongly in $H^{1,0}(\Omega_\gamma)^2$ as n tends to ∞ [32]. Therefore, for an arbitrary continuous functional $G : H(\Omega) \to \mathbb{R}$ we can define a cost functional $J : \mathbf{N} \cup \infty \to \mathbb{R}$ of an optimal control problem with the use of the following equalities $J_G(n) = G(U(n))$, $n \in \mathbf{N}$ where $U(n)$ is the solution of the problem (2.1), $J_G(\infty) = G(U_\omega)$.

As examples of such functionals having important physical sense, we can provide the functional $G_1(W) = \|W - W_0\|_{H(\Omega)}$ characterizing the deviation of the displacement vector from a given function W_0. Under assumptions that $0 < \lambda < 1$ and that the part

$$\gamma_r = \{(x_1, x_2) \,|\, 1 < x_1 < 1 + \lambda, \quad x_2 = g(x_1) + a\}$$

of the crack's curve γ lying inside the elastic medium is a straight line, i.e. $g(x_1) = c$, $c = \text{const}$, the second example could be given by the first derivative of the energy functional with respect to the crack length. It is well-known that Griffith rupture criterion relies on values of the first derivative of the energy functional with respect to the crack length [34, 35]. It is known (see [33]) that the functional of the first derivative of the energy functional with respect to the crack length can be expressed as

$$G(W) \;=\; \int\limits_{\Omega_0} \Big\{ \frac{1}{2}\theta_{,1}\, \sigma_{ij}(W)\varepsilon_{ij}(W) \;-\; \sigma_{ij}(W) w_{i,1}\theta_{,j} \Big\} \;-\; \int\limits_{\Omega_0} (\theta f_i)_{,1}\, w_{i,1}. \qquad (4.1)$$

In the formulae (4.1) the value of the derivative of the energy functional is independent of the choice of function θ satisfying $\mathrm{supp}(\theta) \subset \mathcal{O}_1$ and $\theta = 1$ in $\mathcal{O}_2$. Here $\mathcal{O}_2$ and $\mathcal{O}_1$ are some small neighbourhoods of the point $(1 + \lambda, a + c) \in \mathbf{R}^2$, $\mathcal{O}_2 \subset \mathcal{O}_1 \subset \Omega$, $\mathcal{O}_1 \cap \bar{\omega} = \emptyset$. We can refer to paper [36] for expanded explanations related to the first derivative of the energy functional for a two-dimensional body with a rigid inclusion and a crack.

Consider the optimal control problem:

$$\text{Find} \quad n^* \in \mathbf{N} \cup \infty \quad \text{such that} \quad J_G(n^*) = \sup_{n \in \mathbf{N} \cup \infty} J_G(n). \qquad (4.2)$$

This means that we want to find the optimal inclusion's amount which provides the maximal value for the cost functional. The following is our main existence result.

Theorem 4.1. *There exists a solution of the optimal control problem (4.2).*

Proof. We will distinguish the following two cases:

1. For some $m \in \mathbf{N}$

$$J(m) \geq J(n), \quad \forall n \in \mathbf{N};$$

2. For every $m \in \mathbf{N}$ there exists a number $n \in \mathbf{N}$ such that $n > m$ and

$$J(m) < J(n).$$

Obviously, for the first case we have a solution of (4.2) given by the value $n^* = m$. For the second case we can construct a subsequence $\{n_k\}$ such that

$$J(n_1) < J(n_2) < \ldots < J(n_k) < \ldots,$$

where $J(n_k) = G(U(n_k))$. Since $U_{n_k} \to U_\omega$ converges to U_ω strongly in $H^{1,0}(\Omega_\gamma)^2$ as $k \to \infty$, we have for every fixed number m there exists $n_m \in \mathbf{N}$, such that for all $k \geq n_m$ hold $J(m) < J(n_k)$. Therefore,

$$J(m) \leq \lim_{k\to\infty} J(n_k) = \lim_{k\to\infty} G(U(n_k)) = G(U_\omega).$$

In this case a solution of (4.2) is given by $n^* = \infty$. The theorem is proved. $\square$

Acknowledgment. This work has been supported by the Ministry of Education and Science of the Russian Federation within the framework of the base part of the state task FSRG-2020-0006.

References

1. F. Dal Corso, D. Bigoni and M. Gei, *The stress concentration near a rigid line inclusion in a prestressed, elastic material. Part I. Full-field solution and asymptotics.* J. Mech. Phys. Solids **56** (2008), 815–838.
2. I. I. Il'ina, V. V. Sil'vestrov, *The problem of a thin interfacial inclusion detached from the medium along one side.* Mech. Solids. **40**(3) (2005) 123–133.
3. H. Itou, A. M. Khludnev, E.M. Rudoy and A. Tani, *Asymptotic behaviour at a tip of a rigid line inclusion in linearized elasticity.* Z. Angew. Math. Mech. **92** (2012), 716–730.

4. A. Khludnev, G. Leugering, *On elastic bodies with thin rigid inclusions and cracks.* Math. Method. Appl. Sci. **33** (2010), 1955–1967.
5. N.P. Lazarev, E.M. Rudoy, *Optimal size of a rigid thin stiffener reinforcing an elastic plate on the outer edge.* Z. Angew. Math. Mech. **97** (2017), 1120–1127.
6. N. Lazarev, *Existence of an optimal size of a delaminated rigid inclusion embedded in the Kirchhoff-Love plate.* Bound. Value Probl. **2015** (2015), 180.
7. E.M. Rudoy, *The Griffith formula and Cherepanov-Rice integral for a plate with a rigid inclusion and a crack.* J. Math. Sci. **186** (2012), 511–529.
8. Z.M. Xiao, B.J. Chen, *Stress intensity factor for a Griffith crack interacting with a coated inclusion.* Int. J. Fract. **108** (2001), 193–205.
9. E.M. Rudoy, *Numerical solution of an equilibrium problem for an elastic body with a thin delaminated rigid inclusion.* J. Appl. Ind. Math. **10** (2016) 264–276.
10. A.M. Khludnev, V.A. Kovtunenko, *Analysis of Cracks in Solids*, WIT-Press, Southampton, 2000.
11. A.M. Khludnev, *Elasticity Problems in Nonsmooth Domains*, Fizmatlit, Moscow, 2010. (in Russian).
12. A.M. Khludnev, V.V. Shcherbakov, *A note on crack propagation paths inside elastic bodies.* Appl. Math. Lett. **79** (2018), 80–84.
13. G. Leugering, J. Sokolowski and A. Zochowski *A Control of crack propagation by shape-topological optimization.* Discret. Contin. Dyn. S - Series A. **35** (2015), 2625–2657.
14. N.P. Lazarev, H. Itou and N.V. Neustroeva, *Fictitious domain method for an equilibrium problem of the Timoshenko-type plate with a crack crossing the external boundary at zero angle*, Jpn. J. Ind. Appl. Math. **33** (2016), 63–80.
15. N.P. Lazarev, *Equilibrium problem for a Timoshenko plate with an oblique crack, Journal of Applied Mechanics and Technical Physics*, J. Appl. Mech. Tech. Phys. **54** (2013), 662–671.
16. T. Popova, G.A. Rogerson, *On the problem of a thin rigid inclusion embedded in a Maxwell material.* Z. Angew. Math. Phys. **67** (2016), 105.
17. A.M. Khludnev, A.A. Novotny, J. Sokolowski and A. Zochowski, *Shape and topology sensitivity analysis for cracks in elastic bodies on boundaries of rigid inclusions.* J. Mech. Phys. Solids. **57** (2009), 1718–1732.
18. A.M. Khludnev, G.R. Leugering, *On Timoshenko thin elastic inclusions inside elastic bodies,* Math. Mech. Solids **20** (2015), 495–511.
19. A.M. Khludnev, *Shape control of thin rigid inclusions and cracks in elastic bodies,* Arch. Appl. Mech. **83** (2013), 1493–1509.
20. A. Khludnev, M. Negri, *Crack on the boundary of a thin elastic inclusion inside an elastic body.* Z. Angew. Math. Mech. **92** (2012), 341–354.
21. H. Itou, A.M. Khludnev, *On delaminated thin Timoshenko inclusions inside elastic bodies.* Math. Method. Appl. Sci. **39** (2016), 4980–4993.
22. A.M. Khludnev, V.V. Shcherbakov, *Singular path-independent energy integrals for elastic bodies with Euler-Bernoulli inclusions.* Math. Mech. Solids **22** (2017), 2180–2195.
23. V.V. Shcherbakov, *The Griffith formula and J-integral for elastic bodies with Timoshenko inclusions.* Z. Angew. Math. Mech. **96** (2016), 1306–1317.
24. V.V. Shcherbakov, *Shape optimization of rigid inclusions for elastic plates with cracks.* Z. Angew. Math. Phys. **67** (2016), 71.
25. G. Kassay, V.D. Radulescu, *Equilibrium Problems and Applications. Mathematics in Science and Engineering*, Elsevier/Academic Press, London, 2018.
26. N.A. Kazarinov, E.M. Rudoy, V.Y. Slesarenko and V.V. Shcherbakov, *Mathematical and numerical simulation of equilibrium of an elastic body reinforced by a thin elastic inclusion.* Comput. Math. Math. Phys. **58** (2018), 761–774
27. L. Faella, A. Khludnev, *Junction problem for elastic and rigid inclusions in elastic bodies.* Math. Method. Appl. Sci. **39** (2016), 3381–3390.
28. V.A. Kovtunenko, G. Leugering, *A shape-topological control problem for nonlinear crack-defect interaction: The antiplane variational model.* SIAM J. Control Optim. **54** (2016), 1329–1351.

29. A.M. Khludnev, L. Faella and T.S. Popova *Junction problem for rigid and Timoshenko elastic inclusions in elastic bodies.* Math. Mech. Solids. **22** (2017), 1–14.
30. A.M. Khludnev, T.S. Popova, *Junction problem for Euler-Bernoulli and Timoshenko elastic inclusions in elastic bodies.* Q. Appl. Math. **74** (2016), 705–718.
31. N. Lazarev, V. Everstov, *Optimal location of a rigid inclusion in equilibrium problems for inhomogeneous two-dimensional bodies with a crack.* Z. Angew. Math. Mech. **99** (2019), e201800268.
32. N. Lazarev, G. Semenova, *On the connection between two equilibrium problems for cracked bodies in the cases of thin and volume rigid inclusions. Bound. Value Probl.* **2019** (2019), 87.
33. A.M. Khludnev, *Optimal control of crack growth in elastic body with inclusions.* Eur. J. Mech. A. Solids. **29** (2010), 392–399.
34. G.P. Cherepanov, *Mechanics of Brittle Fracture*, McGraw-Hill, New-York, 1979.
35. V.Z. Parton, E.M. Morozov, *Mechanics of Elastic-Plastic Fracture,* Hemisphere Publishing Corp., Washington, 1989.
36. N.P. Lazarev, *Optimal control of the thickness of a rigid inclusion in equilibrium problems for inhomogeneous two-dimensional bodies with a crack,* Z. Angew. Math. Mech. **96**(4) (2016), 509–518.

Nyurgun Lazarev: Scientific Research Institute of Mathematics, North-Eastern Federal University, Yakutsk, 677000, Russian Federation
E-mail address: nyurgun@ngs.ru

Galina Semenova: Yakutsk branch of the Regional Scientific and Educational Mathematical Center "Far Eastern Center of Mathematical Research", North-Eastern Federal University, Yakutsk, 677000, Russian Federation
E-mail address: sgm.08@yandex.ru

Natalyya Romanova: Institute of Mathematics and Information Science, North-Eastern Federal University, Yakutsk, 677000, Russian Federation
E-mail address: romnatan@mail.ru