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SYMMETRIC LEVY PROCESSES WITH REFLECTION

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Abstract. In this paper we give meaning to the notion of reflecting Levy process in a smooth domain D in terms of the domain of its generator. In contrast with other approaches, we work with the classical Neumann conditions. The main idea is to construct a suitable continuation of an initial function f defined on D to $\mathbb{R}^d$, and then associate with it some Markov semi-group..

1. Introduction and related work

Let $(\xi(t))_{t\geq 0}$ be a d-dimensional Levy process with characteristic function

$$\varphi_t(\mathbf{p}) = \exp(-tL(\mathbf{p})), \quad L(\mathbf{p}) = -\int_{\mathbb{R}^d} \left(e^{i\mathbf{p}\cdot\mathbf{x}} - 1 - i\mathbf{p}\cdot\mathbf{x}\right) d\Pi(\mathbf{x}) \,. \tag{1.1}$$

We shall suppose that the corresponding Levy measure Π is rotation-invariant and has finite second moment.

Our aim is to define a version of this process whose sample paths remain in a smooth bounded domain $D \subset \mathbb{R}^d$ and reflect off the boundary ∂D elastically.

As it is pointed out in [6], from a mathematician's perspective, Levy processes – and α-stable processes in particular – confined to bounded domains are of interest because of their limit counterparts ($\alpha = 2$) being nothing but the well-known probabilistic models for reflecting/absorbing Brownian motion.

The applied side of reflecting processes theory (reflecting Levy processes in particular) resides in the area of stochastic models with restrictions. Such models naturally arise in stochastic control theory and financial mathematics ([28]), models for queues of finite capacity ([4], [10], [11], [12]) and various models for dams and fluids ([1], [27]).

Different versions of ξ with values in D could be defined in terms of a "free" (unrestricted) process semi-group

$$(T^t f)(\mathbf{x}) = \mathbb{E} f(\mathbf{x} + \xi(t)).$$

In these terms, there is a natural way of defining the "version of ξ with paths in D". By this we shall mean the process whose generator A is a specific restriction of generator L of T^t to a certain class of functions defined on D. The process whose restricted domain satisfies Dirichlet conditions is naturally called the absorbing version of ξ, whereas the one with the Neumann conditions imposed is called the reflecting version.

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The generator of Levy process $\xi(t)$ is the non-local operator given by (Theorem 31.5, [24])

$$-Lf(\mathbf{x}) = \int_{\mathbb{R}^d} \Big(f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x}) - f'(\mathbf{x})\cdot\mathbf{y} \Big)\, d\Pi(\mathbf{y}) \tag{1.2}$$

with a kernel $C_c^\infty(\mathbb{R}^d) \subset D(L)$, $-L \geq 0$. Its associate semi-group $(T^t)_{t\geq 0}$ is continuous in the space $C_0(\mathbb{R}^d)$ of continuous functions tending to zero at infinity.

We shall, however, employ a slightly different approach, since the non-local character of L causes trouble with straightforward restriction. Following the work [17], instead of restricting L, we shall construct a special continuation for a given initial function $f \in W_2^2(D)$ to the function $\widetilde{f}$ belonging to the domain $D(L)$. Using this continuation, we shall define the semi-group P^t by setting

$$P^t f = T^t \widetilde{f}.$$

The continuation $f \mapsto \widetilde{f}$ will be chosen so that the boundary conditions be "hard-wired" into it and the plan described above be carried out.

Further study of P^t is a subject of future research. Let us note in passing that the problem of constructing a process with a given semi-group is not trivial, although it can be dealt with analytically. The work [3] presents an example of investigating properties of a process associated with Dirichlet form

$$\mathcal{E}(u,v) = \int_D \nabla u \cdot \overline{\nabla v}\, d\mathbf{x}, \quad D[\mathcal{E}] = W_2^1(D)$$

in Hölder domain D. It is easy to see that this form is nothing but the Laplace-Neumann quadratic form. The corresponding process therefore is the reflecting Brownian motion in our sense. In the work Bass and Pei invoke general theory of correspondence between Dirichlet forms (i.e. closed quadratic forms having the so-called Markovian property) and Hunt processes (quasi-left continuous strictly Markovian processes), developed throughout the '60s in works of Hunt, Dynkin, Beurling, Deny and reached its finished form in the book [14] by Fukushima. Specifically, the general theory states that to each regular Dirichlet form there is associated a Hunt process (Theorem 6.2.1 from [14]). If, moreover, the form possesses the so-called local property, then the process is continuous (Theorem 4.5.1 from [14]). Finally, the work generalizes the representation of Neumann problem $\gamma_1 u = f \in B(\partial D)$ solution u in terms of average over the reflecting Brownian motion paths

$$u(\mathbf{x}) = \lim_{t\to\infty} \frac{1}{2}\mathbb{E}_x \int_0^t f(X_s)\, dL_s,$$

(theorem due to Broslamer [7]). Here, X_s denotes the Brownian motion reflecting in D, L_s is its local time on the boundary (see [5]). The local time is also constructed via means of Dirichlet forms technique (Theorem 5.1.1 from [14]).

The Dirichlet forms technique does not require path continuity and can therefore be applied to jump processes directly. The most general closable Markovian form in $L_2(D)$ possessing $C_0^\infty(D)$ as its kernel is given by the Beurling – Deny formula

(Theorems 2.2.1 and 2.2.2 from [14])

$$\mathcal{E}(u,v) = \int_D u_{x_i}(\mathbf{x})\overline{v_{x_j}(\mathbf{x})}\,\nu_{ij}(d\mathbf{x}) + \\ + \int_{D\times D\setminus d} \Big(u(\mathbf{x}) - u(\mathbf{y})\Big)\overline{\Big(v(\mathbf{x}) - v(\mathbf{y})\Big)}\,J(d\mathbf{x}\times d\mathbf{y}) + \\ + \int_D u(\mathbf{x})\overline{v(\mathbf{x})}\,k(d\mathbf{x})$$

for $u, v \in D(\mathcal{E})$. The first term is easily identified as *the diffusion part.* The family of measures ν_{ij} is symmetric $\nu_{ij} = \nu_{ji}$ and positively defined in the sense that for a given compact set $K \subset D$ and a vector $\xi \in \mathbb{R}^d$ holds

$$\sum_{ij} \xi_i\xi_j\nu_{ij}(K) \geq 0.$$

The second term is interpreted as *the jumps.* The measure $J(dx \times dy)$ is supposed to be positive outside the diagonal d and for each compact set $K \subset D$ satisfy

$$\int_{K\times K\setminus d} |\mathbf{x} - \mathbf{y}|^2\,J(d\mathbf{x}\times d\mathbf{y}) < \infty.$$

The only condition on $k(\mathbf{x})$ is that it should be a positive measure. The last term represents *the absorption part.*

We are interested solely in the second term. However, the space $C_0^\infty(D)$ is not dense in $W_2^1(D)$ and therefore cannot be a kernel of reflecting Levy process Dirichlet form.

It should be noted that in case of diffusion processes reflecting in a sufficiently smooth domain (e.g. having C^3-boundary; see [22]) one can construct the sample paths directly. The method is due to Skorokhod [26]. In the simplest case of reflecting Brownian motion $w(t)$ on $[0,\infty)$ we can make use of the Tanaka formula (see [18] or [**?**])

$$|w(t)| \stackrel{\mathrm{d}}{=} w(t) + \zeta(t).$$

Here $\zeta(t)$ is the local time. The process $|w(t)|$ is easily checked (by making use of the Skorokhod lemma) to be the reflecting version in the generator sense.

In the present work we shall to a certain extent reproduce this semimartingale decomposition (although our results will be essentially weaker, partially because Levy processes do not have local times). For this purpose for a given initial function $f \in W_2^2(D)$ we shall construct not one but two continuations $\overline{f}$ and $\widetilde{f}$, both belonging to $D(L)$. To $\overline{f}$ we associate a semi-group R^t, corresponding to the part of the process inside D (analogue of $w(t)$ in the Tanaka formula). To $\widetilde{f}$ we associate a semi-group P^t, representing the reflecting version (analogue of $|w(t)|$). It is natural to expect that their difference is concentrated on the boundary ∂D. We shall show that this difference can be conveniently rewritten in terms of a specific operator family

$$Q^t \colon W_2^{1/2}(\partial D) \to W_2^2(D).$$

As stated in [17], we can give natural meaning to this operator. Namely, it represents the momentum, accumulated by the boundary to the time t, as a result of reflections. Moreover, we will show that the accumulated momentum can be defined path-wisely (not only on the average) in the $L_2(dx \times dP)$-sense.

There are a few rather general results on reflecting Hunt processes in the Dirichlet form sense in a smooth domain D. All of them are rooted in the papers by Silverstein [25] and Chen [8]. By analogy with $W_2^1(D)$ orthogonal decomposition (see [19], ch. 2, §10, Theorem 4)

$$W_2^1(D) = W_2^{1,0}(D) \oplus G_2^1(D), \tag{1.3}$$

where $G_2^1(D)$ is the space of harmonic functions belonging to $W_2^1(D)$, the authors take a Dirichlet form $\mathcal{E}$ with kernel $C_c^\infty(D) \subset D(\mathcal{E})$, define the notion of harmonicity relative to $\mathcal{E}$, and then define form $\mathcal{E}^{\text{ref}}$ on the new domain

$$D(\mathcal{E}^{\text{ref}}) = D(\mathcal{E}) \oplus \widetilde{G}_2^1(D).$$

The Hunt process associated with $\mathcal{E}^{ref}$ can be called the reflection in the Silverstein – Chen sense.

In some cases it is possible to go further. In the work [6] the authors constructed processes associated to the Dirichlet form of the fractional Laplacian in D

$$\mathcal{E}(u,v) = \frac{1}{c}\int_D\int_D \frac{\big(u(\mathbf{x}) - u(\mathbf{y})\big)\overline{\big(v(\mathbf{x}) - v(\mathbf{y})\big)}}{|\mathbf{x}-\mathbf{y}|^{d+\alpha}}\, d\mathbf{x}\, d\mathbf{y}$$

on the domain $\mathcal{F}$ consisting of functions $u \in L_2(D)$ such that

$$\int_D\int_D \frac{\big(u(\mathbf{x}) - u(\mathbf{y})\big)^2}{|\mathbf{x}-\mathbf{y}|^{d+\alpha}}\, d\mathbf{x}\, d\mathbf{y} < \infty.$$

They have shown that their processes are indeed the reflection processes in the Silverstein – Chen sense. The latter inequality defines the domain of $\mathcal{E}^{\text{ref}}$.

Another approach was proposed in [23]. Namely, the authors consider the fractional Laplacian in $\mathbb{R}^d$

$$\mathcal{E}(u,v) = \frac{1}{c}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\big(u(\mathbf{x}) - u(\mathbf{y})\big)\overline{\big(v(\mathbf{x}) - v(\mathbf{y})\big)}}{|\mathbf{x}-\mathbf{y}|^{d+\alpha}}\, d\mathbf{x}\, d\mathbf{y},$$

with domain satisfying the so-called *non-local Neumann condition*

$$\mathcal{N}_s u(\mathbf{x}) = \int_D \frac{u(\mathbf{x}) - u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+\alpha}}\, d\mathbf{y} = 0 \text{ for } \mathbf{x} \notin D.$$

This set up has several advantages. Among them, a clear probabilistic meaning. The process with the this generator, when leaving D, immediately comes back at *a random point of* D with density proportional to $|\mathbf{x}-\mathbf{y}|^{-d-\alpha}$.

It is worth noting the work [13], where the initial point is the fractional Laplacian in the sense of the spectral theorem.

Finally, in the works [16] and [15] the authors provide constructions for a *deterministic* return of the process inside the domain D upon leaving. They prove that their construction is linked to some Neumann problem for some non-local operators.

Our work succeeds the work [17]. It differs from the results stated above primarily in the fact that we work with the Levy process generator *with the classical Neumann condition.* That is to say, we consider an analogue A^N of the generator L in D with domain $D(A^N) = \mathcal{N}(D)$, where

$$\mathcal{N}(D) = \left\{u \in W_2^2(D) \colon \gamma_1 u = 0\right\},$$

where $\gamma_1 \colon W_2^2(D) \to W_2^{1/2}(D)$ is the normal derivative.

The case $\xi = w$ (the Brownian motion) is the subject of our paper [20]. In this case A^N is nothing but the usual Neumann – Laplace operator, and all our results coincide with the mentioned above. It can be shown that in a simple case of $D = [-1, 1]$ our reflection is exactly the Skorokhod reflection.

To motivate our setup, let us mention that every solution $u \in C^\infty(\mathbb{R}^d)$ of

$$-\Delta u = \kappa^2 u$$

satisfies

$$-Lu = L(\kappa)u,$$

where L is defined by (1.2), $L(p)$ is defined by (1.1) (which in our case depends solely on $p = |\mathbf{p}|$. At that, holds the inequality

$$|L(p)| \le Cp^2$$

for all $p \ge 0$.

Let us emphasize that the generality of results is not in our ambitions. We restrict ourselves to the smooth boundary case and pure-jump symmetric Levy processes with finite second moment. The diffusion part could be easily included and poses no new problems for our method. Levy processes not having finite second moment could also be dealt with, however, in that case the direct approach fails and some additional tricks are required. The symmetry of the process remains an essential restriction on our method.

2. Notation

Let us assume for simplicity that D is a smooth domain in $\mathbb{R}^d$ (3-smoothness is sufficient). Let us denote $x = |\mathbf{x}|$ and $\widehat{\mathbf{x}} = \mathbf{x}/x$ for $\mathbf{x} \in \mathbb{R}^d$. The Lebesgue measure on the boundary ∂D we denote by dS.

The eigenvalues of the Laplace – Neumann operator in D arranged in the increasing order obey the Weyl law (formula (17.3.6) in [**?**])

$$\kappa_m^2 \sim 4\Gamma^{2/d}\left(\frac{d}{2}+1\right)m^{2/d} \text{ as } m \to \infty.$$

Let us denote the $L_2(D)$-normed eigenfunction corresponding to κ_m^2 by s_m.

We call $\mathcal{N}(D)$ the domain of Laplace – Neumann operator in $W_2^2(D)$

$$\mathcal{N}(D) = \left\{u \in W_2^2(D)\colon \gamma_1 u = 0\right\}$$

where $\gamma_1\colon W_2^2(D) \to W_2^{1/2}(\partial D)$ is the normal derivative operator.

It is a well-known fact that for every $f \in W_2^1(D)$ the series

$$f = \sum_{m=0}^{\infty} (f, s_m)_{L_2(D)} s_m \tag{2.1}$$

converges to f in the $W_2^1(D)$-norm. If additionally $f \in W_2^2(D)$, then the latter series converges in $W_2^2(D)$. It should be noted that f belonging to $W_2^2(D)$ is not enough to ensure the convergence.

We denote by $h(\mathbf{x}, \mathbf{z})$ the Green function of the Laplace – Neumann operator in D, that is, the solution of the equation

$$-\Delta_{\mathbf{z}} h(\mathbf{x}, \mathbf{z}) = \delta_{\mathbf{x}} - |D|^{-1},$$

satisfying

$$\int_D h(\mathbf{x}, \mathbf{z})\, d\mathbf{x} = 0.$$

If a function g satisfies the Neumann problem solvability condition (i.e. has vanishing average over the boundary), then

$$f(\mathbf{x}) = \int_{\partial D} h(\mathbf{x}, \mathbf{z})\; g(\mathbf{z})\, dS(\mathbf{z})$$

solves the Neumann problem $-\Delta f = 0$ in D, $\gamma_1 f = g$ on ∂D.

We need some notation concerning the Laplace – Neumann operator in the unit ball $\{x < 1\}$. Let $L_2(S^{d-1})$ denote the space of square-integrable functions on S^{d-1} with respect to dS. The eigenfunctions Y_λ^μ, $\lambda \in \mathbb{Z}_+$, $\mu = 1, \dots, d(\lambda)$ of the Laplace – Beltrami operator on sphere are known as spherical harmonics (the number $d(\lambda)$ is given by (2.46) from [2]). They constitute an orthogonal basis in $L_2(S^{d-1})$. We will omit the summation indices when summing series in spherical harmonics since no confusion can arise.

We set $\alpha = d/2 - 1$. Then the Laplace – Neumann eigenfunctions in the unit ball can be written as

$$j_\lambda^d(\kappa_{\lambda k} x) Y_\lambda^\mu(\widehat{\mathbf{x}}),$$

where j_λ^d is the d-dimensional hyperspherical Bessel function of order λ, and $\kappa_{\lambda k}$, $\kappa \geq 0$ are the zeros of $J'_{\lambda+\alpha}$. They can be shown to be analytic in $\mathbf{x} \in \mathbb{R}^d$ and belonging to $C_0^\infty(\mathbb{R}^d)$.

It will be important for us that the function $u(\mathbf{x}) = j_\lambda^d(\kappa x) Y_\lambda^\mu(\widehat{\mathbf{x}})$ belongs to $D(L)$ (formula (1.2)), and is therefore an eigenfunction of L

$$-Lu = L(\kappa) u. \tag{2.2}$$

The notation $A \Subset B$ means that the set A compactly belongs to B.

3. $W_2^2(D)$ decomposition

Let us define a quadratic form a by

$$a(u, v) = \int_D \Delta u \cdot \overline{\Delta v}\, d\mathbf{x}, \quad D(a) = W_2^2(D).$$

It is easy to see that $a(u, u) = 0$ if and only if the function u belongs to the class $G_2^2(D)$. Therefore, a is a norm in W_2^2 / G_2^2. Since the Neumann problem has a unique solution up to a constant, the form a is non-degenerate in

$$\mathcal{N}^0 = \left\{ u \in \mathcal{N} \colon \int_D u(\mathbf{x})\, d\mathbf{x} = 0 \right\}.$$

We will need the following lemma.

Lemma 3.1. *The space $W_2^2(D)$ has the following a-orthogonal decomposition:*

$$W_2^2(D) = \mathcal{N}^0(D) \oplus BG_2^{2,0}(D).$$

Here

$$BG_2^{2,0}(D) = \left\{ u \in W_2^2(D) \colon \Delta^2 u = 0,\ \gamma_1 \Delta u = 0 \right\}.$$

The form a is non-degenerate on $\mathcal{N}^0(D)$.

One can easily check that the subspace of $BG_2^{2,0}(D)$ on which the form a is non-degenerate is one-dimensional. We take the a-normed vector

$$\chi(\mathbf{x}) = \frac{x^2}{\sqrt{2d|D|}}$$

and define the a-projection onto $\chi\mathbb{C}$ by

$$f_b(\mathbf{x}) = a(f,\chi)\,\chi(\mathbf{x}) = \chi(\mathbf{x}) \int_D \Delta f \cdot \overline{\Delta\chi}\, d\mathbf{y}\,.$$

Lemma 3.2. *The space $W_2^2(D)$ has the following a-orthogonal decomposition:*

$$W_2^2(D) = \mathcal{N}^0(D) \oplus \chi\mathbb{C} \oplus G_2^2(D).$$

The form a is non-degenerate in $\mathcal{N}^0(D)$ and $\chi\mathbb{C}$ and vanishes on $G_2^2(D)$.

4. Two continuations

We begin the section with a short summary of the main idea, and afterwards proceed to the formal presentation. To each point $\mathbf{x} \in D$ we associate a ball neighborhood $\mathbf{x} + D(\mathbf{x}) \Subset D$, where $D(\mathbf{x})$ is the ball of radius $r(\mathbf{x}) > 0$ $f \in W_2^2(D)$ we construct in every neighborhood a sequence of *tangential* functions $\widetilde{f}_M(\mathbf{x},\cdot)$, $M \in \mathbb{N}$, converging to f in this neighborhood. In addition, we choose the sequence f_M so that each function f_M be defined in $\mathbb{C}^d$ (in contrast with f), and, moreover, belongs to $D(L)$. Having defined f_M, we define the operator A by

$$Af(\mathbf{x}) = \lim_{M\to\infty} L_{\mathbf{y}} f_M(\mathbf{x},\mathbf{y})\Big|_{\mathbf{y}=0}.$$

Such tangential families we shall by terminology abuse refer to as the continuations of f. Clearly, there exists quite a lot of such tangential families, and we are only concerned with two specific ones. In our case the role of $\widetilde{f}_M$ and $\overline{f}_M$ will be played by partial sums of some special series.

Now we turn to the formal exposition. We begin with employing Lemma (3.2)

$$f = f_0 + f_b + f_h,$$

where $f_0 \in \mathcal{N}^0(D)$, $f_h \in G_2^2(D)$ and $f_b = a(f,\chi)\chi$. Let us first focus on the harmonic component f_h. For every $\mathbf{y} \in D(\mathbf{x})$ holds

$$f_h(\mathbf{x}+\mathbf{y}) = \sum (f_h, Y_\lambda^\mu) \left(\frac{y}{r(\mathbf{x})}\right)^\lambda Y_\lambda^\mu(\widehat{\mathbf{y}}),$$

where $(\cdot,\cdot)$ denotes the $L_2(\partial(\mathbf{x}+D(\mathbf{x}))$-scalar product. The series converges uniformly, and while the whole sum depends solely on $\mathbf{x}+\mathbf{y}$, the partial sums

$$f_{hM}(\mathbf{x},\mathbf{y}) = \sum_{\lambda\le M} (f_h, Y_\lambda^\mu) \left(\frac{y}{r(\mathbf{x})}\right)^\lambda Y_\lambda^\mu(\widehat{\mathbf{y}}), \quad M \in \mathbb{N},$$

depend both on $\mathbf{x} \in D$ and $\mathbf{y} \in D(\mathbf{x})$. More importantly, $f_{hM}(\mathbf{x},\mathbf{y})$ is analytic in $\mathbb{C}^d$ as a function of $\mathbf{y}$. That said, for $\mathbf{y} \notin D(\mathbf{x})$ the limit does not exist in general. Let us specifically point out that $f_{hM}(\mathbf{x},0) = f_h(\mathbf{x})$ for every $M \in \mathbb{N}$.

Next, we construct a continuation of f_0. For that we employ its series expansion in s_m, and then re-expand the functions s_m in it. As before, for every $\mathbf{y} \in D(\mathbf{x})$ holds

$$s_m(\mathbf{x}+\mathbf{y}) = \sum c_{m\lambda\mu}(\mathbf{x}) j_\lambda^d(\kappa_m y) Y_\lambda^\mu(\widehat{\mathbf{y}})$$

uniformly. Denote the partial sums by $s_{mM}(\mathbf{x},\mathbf{y})$, $\mathbf{x}\in D$ and $\mathbf{y}\in D(\mathbf{x})$

$$s_{mM}(\mathbf{x},\mathbf{y})=\sum_{\lambda\le M} c_{m\lambda\mu}(\mathbf{x}) j^d_\lambda(\kappa_m y) Y^\mu_\lambda(\widehat{\mathbf{y}}).$$

It should be noted that s_{mM} is analytic in its second argument in $\mathbb{C}^d$, and, as before, $s_{mM}(\mathbf{x},0)=s_m(\mathbf{x})$. Since $s_{mM}\in D(L)$ and by virtue of (2.2), the functions s_{mM} are the eigenfunctions of L corresponding to the same eigenvalue $L(\kappa_m)$. It follows that for every $\mathbf{y}\in\mathbb{C}^d$ holds

$$-L_{\mathbf{y}} s_{mM}(\mathbf{x},\mathbf{y})=L(\kappa_m)s_{mM}(\mathbf{x},\mathbf{y}).$$

Define the partial sums $f_{0M}(\mathbf{x},\mathbf{y})$:

$$f_{0M}(\mathbf{x},\mathbf{y})=\sum_{m\le M}(f_0,s_m)s_{mM}(\mathbf{x},\mathbf{y})\quad \mathbf{x}\in D,\mathbf{y}\in\mathbb{R}^d.$$

Given that $s_{mM}(\mathbf{x},0)=s_m(\mathbf{x})$, takes place the convergence

$$f_{0M}(\cdot,0)\to f_0 \text{ in } W_2^2(D) \text{ as } M\to\infty.$$

The conditions under which $f_{0M}(\mathbf{x},\mathbf{y})$ converges to f_0 are not so easy to establish, being intimately related to the eigenfunctions behaviour.

At last, we define our first continuation $\widetilde{f}_M$ by setting for $\mathbf{x}\in D$ and $\mathbf{y}\in\mathbb{C}^d$

$$\widetilde{f}_M(\mathbf{x},\mathbf{y})=f_b(\mathbf{x}+\mathbf{y})+f_{hM}(\mathbf{x},\mathbf{y})+f_{0M}(\mathbf{x},\mathbf{y}).$$

As it was indicated before, the limits of $f_M(\mathbf{x},\mathbf{y})$ may not exist in general if $\mathbf{y}\notin D(\mathbf{x})$. Even the convergence inside $D(\mathbf{x})$ requires some additional knowledge on the eigenfunctions behaviour. However at $\mathbf{y}=0$ it follows from the $f_0\in\mathcal{N}(D)$ and $s_{mM}(\mathbf{x},0)=s_m(\mathbf{x})$ that

$$\widetilde{f}_M(\cdot,0)\to f \text{ as } M\to\infty \text{ in } W_2^2(D).$$

Our second continuation we define by expanding f in s_m and then taking its partial sums. For $\mathbf{x}\in D$ and $\mathbf{y}\in\mathbb{C}^d$ let

$$\overline{f}_M(\mathbf{x},\mathbf{y})=\sum_{m\le M}(f,s_m)s_{mM}(\mathbf{x},\mathbf{y}).$$

As noted earlier, the expansion of f in s_m converges only in $W_2^1(D)$ and not in $W_2^2(D)$:

$$\overline{f}_M(\cdot,0)\to f \text{ as } M\to\infty \text{ in } W_2^1(D).$$

Note that thus defined $\widetilde{f}_M(\mathbf{x},\mathbf{y})$ and $\overline{f}_M(\mathbf{x},\mathbf{y})$ lie in the domain $D(L)$ as functions of $\mathbf{y}$. Making use of the two continuations, we define for $\mathbf{x}\in D$ two corresponding semi-groups

$$(P^t f)(\mathbf{x})=\lim_{M\to\infty}\mathbb{E}\widetilde{f}_M(\mathbf{x},\xi(t)) \text{ and } (R^t f)(\mathbf{x})=\lim_{M\to\infty}\mathbb{E}\overline{f}_M(\mathbf{x},\xi(t)).$$

The generators of the two semi-groups could be expressed in terms of the “free” process generator $-L$ (formula (1.2)) as follows. The generator $-A$ of P^t acts on $D(A)=W_2^2(D)$ by formula

$$(Af)(\mathbf{x})=\lim_{M\to\infty}(L\widetilde{f}_M)(\mathbf{x},0) \text{ as } \mathbf{x}\in D.$$

Hereinafter the operator L acts in the second variable.

The generator $-A^N$ of R^t acts on its domain $D(A^N) = \mathcal{N}(D) \subset W_2^2(D)$ by formula

$$(A^N f)(\mathbf{x}) = \lim_{M\to\infty} (L\overline{f}_M)(\mathbf{x}, 0) \text{ as } \mathbf{x} \in D.$$

5. Boundary operator

In the current section we show that the difference between P^t and R^t is concentrated on the boundary ∂D. This is an analogue of the support lemma for a local time process.

With this idea in mind, let us find a few handy formulas for the difference of the semi-groups.

Lemma 5.1. *If $f \in W_2^2(D)$ and $\mathbf{x} \in D$, then*

$$(P^t f)(\mathbf{x}) - (R^t f)(\mathbf{x}) = -\lim_{M\to\infty} \int_0^t P^\tau L\Big(\widetilde{f}_M - \overline{f}_M\Big)(\mathbf{x}, 0)\, d\tau\,.$$

The limit also exists in the $W_2^2(D)$-sense.

Proof. By virtue of Theorem 2.4 from [21], which is valid for any $f \in W_2^2(D)$ and does not require that f lie in the domain of $-A^N$, we have

$$R^t f - f = -A^N \int_0^t R^\tau f\, d\tau\,.$$

However, it is not possible to put $-A^N$ inside the integral. To remedy this, we suitably approximate f

$$f_M = \sum_{m\le M} (f, s_m) s_m.$$

Since $R^\tau f = (L_2) \lim R^\tau f_M$, we obtain

$$R^t f - f = -A^N \lim_{M\to\infty} \int_0^t R^\tau f_M\, d\tau\,.$$

It follows from the closedness of A^N

$$A^N \lim_{M\to\infty} f_M = \lim_{M\to\infty} A^N f_M \text{ in } L_2(D),$$

therefore

$$R^t f - f = -\lim_{M\to\infty} \int_0^t R^\tau A^N f_M\, d\tau\,.$$

By the definition of A^N and the fact that $\overline{(f_M)}_M(\mathbf{x}, 0) = f_M(\mathbf{x}, 0)$ follows that $A^N f_M(\mathbf{x}, 0) = L f_M(\mathbf{x}, 0)$. We thus proved that

$$(R^t f)(\mathbf{x}) - f(\mathbf{x}) = -\lim_{M\to\infty} \int_0^t R^\tau L f_M(\mathbf{x}, 0)\, d\tau\,.$$

The generator $-A$ is defined on the whole $W_2^2(D)$, and we can write

$$P^t f - f = -\int_0^t P^\tau A f\, d\tau\,.$$

Recall the definition of A to get

$$(P^t f)(\mathbf{x}) - f(\mathbf{x}) = -\lim_{M\to\infty} \int_0^t P^\tau L\widetilde{f}_M(\mathbf{x}, 0)\, d\tau\,.$$

To conclude the proof, it now remains to subtract the formula for P^t from the formula for R^t and note that in the formula for R^t we can interchange R^τ for P^τ. □

Making use of the lemma above, we will find yet another formula for the difference and show that the difference can be expressed in terms of a certain operator acting on $W_2^{1/2}(\partial D)$.

Lemma 5.2. *If $f \in W_2^2(D)$, then*

$$(P^t f)(\mathbf{x}) - (R^t f)(\mathbf{x}) = \int_{\partial D} Q^t(\mathbf{x}, \mathbf{z})(\gamma_1 f)(\mathbf{z})\, dS(\mathbf{z}),$$

where

$$Q^t(\mathbf{x}, \mathbf{z}) = \frac{1}{2}\int_0^t \widetilde{R}^\tau(\mathbf{x}, \mathbf{z})\, d\tau$$

and

$$\widetilde{R}^\tau(\mathbf{x}, \mathbf{z}) = \sum_{l=0}^{\infty} \frac{L(\kappa_l)}{\kappa_l^2} e^{-tL(\kappa_l)} s_l(\mathbf{x})\overline{s_l(\mathbf{z})}.$$

Proof. We begin with calculating the difference of our continuations

$$\widetilde{f}_M(\mathbf{x}) = f_b(\mathbf{x}+\mathbf{y}) + f_{hM}(\mathbf{x}, \mathbf{y}) + f_{0M}(\mathbf{x}, \mathbf{y})$$

and

$$\overline{f}_M(\mathbf{x}, \mathbf{y}) = \overline{(f_b)}_M(\mathbf{x}, \mathbf{y}) + \overline{(f_h)}_M(\mathbf{x}, \mathbf{y}) + \overline{(f_0)}_M(\mathbf{x}, \mathbf{y})$$

under the integral sign. Note that the continuations agree on f_0

$$\widetilde{f}_{0M} - \overline{(f_0)}_{0M} = 0.$$

Next, we omit the f_{hM} terms since they do not have any effect on the semi-group difference $L\,\widetilde{f}_{hM}(\mathbf{x}, 0) = 0$.

It remains to find a suitable expression for the difference $f_b - \overline{(f_b)}_M - \overline{(f_h)}_M$. Let us rewrite the three functions in terms of integrals over the boundary. For f_h we have

$$f_h(\mathbf{x}) = \int_{\partial D} dS(\mathbf{z})(\gamma_1 f)(\mathbf{z}) \left[h(\mathbf{x}, \mathbf{z}) - \int_{\partial D} dS(\mathbf{w})(\gamma_1\chi)(\mathbf{w})h(\mathbf{x}, \mathbf{w})\right].$$

Thus,

$$\overline{(f_h)}_M(\mathbf{x}, \mathbf{y}) = \int_{\partial D} dS(\mathbf{z})(\gamma_1 f)(\mathbf{z})\varphi_M(\mathbf{x}, \mathbf{y}, \mathbf{z}),$$

where

$$\varphi_M(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{l=0}^{M} \left(h(\cdot, \mathbf{z}) - \int_{\partial D} dS(\mathbf{w})\, h(\cdot, \mathbf{w})(\gamma_1\chi)(\mathbf{w}), s_l\right) s_{lM}(\mathbf{x}, \mathbf{y}).$$

For f_b we have

$$\overline{(f_b)}_M(\mathbf{x}, \mathbf{y}) = \int_{\partial D} dS(\mathbf{z})(\gamma_1 f)(\mathbf{z})\psi_M(\mathbf{x}, \mathbf{y}),$$

where

$$\psi_M(\mathbf{x}, \mathbf{y}) = \sum_{l=0}^{M} (\chi, s_l) s_{lM}(\mathbf{x}, \mathbf{y}).$$

Our problem is now reduced to the calculation of

$$\chi(\mathbf{x}+\mathbf{y}) - \varphi_M(\mathbf{x}, \mathbf{y}, \mathbf{z}) - \psi_M(\mathbf{x}, \mathbf{y}).$$

Let us rewrite $\varphi_M + \psi_M$ as follows:

$$\varphi_M(\mathbf{x},\mathbf{y},\mathbf{z}) + \psi_M(\mathbf{x},\mathbf{y}) =$$

$$= \sum_{l=0}^{M} \Big(\chi(\cdot) - \int_{\partial D} dS(\mathbf{w})(\gamma_1\chi)(\mathbf{w})h(\cdot,\mathbf{w}), s_l\Big)\, s_{lM}(\mathbf{x},\mathbf{y}) +$$

$$+ \sum_{l=0}^{M} \Big(h(\cdot,\mathbf{z}), s_l\Big) s_{lM}(\mathbf{x},\mathbf{y}).$$

The function

$$\theta(\mathbf{x}) = \chi(\mathbf{x}) - \int_{\partial D} dS(\mathbf{w})(\gamma_1\chi)(\mathbf{w})h(\mathbf{x},\mathbf{w})$$

lies in $\mathcal{N}(D)$. It follows that its series in s_l converges in $W_2^2(D)$. Keeping this in mind, we get

$$\sum_{l=0}^{M} (\theta, s_l)\, s_{lM}(\mathbf{x},\mathbf{y}) = \chi(\mathbf{x}) - \int_{\partial D} dS(\mathbf{w})(\gamma_1\chi)(\mathbf{w})h(\mathbf{x},\mathbf{w}) + r_M^{(0)}(\mathbf{x}) + r_M^{(1)}(\mathbf{x},\mathbf{y})$$

where the residual terms $r_M^{(0,1)}$ are

$$r_M^{(0)}(\mathbf{x}) = \sum_{l=M+1}^{\infty} (\theta, s_m)s_m(\mathbf{x}), \quad r_M^{(1)}(\mathbf{x},\mathbf{y}) = \sum_{l=0}^{M} (\theta, s_m)\Big(s_{lM}(\mathbf{x},\mathbf{y}) - s_l(\mathbf{x})\Big).$$

In order to evaluate the last term we make use of the fact that h is the Laplace – Neumann Green function:

$$\Big(h(\cdot,\mathbf{z}), s_l\Big) = \frac{1}{\kappa_l^2}\,\overline{s_l(\mathbf{z})}.$$

So far we have obtained

$$\chi(\mathbf{x}+\mathbf{y}) - \varphi_M(\mathbf{x},\mathbf{y},\mathbf{z}) - \psi_M(\mathbf{x},\mathbf{y}) =$$

$$= \sum_{l=0}^{M} \frac{1}{\kappa_l^2} s_l(\mathbf{x})\overline{s_l(\mathbf{z})} + \int_{\partial D} dS(\mathbf{w})(\gamma_1\chi)(\mathbf{w})h(\mathbf{x},\mathbf{w}) +$$

$$+ r_M^{(0)}(\mathbf{x}) + r_M^{(1)}(\mathbf{x},\mathbf{y}) + r_M^{(2)}(\mathbf{x},\mathbf{y},\mathbf{z})$$

where the residual term $r_M^{(2)}$ is given by

$$r_M^{(2)}(\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{l=0}^{M} \frac{1}{\kappa_l^2}\Big(s_{lM}(\mathbf{x},\mathbf{y}) - s_l(\mathbf{x})\Big)\overline{s_l(\mathbf{z})}.$$

Applying $-L$ and taking $\mathbf{y} = 0$, we get

$$- L_{\mathbf{y}}\Big(\chi(\mathbf{x}+\mathbf{y}) - \varphi_M(\mathbf{x},\mathbf{y},\mathbf{z}) - \psi_M(\mathbf{x},\mathbf{y})\Big)\Big|_{\mathbf{y}=0} =$$

$$= \sum_{l=0}^{M} \frac{L(\kappa_l)}{\kappa_l^2} s_l(\mathbf{x})\overline{s_l(\mathbf{z})} + Lr_M^{(0)}(\mathbf{x}).$$

We use the fact that the second and the third residual terms are zero, since $s_{lM}(\mathbf{x},0) = s_l(\mathbf{x})$.

This establishes the formula

$$(P^t f)(\mathbf{x}) - (R^t f)(\mathbf{x}) =$$

$$= (W_2^2) \lim_{M\to\infty} \frac{1}{2}\int_0^t d\tau \int_{\partial D} dS(\mathbf{z})\; (\gamma_1 f)(\mathbf{z})\; P^\tau \left(\sum_{l=0}^{M} \frac{L(\kappa_l)}{\kappa_l^2} s_l(\mathbf{x})\overline{s_l(\mathbf{z})}\right), \quad (5.1)$$

since the last residual term

$$R_M(\mathbf{x}) = \int_0^t d\tau \int_{\partial D} dS(\mathbf{z})(\gamma_1 f)(\mathbf{z}) P^\tau L r_M^{(0)}(\mathbf{x})$$

tends to zero in $W_2^2(D)$. Indeed,

$$R_M(\mathbf{x}) = C_f \sum_{l=M+1}^{\infty} c_l(\theta, s_l) s_l(\mathbf{x}),$$

where $c_l = 1 - \exp(-tL(\kappa_l)) \leq 1$, while the θ series converges in $W_2^2(D)$. □

We now define the operator family Q^t for $g \in W_2^{1/2}(D)$ and $t > 0$ by

$$(Q^t g)(\mathbf{x}) = \int_{\partial D} Q^t(\mathbf{x},\mathbf{y}) g(\mathbf{y})\, dS(\mathbf{y})\,.$$

Another handy formula for this operator follows from the previous lemma. We take $g \in W_2^{1/2}(\partial D)$ and form a function $G_b \in \chi\mathbb{C}$ from it as follows

$$G_b(\mathbf{x}) = \chi(\mathbf{x}) \int_{\partial D} g(\mathbf{z})\, dS(\mathbf{z}) \quad (5.2)$$

and $G_h \in G_2^2(D)$

$$G_h(\mathbf{x}) = \int_{\partial D} h(\mathbf{x},\mathbf{z}) g(\mathbf{z})\, dS(\mathbf{z})\,. \quad (5.3)$$

Then $G = G_b + G_h \in W_2^2(D)$, and Q^t acts on g as

$$(Q^t g)(\mathbf{x}) = \lim_{M\to\infty} \int_0^t P^\tau L\Big(\widetilde{G}_M - \overline{G}_M\Big)(\mathbf{x},0)\, d\tau\,. \quad (5.4)$$

From what we have proved follow three theorems below (5.3, 5.4 and 5.5).

Theorem 5.3. *The operator families R^t and Q^t satisfy the following evolution relations*

$$R^{t+s} = R^t R^s,$$
$$Q^{t+s} = Q^t + \widetilde{R}^t Q^s.$$

At that, $R^0 = I$ and $Q^0 = 0$.

Theorem 5.4. *If $f \in L_2(D)$ and $t > 0$, then*

$$\frac{\partial}{\partial t} R^t f = \frac{1}{2} A^N R^t f.$$

Theorem 5.5. *If $g \in W_2^{1/2}(\partial D)$ and $t > 0$, then*

$$\frac{\partial}{\partial t} Q^t g = \frac{1}{2}\int_{\partial D} \widetilde{R}^t(\mathbf{x},\mathbf{y}) g(\mathbf{y})\, dS(\mathbf{y})\,.$$

6. Pathwise accumulated momentum

In the previous section we have constructed two operator families R^t and Q^t. Now we will show that there is a natural way to define them as the average of some random operators defined pathwisely.

Let us define a random operator $\mathcal{P}^\tau = \mathcal{P}^\tau[\xi(\cdot)]$ on $W_2^2(D) = \mathcal{N}^0(D) \oplus BG_2^{2,0}(D)$. Set for $f \in \mathcal{N}^0(D)$

$$(\mathcal{P}^\tau f)(\mathbf{x}) = \sum e^{i\kappa_m \xi_1(\tau)} (f, s_m) s_m(\mathbf{x}),$$

and

$$(\mathcal{P}^\tau f)(\mathbf{x}) = f(\mathbf{x} + \xi(\tau))$$

for $f \in BG_2^{2,0}(D)$. Obviously, $P^t = \mathbb{E}\mathcal{P}^t$. We could have defined $\mathcal{P}^\tau$ as the shift along $\xi(\tau)$ still having the same average, since

$$\mathbb{E} e^{i\kappa_m \xi_1(\tau)} s_m(\mathbf{x}) = e^{-\tau L(\kappa_m)} s_m(\mathbf{x}) = \mathbb{E} s_m(\mathbf{x} + \xi(\tau)).$$

Our definition turns out to be more computationally advantageous. Note that only the first coordinate $\xi_1(\tau)$ of $\xi(\tau)$ is involved in the definition. This is due to rotation symmetry of ξ.

We proceed by defining one more random operator $\mathcal{Q}^t = \mathcal{Q}^t[\xi(\cdot)]$, by employing the formula (5.4)

$$(\mathcal{Q}^t g)(\mathbf{x}) = \lim_{M \to \infty} \int_0^t \mathcal{P}^\tau L\Big(\widetilde{G}_M - \overline{G}_M\Big)(\mathbf{x}, 0)\, d\tau, \tag{6.1}$$

where $G = G_b + G_h \in W_2^2(D)$, while G_b and G_h are defined by (5.2), (5.3).

Theorem 6.1. *The limit on the right-hand side of* (6.1) *exists in* $L_2(\mathcal{H}, \mu)$, *where* $\mathcal{H} = D \times \Omega$ *and* $d\mu = d\mathbf{x} \times d\mathbf{P}$.

Proof. The proof is based on the formula (5.4), in which we replace P^τ with $\mathcal{P}^\tau$:

$$(\mathcal{Q}^t g)(\mathbf{x}) = \frac{1}{2} \lim_{M \to \infty} \int_{\partial D} \int_0^t g(\mathbf{z}) \mathcal{P}^\tau \sum_{l=0}^{M} \frac{L(\kappa_l)}{\kappa_l^2} s_l(\mathbf{x}) \overline{s_l(\mathbf{z})}\, dS(\mathbf{z})\, d\tau\,.$$

It suffices to prove that for each $t > 0$ the sequence

$$\Psi_m(\mathbf{x}, \xi(\cdot)) = \int_{\partial D} \int_0^t g(\widehat{\mathbf{y}}) \mathcal{P}^\tau \sum_{l=0}^{M} \frac{L(\kappa_l)}{\kappa_l^2} s_l(\mathbf{x}) \overline{s_l(\mathbf{z})}\, dS(\mathbf{z})\, d\tau$$

is a Cauchy sequence in $L_2(\mathcal{H}, \mu)$.

Let us estimate the norm of the difference $\Psi_m - \Psi_n$, $m > n$

$$\|\Psi_m - \Psi_n\|^2_{L_2(\mathcal{H},\mu)} =$$

$$= \int_D \mathbb{E} \left| \int_{\partial D} \int_0^t g(\mathbf{z}) \sum_{l=n+1}^{m} \frac{L(\kappa_l)}{\kappa_l^2} \mathcal{P}^\tau s_l(\mathbf{x}) \overline{s_l(\mathbf{z})}\, dS(\mathbf{z})\, d\tau \right|^2 d\mathbf{x} =$$

$$= \int_D \mathbb{E} \left| \int_0^t \sum_{l=n+1}^{m} e^{i\kappa_l \xi_1(\tau)} \frac{L(\kappa_l)}{\kappa_l^2} g_l s_l(\mathbf{x})\, d\tau \right|^2 d\mathbf{x},$$

where

$$g_l = \int_{\partial D} g(\mathbf{z}) \overline{s_l(\mathbf{z})}\, dS(\mathbf{z})\,.$$

It is easy to check that

$$\left|\int_0^t \varphi(\tau)\,d\tau\right|^2 = 2\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \,\mathrm{Re}\left(\overline{\varphi(\tau_1)\varphi(\tau_2)}\right)$$

for any $\varphi \in L_1[0,t]$. By making use of it, we get

$$\int_D \mathbb{E}\left|\int_0^t \sum_{l=n+1}^m e^{i\kappa_l \xi_1(\tau)} \frac{L(\kappa_l)}{\kappa_l^2} g_l s_l(\mathbf{x})\,d\tau\right|^2 d\mathbf{x} = \tag{6.2}$$

$$= 2\mathbb{E}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \,\mathrm{Re}\left(\sum_{l=n+1}^m e^{i\kappa_l(\xi_1(\tau_2)-\xi_1(\tau_1))}\frac{L^2(\kappa_l)}{\kappa_l^4}|g_l|^2\right) = \tag{6.3}$$

$$= 2\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \,\mathrm{Re}\left(\sum_{l=n+1}^m e^{-(\tau_2-\tau_1)L(\kappa_l)}\frac{|L(\kappa_l)|^2}{\kappa_l^4}|g_l|^2\right) = \tag{6.4}$$

$$\leq Ct \sum_{l=n+1}^m |g_l|^2 \frac{|L(\kappa_l)|}{\kappa_l^4} \tag{6.5}$$

In the last inequality we use the fact that $\mathrm{Re}\, L(\kappa_l) \geq 0$. Since $|L(\kappa_l)| \leq \kappa_l^2$, it remains to show that the sequence

$$\sum_{l=n+1}^m \frac{|g_l|^2}{\kappa_l^2}$$

tends to zero, which in its turn follows from the fact that $D(\sqrt{-\Delta_N}) = W_2^1(D)$ and therefore $\sqrt{-\Delta_N}G \in L_2(D)$. Indeed, by Green's identity

$$g_l = \int_{\partial D} (\gamma_1 G)(\mathbf{z})\overline{s_l(\mathbf{z})}\,dS(\mathbf{z}) = \int_D \nabla G \cdot \nabla\overline{s_l}\,d\mathbf{x} =$$
$$= \left(\sqrt{-\Delta_N}G, \sqrt{-\Delta_N}s_l\right) = \kappa_l\left(\sqrt{-\Delta_N}G, s_l\right),$$

thus

$$\sum_{l=1}^\infty \frac{|g_l|^2}{\kappa_l^2} = \sum_{l=1}^\infty \left|\left(\sqrt{-\Delta_N}G, s_l\right)\right|^2 < \infty.$$

□

We now show that the average of $\mathcal{Q}^t$ over the paths of $\xi(\cdot)$ equals Q^t.

Theorem 6.2. *If* $g \in W_2^{1/2}(\partial D)$, *then*

$$\mathbb{E}(\mathcal{Q}^t g)(\mathbf{x}) = (Q^t g)(\mathbf{x}).$$

Proof. Once again making use of (5.1), in which we replace P^τ for $\mathcal{P}^\tau$, we get

$$\mathbb{E}(\mathcal{Q}^t g)(\mathbf{x}) = \mathbb{E}\frac{1}{2}\lim_{M\to\infty}\int_{\partial D}\int_0^t g(\mathbf{z})\mathcal{P}^\tau \sum_{l=0}^{M}\frac{L(\kappa_l)}{\kappa_l^2} s_l(\mathbf{x})\overline{s_l(\mathbf{z})}\, d\mathbf{z}\, d\tau =$$

$$= \frac{1}{2}\lim_{M\to\infty}\int_{\partial D}\int_0^t g(\mathbf{z})\sum_{l=0}^{\infty}\mathbb{E}e^{i\kappa_l\xi_1(\tau)}\frac{L(\kappa_l)}{\kappa_l^2} s_l(\mathbf{x})\overline{s_l(\mathbf{z})}\, dS(\mathbf{z})\, d\tau =$$

$$= \frac{1}{2}\int_{\partial D}\int_0^t \widetilde{R}^t(\mathbf{x},\mathbf{z})g(\mathbf{z})\, dS(\mathbf{z})\, d\tau\,.$$

□

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