

NON-STATIONARY ADIABATIC FILTRATION OF GASES IN POROUS MEDIA

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ABSTRACT. A non-stationary isentropic filtration of gases in porous media is considered. Thermodynamics in terms of contact and symplectic geometries is briefly discussed. Algebra of symmetries for the PDE system is found, and the classification of media with respect to admissible symmetries is given. Solution for one class of media is found and the phase transitions for this solution are studied.

1. Introduction

In this paper we consider an isentropic filtration process of gases in a porous medium with constant porosity. Unlike [4], where steady filtration of gases was studied, non-stationary processes are considered here.

The system of differential equations describing such processes consists of the following equations:

- the Darcy law

$$\mathbf{u} = -\mu(v, T)\text{grad}p, \quad (1.1)$$

where the vector $\mathbf{u}(t, x, y, z) = (u_1, u_2, u_3)$ is the gas volumetric flow, $p(t, x, y, z)$ is the pressure, $T(t, x, y, z)$ is the temperature, $v(t, x, y, z)$ is the specific volume of the gas, the function $\mu(v, T)$ depends on the gas viscosity and the medium permeability;

- the mass conservation law

$$q v_t + \mathbf{u} \cdot \text{grad}v = v \text{div} \mathbf{u}, \quad (1.2)$$

where constant q is the medium porosity;

- the energy conservation law, which in the case of isentropic process, has the form

$$s_t + \mathbf{u} \cdot \text{grad}s = 0, \quad (1.3)$$

where $s(t, x, y, z)$ is the specific entropy.

See also [1] for details.

The paper is organized as follows. Section 2 briefly reminds the thermodynamical principles in terms of contact and symplectic geometries. In Section 3, we find point symmetries of the isentropic filtration PDE system. The classification of

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porous media with respect to admissible symmetries groups is given. In Section 4, we find a solution that is invariant with respect to a certain symmetries subgroup. This leads to an ODE system, which can be solved explicitly. We find solution for the ideal gas model and a certain class of media. Then the ideal gas solution is used to construct a solution for real gases described by van der Waals equations. Finally, possible phase transitions are studied.

Many of the computations in this paper were done in Maple with the Differential Geometry package by I. Anderson and his team. Maple files with the most important computations in this paper can be found on the web-site <http://d-omega.org>.

2. Thermodynamics

Here we briefly recall the thermodynamical principles expressed in terms of contact and symplectic geometries that we need for the further discussion (for details see [3], [4], [5]).

Consider a 5-dimensional contact manifold \mathbb{R}^5 equipped with the coordinates (p, T, ϵ, v, s) and the contact 1-form

$$\theta = ds - T^{-1}d\epsilon - T^{-1}pdv,$$

here ϵ is the specific energy, and $v = \rho^{-1}$ is the specific volume.

The thermodynamical states are a 2-dimensional Legendrian manifold L , i. e. such surface L that the first law of thermodynamics $\theta|_L = 0$ holds.

Using the projection $\pi : \mathbb{R}^5 \rightarrow \mathbb{R}^4$, $\pi : (p, T, \epsilon, v, s) \mapsto (p, T, \epsilon, v)$, eliminate the specific entropy s from the description of the thermodynamic states. The restriction of this projection on the state surface L leads to a Lagrangian manifold \bar{L} in the 4-dimensional symplectic space \mathbb{R}^4 equipped with the structure form

$$\Omega = -d\theta = T^{-1}dp \wedge dv - T^{-2}dT \wedge (d\epsilon + pdv).$$

Therefore, the thermodynamic states can be considered as the Lagrangian submanifolds in the symplectic space (\mathbb{R}^4, Ω) , and it can be defined by the equations

$$\begin{cases} f(p, T, \epsilon, v) = 0, \\ g(p, T, \epsilon, v) = 0 \end{cases} \quad (2.1)$$

if

$$[f, g] = 0 \text{ on } \bar{L}, \quad (2.2)$$

where $[f, g]$ is the Poisson bracket with respect to the symplectic form Ω .

In order to find the state manifolds \bar{L} for real gases, we consider the following two equations:

$$\begin{cases} f(p, T, \epsilon, v) = p - A(v, T), \\ g(p, T, \epsilon, v) = \epsilon - B(v, T). \end{cases}$$

The first equation is called thermic equation of state, and the second one is called caloric equation of state. Then the compatibility condition (2.2) for them has the form

$$(T^{-2}B)_v = (T^{-1}A)_T.$$

Moreover, the following theorem is valid.

Theorem 2.1. *Thermodynamical states of real gases are defined by Massieu-Planck potential function $\phi(v, T)$ and have the following form:*

$$p = RT\phi_v, \quad \epsilon = RT^2\phi_T, \quad s = R(\phi + T\phi_T), \quad (2.3)$$

where function ϕ has the following expression in terms of virial coefficients A_k ,

$$\phi(v, T) = \frac{n}{2} \ln T + \ln v - A_1(T)v^{-1} - \frac{1}{2}A_2(T)v^{-2} - \dots - \frac{1}{k}A_k(T)v^{-k} - \dots \quad (2.4)$$

The domain of applicable states on the plane (v, T) is given by inequalities

$$\phi_{vv} < 0, \quad T\phi_{TT} + 2\phi_T > 0. \quad (2.5)$$

Phase transitions occur near the curve

$$\phi_{vv} = 0.$$

Thus, by the system \mathcal{E} of differential equations describing the isentropic filtration process of gases we mean the differential equations (1.1)–(1.3) and the equations of state (2.3).

3. Symmetry Lie algebra

By a symmetry of the system \mathcal{E} we mean a point symmetry, i.e. a vector field X on the 0-jet space such that its second prolongation $X^{(2)}$ is tangent to the submanifold $\mathcal{E}^{(2)} \subset \mathbf{J}^2(4, 7)$.

Using the standard techniques of the symmetries computations we obtain (see the Maple file) the following result.

Theorem 3.1. *The Lie algebra \mathfrak{g} of point symmetries of the system \mathcal{E} of differential equations describing the isentropic filtration process of real gases in an arbitrary porous medium is generated by the vector fields*

$$\begin{aligned} X_1 &= \partial_x, & X_5 &= y\partial_x - x\partial_y, \\ X_2 &= \partial_y, & X_6 &= z\partial_x - x\partial_z, \\ X_3 &= \partial_z, & X_7 &= z\partial_y - y\partial_z, \\ X_4 &= \partial_t, & X_8 &= 2t\partial_t + x\partial_x + y\partial_y + z\partial_z. \end{aligned} \quad (3.1)$$

So, transformations corresponding to elements of the algebra \mathfrak{g} are compositions of the translations, the rotations $SO(3)$ and the scale transformation X_8 .

Consider the case when the gas satisfies ideal gas model, in other words, the thermodynamic states are given by the potential function

$$\phi(v, T) = \frac{n}{2} \ln T + \ln v.$$

Denote the corresponding system of differential equations \mathcal{E}_{id} .

Depending on the properties of the gas and the rigid medium, we use different functions $\mu(v, T)$ and, accordingly, the algebra of point symmetries has different additional symmetries.

$\mu(v, T) = f(v)T^\alpha$	$X_9 = (1 + \alpha) t \partial_t - T \partial_T$
$\mu(v, T) = f(T)v^\alpha$	$X_9 = (1 - \alpha) t \partial_t + v \partial_v$
$\mu(v, T) = \alpha v^\beta T^\gamma$	$X_9 = t \partial_t - \frac{1}{\beta + \gamma} (v \partial_v + T \partial_T)$ $X_{10} = (1 + \gamma) v \partial_v + (1 - \beta) T \partial_T$
$\mu(v, T) = \alpha \left(\frac{T}{v} \right)^\beta$	$X_9 = (1 + \beta) t \partial_t + v \partial_v$
$\mu(v, T) = f(vT)v^{3-q}$	$X_9 = (q - 1) t \partial_t + v \partial_v - T \partial_T$

Summarizing, we get the following result.

Theorem 3.2. *The Lie algebra \mathfrak{g}_{id} of point symmetries of the system \mathcal{E}_{id} of differential equations describing the isentropic filtration of ideal gases is generated by the vector fields X_1, X_2, \dots, X_8 and by one or two additional symmetries which form depends on the particular properties of a medium, i.e. depends on the function $\mu(v, T)$.*

4. Invariant solutions

In this section we find some invariant solutions of the system \mathcal{E} .

In order to find these solutions consider the subalgebra

$$\mathfrak{g}_4 = \mathfrak{so}(3) \oplus \langle X_8 \rangle \subset \mathfrak{g}.$$

The corresponding Lie group has three-dimensional orbits and the invariant of its action has the form

$$r^2 = \frac{x^2 + y^2 + z^2}{t}.$$

We find a solution that is invariant with respect to the action of this group.

4.1. Ideal gas solution. First, we consider the case of ideal gas. Then reduction of system \mathcal{E}_{id} with respect to the action of \mathfrak{g}_4 leads to the system of ODEs

$$\begin{cases} (2R\mu(v, T) (v_r T - v T_r) - r v^2) (2v_r T + n v T_r) = 0, \\ 2R\mu(v, T) (r v (v_{rr} T - v T_{rr}) + (3r v_r - 2v) (v_r T - v T_r)) + \\ \quad r v (2R\mu_r(v, T) (v_r T - v T_r) + q r v v_r) = 0. \end{cases}$$

Note that the second factor in the first equation corresponds the case when the entropy s is constant.

The proof of the following theorem can be found in the Maple file.

Theorem 4.1. *The \mathfrak{g}_4 -invariant solution of the system \mathcal{E}_{id} for the case of ideal gas depends on the properties of a porous medium and has the form*

$$\begin{aligned} v(r) &= RC_1 r^{\frac{3}{1-q}}, \\ p(r) &= -\frac{1}{2} \int \frac{r}{\mu(v, T)} dr, \\ T(r) &= \frac{p(r)v(r)}{R}, \end{aligned}$$

where $r = \sqrt{\frac{x^2 + y^2 + z^2}{t}}$, $t > 0$ and $C \in \mathbb{R}$.

Consider some cases of the function $\mu(v, T)$ from the table above and corresponding expression for the pressure.

$$\begin{aligned} (1) \quad \mu(v, T) &= \alpha \left(\frac{T}{v}\right)^\beta, \quad \beta \neq -1, \quad p(r) = R \left(\frac{(1+\beta)(C_2 - r^2)}{4\alpha R}\right)^{\frac{1}{1+\beta}} \\ (2) \quad \mu(v, T) &= \alpha \frac{v}{T}, \quad p(r) = C_2 \exp\left(-\frac{r^2}{4\alpha R}\right) \\ (3) \quad \mu(v, T) &= \alpha v^\beta T^\gamma, \quad p(r) = \left(C_2 + \frac{C_1^{-\beta-\gamma}(1+\gamma)(1-q)}{2\alpha R^\beta(3\gamma+3\beta+2q-2)} r^{\frac{3\gamma+3\beta+2q-2}{q-1}}\right)^{\frac{1}{1+\gamma}} \\ (4) \quad \mu(v, T) &= \alpha v^\beta T^{-1}, \quad p(r) = C_2 \exp\left(\frac{C_1^{1-\beta}(1-q)}{2\alpha R^\beta(3\beta+2q-5)} r^{\frac{3\beta+2q-5}{q-1}}\right) \end{aligned}$$

Example. Let us write a solution for a certain gas and medium for the second case, where $\mu(v, T) = \alpha \frac{v}{T}$. This gives us understanding when the solution is applicable and has physical sense.

For example, we consider methane as the gas, and the values of parameters are the following $q = 0,55$, $\alpha \approx 5 \cdot 10^{-4}$, $C_1 \approx 2,7 \cdot 10^{-3}$, $C_2 \approx 3 \cdot 10^5$.

Instead of a domain in terms of the invariant r , we present it as a region on the plane of distance $d = \sqrt{x^2 + y^2 + z^2}$ (vertical axis) and time t (horizontal axis).

In the Figure 1 the gray-filled region shows where the gas density is sufficiently small to fit into the ideal gas model. Since the maximum pressure equals C_2 , it can be set small enough. But the region with negligibly small pressure should also be excluded as physically impossible. The Figure 2 depicts this region. The similar restrictions are imposed on the gas temperature. The grey-filled region in the Figure 3 is where the solution for temperature is in certain bounds, for example between the melting and kindling points.

The intersection of all three regions is presented in the Figure 4.

4.2. van der Waals gas. In this section we find an asymptotic solution of the system \mathcal{E} for a real gas model. We use asymptotic in terms of virial coefficients and, for example, consider the model of van der Waals gas.

Recall that thermodynamic state of a van der Waals is given by the Massieu-Planck potential of the form:

$$\phi = \frac{n}{2} \ln T + \ln(v - b) + \frac{a}{RvT} = \frac{n}{2} \ln T + \ln v - \frac{b}{v} - \frac{a}{RvT} + o_2,$$

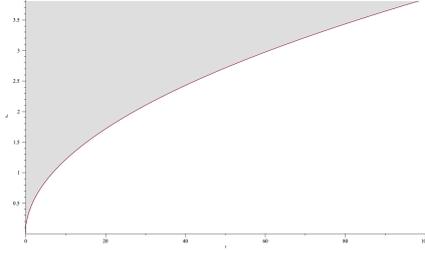


FIGURE 1

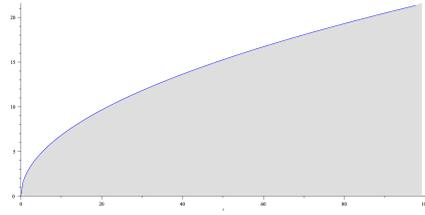


FIGURE 2

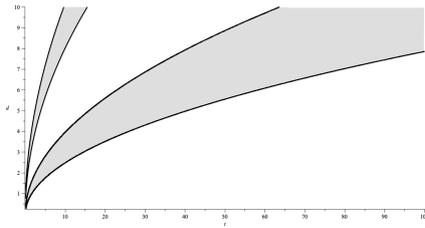


FIGURE 3

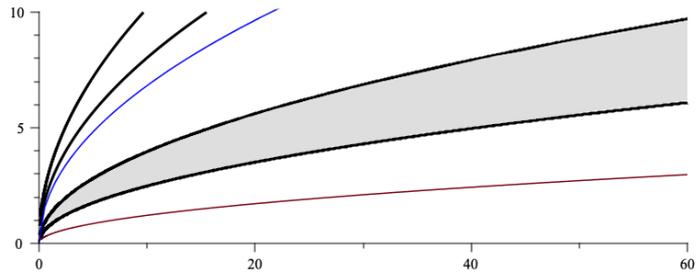


FIGURE 4

where a, b are the gas parameters, and o_2 is the terms of the second order of a and b . Denote by \mathcal{E}_w the corresponding system of differential equations.

Note the value of function ϕ , if $a = b = 0$, corresponds to the ideal gas potential, so it is a zero order approximation for real gases.

Further we look for an asymptotic solution for \mathcal{E}_w in the form

$$\begin{cases} T = T_0 + aT_1 + bT_2 + o_2, \\ p = p_0 + ap_1 + bp_2 + o_2, \\ v = v_0 + av_1 + bv_2 + o_2, \\ s = s_0 + as_1 + bs_2 + o_2, \\ \mathbf{u} = \mathbf{u}_0 + a\mathbf{u}_1 + b\mathbf{u}_2 + o_2, \end{cases} \quad (4.1)$$

where functions T_0, p_0, v_0, s_0 , and \mathbf{u}_0 are the solution of the system \mathcal{E}_{id} , and $T_1, T_2, p_1, \dots, \mathbf{u}_2$ are the first-order corrections.

For this we write function μ in the form

$$\mu(v, T) = \mu(v_0, T_0) + \mu_v(v_0, T_0)(av_1 + bv_2) + \mu_T(v_0, T_0)(aT_1 + bT_2) + o_2$$

and substitute this expression and (4.1) into the system \mathcal{E} . Collecting coefficients for a and b we get the following system

$$\begin{cases} \mathbf{u}_1 + \mu(v_0, T_0)\text{grad}p_1 + \text{grad}p_0(v_1\mu_v(v_0, T_0) + T_1\mu_T(v_0, T_0)) = 0, \\ \mathbf{u}_2 + \mu(v_0, T_0)\text{grad}p_2 + \text{grad}p_0(v_2\mu_v(v_0, T_0) + T_2\mu_T(v_0, T_0)) = 0, \\ q(v_1)_t + \mathbf{u}_1 \cdot \text{grad}v_0 + \mathbf{u}_0 \cdot \text{grad}v_1 - v_0\text{div}\mathbf{u}_1 - v_1\text{div}\mathbf{u}_0 = 0, \\ q(v_2)_t + \mathbf{u}_2 \cdot \text{grad}v_0 + \mathbf{u}_0 \cdot \text{grad}v_2 - v_0\text{div}\mathbf{u}_2 - v_2\text{div}\mathbf{u}_0 = 0, \\ (s_1)_t + \mathbf{u}_0 \cdot \text{grad}s_1 + \mathbf{u}_1 \cdot \text{grad}s_0 = 0, \\ (s_2)_t + \mathbf{u}_0 \cdot \text{grad}s_2 + \mathbf{u}_2 \cdot \text{grad}s_0 = 0. \end{cases}$$

So in order to find the first order corrections, we need to solve the linear system of ODEs on the functions $T_1, T_2, p_1, \dots, \mathbf{u}_2$.

For example, consider the invariant solution for the function

$$\mu(v, T) = \alpha \frac{v}{T}.$$

As in the previous section, we are looking for a \mathfrak{g}_4 -invariant solution, i.e. the first-order corrections $T_1, T_2, p_1, \dots, \mathbf{u}_2$ also depend on the invariant r .

Substituting the expression for the function $\mu(v, T)$ and the solution of the system \mathcal{E}_{id} into the our system we get a linear system of ODEs that can be found in the Maple file. Solution of this system delivers the first-order corrections for volume and temperature

$$\begin{aligned} v_1(r) &= 0, \quad v_2(r) = 0, \\ T_1(r) &= \left(\frac{6}{C_1 R^2 (q-1)} \int \exp\left(\frac{r^2}{4\alpha R}\right) r^{\frac{q-7}{1-q}} dr + C_3 \right) r^{\frac{3}{1-q}} \exp\left(-\frac{r^2}{4\alpha R}\right), \\ T_2(r) &= \left(\frac{C_2(q-1)r^2}{2(2q+1)\alpha R^2} - \frac{C_2}{R} + C_4 r^{\frac{3}{1-q}} \right) \exp\left(-\frac{r^2}{4\alpha R}\right). \end{aligned}$$

Example. Let us draw plots for the temperature first-order corrections for the example we considered above. Given values for constants $a \approx 9 \cdot 10^{-5}$, $b \approx 3 \cdot 10^{-3}$, $C_3 = 0$, $C_4 = 0$ the graphics of the functions T_1 and T_2 are presented on the Figures 5, 6 correspondingly.

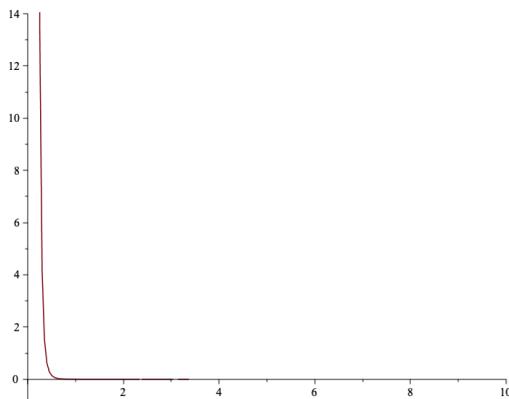


FIGURE 5

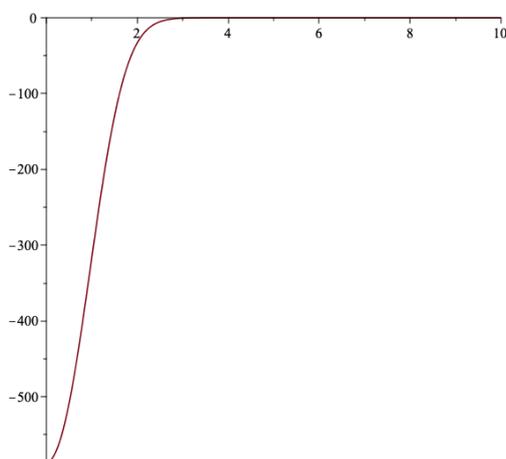


FIGURE 6

The last question we investigate is whether phase transitions occur in the filtration process described by the solution.

In terms of the Massieu-Planck potential the phase coexistence curve on the plane (v, T) is given by equations [3]

$$\begin{cases} \phi_v(v_1, T) - \phi_v(v_2, T) = 0, \\ \phi(v_2, T) - \phi(v_1, T) + v_1\phi_v(v_1, T) - v_2\phi_v(v_2, T) = 0, \end{cases}$$

where v_1 and v_2 are the specific volumes of phases. Substituting the solution into these equations we obtain four phase transition curves on the plane (d, t) , see the orange curves in the Figure 7. Note that, though, one of these curves is practically indistinguishable from the t -axis it is also a parabola.

ADIABATIC FILTRATION IN POROUS MEDIA

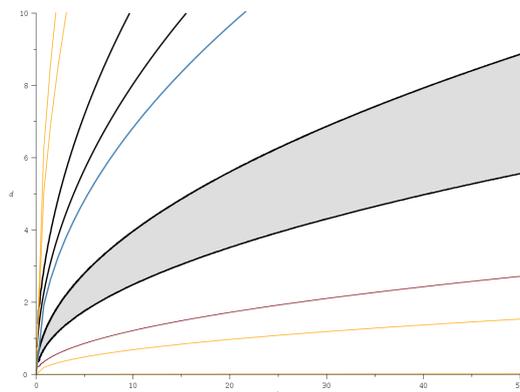


FIGURE 7

We see that for this solution phase transitions do not occur in the domain when it has physical sense.

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