Global and Stochastic Analysis, Vol.9 No. 2 (March, 2022)
Special Issue: Modern Problems of Equations of Mathematical Physics and its Applications

THE CAUCHY PROBLEM OF COUPLE-STRESS ELASTICITY IN $\mathbb{R}^3$

IQBOL ERGASHEVICH NIYOZOV

ABSTRACT. In this paper, we consider the problem of the analytical continuation of the solution of the system of equations of the moment theory of elasticity in the spatial domain in terms of its values and the values of its stresses on a part of the boundary of this domain, i.e. the Cauchy problem. The conditions for the solvability of this problem are considered.

1. Introduction

In this paper, we propose an explicit formula for reconstructing the solution of of the moment theory of elasticity systems in a spatial domain based on its values and the values of stresses given only on part of the domain boundary. Two directions are considered: the search for reasonable conditions of solvability and the derivation of formulas for solutions as well as criteria for the solvability of the problem.

A solution of the Cauchy problem for the one-dimensional system of CauchyRiemann equations was first obtained in 1926 by Carleman [2]. He proposed the idea of introducing an additional function into the Cauchy integral formula, which allows one to use the passage to the limit in order to damp the influence of integrals over that part of the boundary where the values of the function to be continued are not given. Carlemans idea was developed in 1933 by Goluzin and Krylov [5], who found a general way to obtain Carlemans formulas for the one-dimensional system of CauchyRiemann equations.

Based on the results of Carleman and GoluzinKrylov, Lavrentev introduced the concept of the Carleman function for the one-dimensional system of CauchyRiemann equations. Lavrentevs method [9] consists in approximating the Cauchy kernel on the additional part of the domain boundary outside the support of the data of the Cauchy problem.

The Carleman function of the Cauchy problem for the Laplace equation is a fundamental solution that depends on a positive numerical parameter and tends to zero together with its normal derivative on the part of the domain boundary outside the Cauchy data support as the parameter tends to infinity. Using the Carleman function and Greens integral formula, a Carleman formula is produced that gives an exact solution of the Cauchy problem when the data are specified exactly. Having constructed the Carleman function also allows one to construct a

1991 *Mathematics Subject Classification.* Primary 35J46; Secondary 35J56.
Key words and phrases. The Cauchy problem, regularization, factorization, regular solution, fundamental solution.

regularization if the Cauchy data are given approximately. The existence of the Carleman function follows from the Mergelyan approximation theorem [17].

In 1959, Fock and Kuni [4] found an application of the Carleman formula to the one-dimensional system of CauchyRiemann equations. For the case in which part of the domain boundary is a segment of the real axis, they used the Carleman formula to establish a criterion for the solvability of the Cauchy problem for the system of CauchyRiemann equations on the plane. An analog of the Carleman formula and criteria for the solvability of the Cauchy problem were obtained in [6], [8] for analytic functions of several variables, in [26]-[28], [24] for harmonic functions, and also in the papers [18]-[23] by the present authors. The Cauchy problem for matrix factorizations of the Helmholtz equation is considered in papers D.A. Juraev (see, for instance [11], [12], [13], [14], [15], [16] and [30])

The Cauchy problem for solutions of elliptic equations has been studied since the 1950s when it entered geophysics. If the Cauchy data are posed on an open part of the boundary, then the Cauchy problem has at most one solution. However, the solution fails to depend continuously on the Cauchy data, unless they are controlled on the whole boundary. Thus, in a natural setting, the Cauchy problem for elliptic equations is ill posed; and the character of instability is similar to that in the problem of analytic continuation.

The monographs [9], [25], [10], [1] are a fairly complete survey regarding Carlemans formulas.

In the present paper, a regularized solution of the Cauchy problem for the system of Couple-Stress Elasticity equations is constructed on the basis of the Carleman function method. [9].

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be points in real Euclidean space $\mathbb{R}^3$, let D be a bounded simply connected domain in $\mathbb{R}^3$ with piecewise smooth boundary ∂D, and let S be the smooth part of ∂D.

Let a six-component vector function

$$U(x) = (u_1(x), u_2(x), u_3(x), v_1(x), v_2(x), v_3(x))^* = (u(x), v(x))^*,$$

where from now on $*$ denotes the operation of transposition, satisfy the system of Couple-Stress Elasticity equations [7].

$$\begin{cases} (\mu+\alpha)\Delta u + (\lambda+\mu-\alpha)\text{graddiv}\, u + 2\alpha\, rot v + \rho\omega^2 u = 0, \\ (\nu+\beta)\Delta v + (\varepsilon+\nu-\beta)\text{graddiv}\, v + 2\alpha\, rot u - 4\alpha v + \theta\omega^2 v = 0, \end{cases} \tag{1.1}$$

where Δ is the Laplace operator, i is the imaginary unit, i.e., $i^2 = -1$, the coefficients $\lambda, \mu, \rho, \omega$, and θ-the characteristics of the medium satisfy the conditions $\mu > 0,\ 3\lambda + 2\mu > 0,\ \alpha > 0,\quad 3\varepsilon + 2\nu > 0,\ \beta > 0$ and $\gamma\eta > 0$, and ω is some real number called the vibration frequency, ρ- density of the medium.

For brevity of presentation, in what follows, system (1.1) is conveniently written in matrix form. For this purpose, we introduce the matrix differential operator

$$M = M(\partial_x) = \begin{Vmatrix} M^{(1)} & M^{(2)} \\ M^{(3)} & M^{(4)} \end{Vmatrix},$$

where

$$M^{(q)} = \left\| M_{kj}^{(q)} \right\|_{3\times 3}, q = 1,2,3,4,$$

$$M_{kj}^{(1)} = \delta_{kj}(\mu+\alpha)(\Delta + \omega_1^2) + (\lambda + \mu - \alpha)\frac{\partial^2}{\partial x_k \partial x_j},$$

$$M_{kj}^{(2)} = M_{kj}^{(3)} = -2\alpha \sum_{p=1}^{3} \varepsilon_{kjp} \frac{\partial}{\partial x_p},$$

$$M_{kj}^{(4)} = \delta_{kj}(\nu+\beta)(\Delta + \omega_2^2) + (\varepsilon + \nu - \beta)\frac{\partial^2}{\partial x_k \partial x_j},$$

$\omega_1^2 = \dfrac{\rho\omega^2}{\mu+\alpha}$, $\omega_2^2 = \dfrac{\theta\,\omega^2 - 4\alpha}{\nu+\beta}$, $\delta_{kj} = \begin{cases} 1, & \text{if } k = j \\ 0, & \text{if } k \neq j, \end{cases}$ ε_{kjp} so-called $\varepsilon-$tensor or Levi-Civita's symbol, which defend following equaliti's

$$\varepsilon_{kjp} = \begin{cases} 0, & \text{if at least two of three-subscripts } \; k,j,p \; \text{ are equal,} \\ 1, & \text{if } \; (k,j,p) \; \text{ is an even permutation,} \\ -1, & \text{if } \; (k,j,p) \; \text{ is an odd permutation.} \end{cases}$$

Then system (1.1) maybe write in matrix from in the following way:

$$M(\partial_x)U(x) = 0, \tag{1.2}$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix}$.

Now let us introduce the stress operator of the dynamic elasticity equations. Let $x = (x_1, x_2, x_3)$ be a point of the medium, and let $n(x) = (n_1(x), n_2(x), n_3(x))$ be an arbitrary unit vector at the point x. We introduce the matrix differential operator

$$T(\partial_x, n(x)) = \left\| \begin{matrix} T^{(1)}(\partial_x, n(x)) & T^{(2)}(\partial_x, n(x)) \\ T^{(3)}(\partial_x, n(x)) & T^{(4)}(\partial_x, n(x)) \end{matrix} \right\|,$$

$$T^{(i)}(\partial_x, n(x)) = \left\| T_{kj}^{(i)}(\partial_x, n(x)) \right\|_{3\times 3}, \; i = 1,2,3,4,$$

$$T_{kj}^{(1)}(\partial_x, n(x)) = \lambda n_k \frac{\partial}{\partial x_j} + (\mu - \alpha) n_j(x) \frac{\partial}{\partial x_k} + (\mu + \alpha)\delta_{kj} \frac{\partial}{\partial n(x)}, \; k,j = 1,2,3,$$

$$T_{kj}^{(2)}(\partial_x, n(x)) = -2\alpha \sum_{p=1}^{3} \varepsilon_{kjp} n_p(x), \quad T_{kj}^{(3)}(\partial_x, n(x)) = 0, \quad k,j = 1,2,3,$$

$$T_{kj}^{(4)}(\partial_x, n(x)) = \varepsilon\, n_k(x) \frac{\partial}{\partial x_j} + (\nu - \beta) n_j(x) \frac{\partial}{\partial x_k} + (\nu + \beta) \frac{\partial}{\partial n(x)}, \; j = 1,2,3.$$

The main purpose of the couple-stress elasticity theory is to determine the elastic oscillation state. The state should continuously depend on the boundary data. The matter is that these data are obtained by measurements and therefore they always differ from their exact values. Hence the concrete state which has been found by such approximate data will be of practical importance if it differs from the true state to the same extent as the data differ from their exact values. The problem thus posed is called correct.

In the couple-stress elasticity theory four main problems of oscillation are considered. They consist in finding the elastic oscillation state of the medium if on the boundary we are given the displacements and rotations in the first problem, the force- and couple-stresses in the second problem, the displacements and couplestresses in the third problem, the rotations and force-stresses in the fourth problem, [7], Ch. IX.

2. Fundamental solution and the Somiliana formula

The homogeneous equation of steady-state oscillations of the couple-stress theory has the form $MU = 0$. By the (two-sided) fundamental solution of convolution type for M is meant any (6×6) -matrix Ψ whose entries are distributions in all of $\mathbb{R}^3$, such that $M(\Psi * U) = U$ and $\Psi * (MU) = U$ holds for each C^∞ function U with compact support and values in $\mathbb{R}^6$. A familiar argument shows that this just amounts to saying that

This definition implies

$$M(\partial_x)\Psi(x-y) = \delta(x-y)E_6,$$

$$M'(\partial_y)(\Psi(x-y))^\top = \delta(x-y)E_6,$$

for all $(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3$,where M' stands for the transposed differential operator of M, $\Psi^\top$ for the transposed matrix of Ψ, and δ - for the Dirac functional supported at the origin.

Such a fundamental solution can be obtained by the formula $\Psi = \mathcal{M}\varphi$, where $\mathcal{M}-$ is the complementary matrix of M, i.e., the matrix satisfying the equations $\mathcal{M}M = M\mathcal{M} = (\det M)E_6$, and φ a fundamental solution of convolution type for the scalar differential operator $\det M$. An elementary, though cumbersome, computation shows that

We denote the algebraic complement of the element $M_{kj}(\partial_x)(k,j = \overline{1,6})$ in $\det M(\partial_x)$, by $\mathcal{M}_{kj}(\partial_x)$. An elementary, though cumbersome, computation for element's $\mathcal{M}_{kj}(\partial_x)$ the matrix

$$\mathcal{M}(\partial_x) = \left\| \begin{array}{cc} \mathcal{M}^{(1)}(\partial_x) & -\mathcal{M}^{(3)}(\partial_x) \\ -\mathcal{M}^{(2)}(\partial_x) & \mathcal{M}^{(4)}(\partial_x) \end{array} \right\|_{6\times 6},$$

get

$$\mathcal{M}_{kj}^{(1)} = \alpha_0 \left\{ \frac{\delta_{kj}(\Delta + k_1^2)(\Delta + \omega_2^2)}{\mu + \alpha} - \frac{1}{\lambda + 2\mu} \left[\frac{(\lambda + \mu - \alpha)(\Delta + \omega_2^2)}{\mu + \alpha} - \right.\right.$$

$$\left.\left. - \frac{4\alpha^2}{(\mu + \alpha)(\nu + \beta)} \right] \frac{\partial^2}{\partial x_k \partial x_j} \right\} (\Delta + k_2^2)(\Delta + k_3^2)(\Delta + k_4^2),$$

$$\mathcal{M}_{kj}^{(2)} = \mathcal{M}_{kj}^{(3)} =$$

$$= \frac{2\alpha\alpha_0}{(\mu + \alpha)(\nu + \beta)} \sum_{q=1}^{3} \varepsilon_{kjq} \frac{\partial}{\partial x_q} (\Delta + k_1^2)(\Delta + k_2^2)(\Delta + k_3^2)(\Delta + k_4^2), \tag{2.1}$$

$$\mathcal{M}_{kj}^{(4)} = \alpha_0 \left\{ \frac{\delta_{kj}(\Delta + k_2^2)(\Delta + \omega_1^2)}{\nu + \beta} - \frac{1}{\varepsilon + 2\nu} \left[\frac{(\nu + \varepsilon - \beta)(\Delta + \omega_1^2)}{\nu + \beta} - \right.\right.$$

$$\left.\left. - \frac{4\alpha^2}{(\mu + \alpha)(\nu + \beta)} \right] \frac{\partial^2}{\partial x_k \partial x_j} \right\} (\Delta + k_1^2)(\Delta + k_3^2)(\Delta + k_4^2),$$

where $k, j = 1, 2, 3$, $k_1^2 = \dfrac{\rho\omega^2}{\lambda + 2\mu}$, $k_2^2 = \dfrac{\theta\omega^2 - 4\alpha}{\epsilon + 2\nu}$, k_3^2 and k_4^2 satisfy the conditions

$$k_3^2 + k_4^2 = \omega_3^2 + \omega_4^2 + \frac{4\alpha^2}{(\mu + \alpha)(\nu + \beta)}, \quad k_3^2 k_4^2 = \omega_3^2 \omega_4^2,$$

$$\alpha_0 = (\mu + \alpha)^2 (\nu + \beta)^2 (\lambda + 2\mu)(\epsilon + 2\nu) > 0.$$

It is easy to see that $\mathcal{M}(\partial_x)$ (as $M(\partial_x)$) is a self-adjoint operator, i.e. $\mathcal{M}(\partial_x) = (\mathcal{M}(-\partial_x))'$, where the prime denotes the transposition operation. Substituting in $M(\partial_x)U = 0$ instead of U matrix

$$U = (M(\partial_x))' \varphi = \left\| \begin{matrix} \mathcal{M}^{(1)}(\partial_x) & \mathcal{M}^{(2)}(\partial_x) \\ \mathcal{M}^{(3)}(\partial_x) & \mathcal{M}^{(4)}(\partial_x) \end{matrix} \right\| \varphi \tag{2.2}$$

where $\varphi-$ the required scalar function, then we get

$$\det M(\partial_x)\varphi = \alpha_0 \left(\triangle + k_1^2\right) \left(\triangle + k_2^2\right) \left(\triangle + k_3^2\right)^2 \left(\triangle + k_4^2\right)^2 \varphi = 0.$$

In (2.2), each element contains the factor $\alpha_0 \left(\triangle + k_3^2\right) \left(\triangle + k_4^2\right) \varphi$, therefore, it suffices to find exactly the function

$$\psi = \alpha_0 \left(\triangle + k_3^2\right) \left(\triangle + k_4^2\right) \varphi.$$

To determine it, we have the equation

$$\left(\triangle + k_1^2\right) \left(\triangle + k_2^2\right) \left(\triangle + k_3^2\right) \left(\triangle + k_4^2\right) \psi = 0.$$

For the matrix of solutions (2.2) to be fundamental, we have to find a solution of the last equation, the sixth order partial derivatives of which have singularities only of the form $|x|^{-1}$. Such a solution, if it exists, should satisfy the conditions

$$\left(\triangle + k_{q+1}^2\right) \left(\triangle + k_{q+2}^2\right) \left(\triangle + k_{q+3}^2\right) \psi = (2\pi|x|)^{-1} \exp\left(ik_q|x|\right), \quad q = 1, 2, 3, 4,$$

$$k_5 = k_1, \ \ k_6 = k_2, \ \ k_7 = k_3.$$

Considering these relations as a system of equations for ψ, $\triangle\,\psi$, $\triangle^2\,\psi$, $\triangle^3\,\psi$ find

$$\psi = \sum_{q=1}^{4} A_q \left(2\pi|x|\right)^{-1} \exp\left(ik_q|x|\right), \tag{2.3}$$

where $A_q = \prod_{s=1}^{3} \dfrac{1}{\left(k_{q+s}^2 - k_q^2\right)}$.

Relations (2.3) make it easy to check that ψ satisfy all of the above conditions.

Now taking into account that $\left(\triangle + k_3^2\right)\left(\triangle + k_4^2\right)\varphi = \alpha_0^{-1}\psi$ and given the importance ψ, of (2.2) and (2.1), we obtain the matrix of fundamental solutions $U(x)$ of (2.2) which we denote by Ψ. Thus, the block-symmetric matrix fundamental solutions has the form

$$\Psi(x) = \left\| \begin{array}{cc} \Psi^{(1)}(x) & \Psi^{(2)}(x) \\ \Psi^{(3)}(x) & \Psi^{(4)}(x) \end{array} \right\|_{6\times 6}$$

where

$$\Psi^{(i)} = \left\| \Psi_{kj}^{(i)}(x) \right\|_{3\times 3}, \quad i = 1, 2, 3, 4,$$

$$\Psi_{kj}^{(1)}(x) = \sum_{q=1}^{4} \left(\delta_{kj} a_q + b_q \frac{\partial^2}{\partial x_k \partial x_j} \right) \left(-\frac{1}{4\pi} \cdot \frac{\exp\left(ik_q|x|\right)}{|x|} \right),$$

$$\Psi_{kj}^{(2)}(x) = \Psi_{kj}^{(3)}(x) = \frac{2\alpha}{\mu + \alpha} \sum_{q=1}^{4} \sum_{m=1}^{3} \varepsilon_q \varepsilon_{kjm} \frac{\partial}{\partial x_m} \left(-\frac{1}{4\pi} \cdot \frac{\exp\left(ik_q|x|\right)}{|x|} \right),$$

$$\Psi_{kj}^{(4)}(x) = \sum_{q=1}^{4} \left(\delta_{kj} c_q + d_q \frac{\partial^2}{\partial x_k \partial x_j} \right) \left(-\frac{1}{4\pi} \cdot \frac{\exp\left(ik_q|x|\right)}{|x|} \right),$$

for $k, j = 1, 2, 3$.

Theorem 2.1. *Each column of the matrix $\Psi(x)$ considered as a vector, satisfies system (2.1) at all points of the space $\mathbb{R}^3$ except for the origin.*

For the matrix of fundamental solutions, the following equalities hold: [22]

$$\Psi^{\top}(x - y) = \Psi(y - x).$$

The Somiliana formula holds [22].

Theorem 2.2. *For any function $U \in C^1(\overline{D})$ with values in $\mathbb{R}^6$ such that $M(\partial_x)U \in L_1(D)$ holds*

$$\int_{\partial D} \left(\left\{ T\left(\partial_y, n(y)\right) \Psi(y - x) \right\}^{\top} U(y) - \Psi(x - y) \left\{ T\left(\partial_y, n(y)\right) U(y) \right\} \right) ds_y + \\ + \int_{D} \Psi(x - y) M(\partial_y) U(y) dy = \begin{cases} U(x), & x \in D \\ 0, & x \notin \overline{D}. \end{cases} \tag{2.4}$$

where "$\top$" y of the matrix means the operation transposing.

3. Solvability criterion in the language of Carleman matrix

Let us introduce the notation:

$$x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3, x' = (0, x_2, x_3), y' = (0, y_2, y_3),$$

$$s = \alpha^2 = |x' - y'|^2, r^2 = |y - x|^2 = s + (y_1 - x_1)^2,$$

$$G_\rho = \{y \in \mathbb{R}^3 : |y'| < \tau y_1, y_1 > 0, \tau = tg\frac{\pi}{2\rho}, \rho > 1\},$$

$$\partial G_\rho = \{y \in \mathbb{R}^3 : |y'| = \tau y_1, y_1 > 0, \tau = tg\frac{\pi}{2\rho}, \rho > 1\},$$

$$\overline{G_\rho} = G_\rho \bigcup \partial G_\rho,$$

$\varepsilon, \varepsilon_1, \varepsilon_2-$ denote sufficiently small constant positive numbers,

$$G_\rho^\varepsilon = \{y \in \mathbb{R}^3 : |y'| < \tau(y_1 - \varepsilon), y_1 > \varepsilon, \tau = tg\frac{\pi}{2\rho}, \rho > 1\},$$

$$\partial G_\rho^\varepsilon = \{y \in \mathbb{R}^3 : |y'| = \tau(y_1 - \varepsilon), y_1 > \varepsilon, \tau = tg\frac{\pi}{2\rho}, \rho > 1\},$$

$$\overline{G_\rho^\varepsilon} = G_\rho^\varepsilon \bigcup \partial G_\rho^\varepsilon, \tau_1 = sin\frac{\pi}{2\rho}, \rho > 1,$$

$\mathbb{C}$ - complex plane, D_ρ - bounded simply connected domain with boundary ∂D_ρ, consisting of a part of the cone surface ∂G_ρ (in the two-dimensional case of ray segments with a common origin) and smooth surface S (smooth curve), lying inside the cone (of the corner) $\overline{G_\rho}$. The case $\rho = 1$ is limit. In this case G_1 - half-space $y_1 > 0$ and ∂G_1 - hyperplane $y_1 = 0$, D_1 - bounded simply connected domain with a boundary, consisting of a compact connected part of the hyperplane $y_1 = 0$ (in the two-dimensional case, segment $a \leq y_2 \leq b$) and smooth surface S (smooth curve), lying and half-space $y_1 > 0$, $\overline{D_\rho} = D_\rho \bigcup \partial D_\rho, S_0-$ set of interior points S.

We will construct a solution to the problem in the region D_ρ, when the Cauchy data are given on the part S of the boundary ∂D_ρ.

We will assume that the solution to the problem exists (then it is unique) and is continuously differentiable in a closed domain and the Cauchy data are specified exactly. For this case, an explicit continuation formula is established. The found formula allows us to formulate a simple and convenient criterion for the solvability of the Cauchy problem. If, under the indicated conditions, instead of the Cauchy data, their continuous approximations are given with a given error (deviation) in the uniform metric, then an explicit regularization formula is proposed.

The result established here is a multidimensional analogue of the theorem and a version of the Carleman formula obtained by G.M. Goluzin, V.I. Krylov, V.A. Fock, F.M. Cooney in the theory of holomorphic functions of one variable [5], [4].

The continuation formulas proved below are explicitly expressed in terms of the entire Mittag-Leffler function; therefore, we present its main properties without proof. They are given in (see, [3], Ch. 3, 2) with detailed proofs. The entire Mittag-Leffler function is defined by the series

$$E_\rho(w)=\sum_{n=0}^{\infty}\frac{w^n}{\Gamma(1+\frac{n}{\rho})},\rho>0,w\in C,E_1(w)=\exp w,$$

where $\Gamma(\cdot)$ Euler's gamma function. Throughout what follows, we will assume $\rho>1$.

We choose β subject to the condition $0<\beta<\dfrac{\pi}{\rho},\rho>1,$. We let $\gamma=\gamma(1,\beta)$ be a contour in the complex z-plane traversed in the direction of nondecreasing $\arg z$ and consisting of the following parts: 1) $\arg z=-\beta$, 2) the arc $-\beta\le\arg z\le\beta$ of the circle $|z|=1$, and 3) the ray $\arg z=\beta$. The contour divides the z -plane into two unbounded, simply-connected domains D^- and D^+ which lie to the left and the right of $\gamma=\gamma(1,\beta)$, respectively. We put

We will assume that $\dfrac{\pi}{2\rho}<\beta<\dfrac{\pi}{\rho},\rho>1.$

Under these conditions, the following integral representations are valid:

$$E_\rho(w)=\exp(w^\rho)+\Psi_\rho(w),w\in D^+, \tag{3.1}$$

$$E_\rho(w)=\Psi_\rho(w),w\in D^+,\ E_\rho'(w)=\Psi_\rho'(w),w\in D^-, \tag{3.2}$$

where

$$\Psi_\rho(w)=\frac{\rho}{2\pi i}\int\limits_\gamma\frac{\exp\varsigma^\rho d\varsigma}{\varsigma-w},\ \Psi_\rho'(w)=\frac{\rho}{2\pi i}\int\limits_\gamma\frac{\exp\varsigma^\rho d\varsigma}{(\varsigma-w)^2}. \tag{3.3}$$

Since $E_\rho(w)$ is real for real w, we have

$$\mathrm{Re}\Psi_\rho(w)=\frac{\Psi_\rho(w)+\Psi_\rho(\overline{w})}{2}=\frac{\rho}{2\pi i}\int\limits_\gamma\frac{(\varsigma-\mathrm{Re}w)\exp(\varsigma^\rho)d\varsigma}{(\varsigma-w)(\varsigma-\overline{w})}, \tag{3.4}$$

$$\mathrm{Im}\Psi_\rho(w)=\frac{\Psi_\rho(w)-\Psi_\rho(\overline{w})}{2i}=\frac{\rho\mathrm{Im}w}{2\pi i}\int\limits_\gamma\frac{\exp(\varsigma^\rho)d\varsigma}{(\varsigma-w)(\varsigma-\overline{w})}, \tag{3.5}$$

$$\mathrm{Im}\frac{\Psi_\rho'(w)}{Imw}=\frac{\rho}{2\pi i}\int\limits_\gamma\frac{2(\varsigma-\mathrm{Re}w)\exp(\varsigma^\rho)d\varsigma}{(\varsigma-w)^2(\varsigma-\overline{w})^2}. \tag{3.6}$$

Throughout what follows, in the definition of the contour $\gamma(1,\beta)$, we will take $\beta=\dfrac{\pi}{2\rho}+\dfrac{\varepsilon_2}{2},\rho>1$. It is clear that if

$$\frac{\pi}{2\rho}+\varepsilon_2\le|\arg w|\le\pi, \tag{3.7}$$

then $w\in D_\rho^-$ and $E_\rho(w)=\Psi_\rho(w)$.

At $\dfrac{\pi}{2\rho}+\varepsilon_2\le|\arg w|\le\pi$ inequalities hold

$$|E_\rho(w)|\le\frac{M_1}{1+|w|},\ |E_\rho'(w)|\le\frac{M_2}{1+|w|^2}, \tag{3.8}$$

$$|E_\rho^{(k)}(w)|\le\frac{M_3}{1+|w|^{k+1}},\ k=0,1,2,.... \tag{3.9}$$

where M_1, M_2, M_3 - constant, independent of w.

Consider the Cauchy problem for system (2.1): We need to find a solution to the system (2.1) $U(x)$ in D_ρ according to $U(y) = U_0(y),\ T(\partial_y, n(y))U(y) = U_1(y),\ y \in S$ i.e.

$$\begin{cases} M\left(\partial_x\right) U(x) = 0\,,\ x \in D_\rho \\ U(y) = U_0(y)\,, \qquad\quad y \in S \\ T(\partial_y, n(y))U(y) = U_1(y),\ y \in S, \end{cases} \tag{3.10}$$

where $U_0 \in C^1(S) \cap L_1(S),\ U_1 \in C(S) \cap L_1(S)$.

To solve this problem for a given simply connected domain, the method of the Carleman function is used, i.e. the matrix is built Carleman and with the help of this matrix a formula for finding a solution inside a domain is given.

Definition 3.1. By the Carleman matrix of the domain D_ρ and surface S, we mean an 6×6 matrix $\Pi(y, x, \sigma)$ depending on the two points $y, x \in \overline{D}_\rho$ and positive numerical number parameter σ, satisfying the following two conditions:

$$1) \quad \Pi(y, x, \sigma) = \Psi(y - x) + G(y, x, \sigma),$$

where matrix $G(y, x, \sigma)$ satisfies system (1) with respect to the variable y in the domain D, and $\Psi(y - x)$ is a matrix of the fundamental solutions of system (2.1);

$$2) \qquad \int\limits_{\partial D_\rho \setminus S} \left(|\Pi(y, x, \sigma)| + |T(\partial_y, n)\Pi(y, x, \sigma)|\right) ds_y \le \varepsilon(\sigma),$$

where $\varepsilon(\sigma) \to 0$, as $\sigma \to \infty$; here $|\Pi|$ is the Euclidean norm of the matrix $\Pi = ||\Pi_{ij}||_{6\times 6}$, i.e., $|\Pi| = (\sum\limits_{i,j=1}^{6} \Pi_{ij}^2)^{\frac{1}{2}}$. In particular, $|U| = \left(\sum\limits_{m=1}^{3} (u_m^2 + w_m^2)\right)^{\frac{1}{2}}$.

It is well known, that for the regular vector functions $V(y)$ and $U(y)$ holds formula [7]:

$$\int\limits_{D_\rho} \left[V(y)\{M(\partial_y)U(y)\} - U(y)\{M(\partial_y)V(y)\}\right] dy =$$

$$= \int\limits_{\partial D_\rho} \left[V(y)\{T(\partial_y, n(y))U(y)\} - U(y)\{T(\partial_y, n(y))V(y)\}\right] ds_y.$$

Substituting in this equality $V(y) = G(y, x, \sigma)$ and $U(y) = U(y)$ is solution system (2.1), we have

$$0 = \int\limits_{\partial D_\rho} \left[G(y, x, \sigma)\{T(\partial_y, n(y))U(y)\} - \{T(\partial_y, n(y))G(y, x, \sigma)\}^\top U(y)\right] ds_y.$$

Now, taking into account Theorem 2.2 for $M(\partial_y)U(y) = 0$, we have

Theorem 3.2. *Any regular solution $U(x)$ of system (2.1) in the domain D_ρ is specified by the formula*

$$\int\limits_{\partial D_\rho} \left(\{T(\partial_y, n(y))\Pi(y,x,\sigma)\}^\top U(y) - \Pi(y,x,\sigma)\{T(\partial_y, n(y))U(y)\}\right) ds_y =$$
$$= \begin{cases} U(x), & x \in D_\rho \\ 0, & x \notin \overline{D_\rho}. \end{cases} \tag{3.11}$$

where $\Pi(y,x,\sigma)$ *is matrix Carleman.*

Using the Carleman matrix, it is easy to derive an estimate for the stability of the solution to the Cauchy problem (3.10), and also indicate a method for effectively solving this problem.

In order to construct an approximate solution to problem (3.10) we construct the Carleman matrix as follows:

$$\Pi(y,x,\sigma) = \left\| \begin{array}{cc} \Pi^{(1)}(y,x,\sigma) & \Pi^{(2)}(y,x,\sigma) \\ \Pi^{(3)}(y,x,\sigma) & \Pi^{(4)}(y,x,\sigma) \end{array} \right\|,$$

$$\Pi^{(i)}(y,x,\sigma) = \left\|\Pi^{(i)}_{kj}(y,x,\sigma)\right\|_{3\times 3}, \quad i = 1,2,3,4,$$

$$\begin{gathered} \Pi^{(1)}_{kj}(y,x,\sigma) = \sum_{q=1}^{4}\left(\delta_{kj}a_q + b_q\frac{\partial^2}{\partial x_k \partial x_j}\right)\cdot \Phi_\tau(y,x,i\lambda_q), \quad k,j = 1,2,3, \\ \Pi^{(2)}_{kj}(y,x,\sigma) = \Pi^{(3)}_{kj}(y,x,\sigma) = \\ = \frac{2\alpha}{\mu+\alpha}\sum_{q=1}^{4}\sum_{m=1}^{3}\varepsilon_q\varepsilon_{kjs}\frac{\partial}{\partial x_m}\cdot\Phi_\sigma(y,x,i\lambda_q), \quad k,j = 1,2,3, \\ \Pi^{(4)}_{kj}(y,x,\sigma) = \sum_{q=1}^{4}\left(\delta_{kj}c_q + d_q\frac{\partial^2}{\partial x_k \partial x_j}\right)\cdot \Phi_\sigma(y,x,i\lambda_q), \quad k,j = 1,2,3, \end{gathered} \tag{3.12}$$

where

$$\Phi_\sigma(y,x,\lambda) = \Phi_\sigma(y-x,\lambda) = \frac{1}{-2\pi^2}\int\limits_0^\infty \mathrm{Im}\left[\frac{K(w)}{w}\right]\frac{cos(\lambda u)\,du}{\sqrt{u^2+\alpha^2}}, \tag{3.13}$$

and

$$K(w) = \exp[w^2]E_\rho(\sigma w), w = i\sqrt{u^2+\alpha^2} + y_1 - x_1.$$

From the results of [27] it follows

Lemma 3.3. *The function $\Phi_\sigma(y-x,\lambda)$ is Carleman's function for the Helmholtz equation, i.e., it has the following two properties:*

$$1)\ \Phi_\sigma(y-x,\lambda) = \frac{\exp(i\lambda r)}{4\pi r} + \varphi_\sigma(y-x,\lambda), \quad r = |y-x|, \tag{3.14}$$

where $\varphi_\sigma(y,\lambda)$ is a function that is defined for all y and x satisfies Helmholtz equation $\Delta(\partial_y)\varphi_\sigma + \lambda^2\varphi_\sigma = 0$,

$$2)\ \int\limits_{\partial D_\rho \setminus S}\left(|\Phi_\sigma| + |\frac{\Phi_\sigma}{\partial n})|\right) ds_y \le \frac{C(\lambda, D_\rho)}{1+\sigma} \tag{3.15}$$

where $C(\lambda, D_\rho)$ *some function bounded inside* D_ρ*, independent of* σ*,* $\Delta(\partial_y) = \dfrac{\partial^2}{\partial y_1^2} + \dfrac{\partial^2}{\partial y_2^2} + \dfrac{\partial^2}{\partial y_3^2}$.

The function $\Phi_\sigma(y-x,\lambda)$ is called the Carleman function for the Helmholtz equation. Here are some properties of the Carleman function.

Introduce the notation

$$F_\sigma(y-x,\lambda) = \frac{\partial}{\partial\sigma}\Phi_\sigma(y-x,\lambda)$$

The following lemma holds [28].

Lemma 3.4. *Let* K *be compact in* G_ρ*,* δ *- distance from* K *to* ∂G_ρ*. Then, for* $\sigma \geq 0$ *for* $x \in K$*,* $y \in R^3 \setminus G_\rho$ $(|y^{'}| \geq \tau y_1)$*, the following inequalities hold:*

$$|\Phi_\sigma(y-x,\lambda)| + \left|\frac{\partial \Phi_\sigma(y-x,\lambda)}{\partial y_k}\right| \leqslant C_1(\rho,\delta) r (1+\sigma\delta)^{-1}, \tag{3.16}$$

$$|F_\sigma(y-x,\lambda)| + \left|\frac{\partial F_\sigma(y-x,\lambda)}{\partial y_k}\right| \leqslant C_2(\rho,\delta) r (1+\sigma^2\delta^2)^{-1}, r \geq \delta > 0 \tag{3.17}$$

where the constants C_1, C_2 *do not depend on* x, y *and* σ*.*

From Lemma 3.3 and Lemma 3.4 we obtain

Lemma 3.5. *The matrix* $\Pi(y,x,\sigma)$ *defined by formulas (3.12), (3.13) is the Carleman matrix for the domain* D_ρ *and surface* S*.*

Proof. From (3.12), (3.13) and lemma 3.2 we have

$$\Pi(y,x,\sigma) = \Psi(y-x) + G(y-x,\sigma),$$

where

$$G(y-x,\sigma) = \left\| \begin{array}{ll} G^{(1)}(y-x,\sigma) & G^{(2)}(y-x,\sigma) \\ G^{(3)}(y-x,\sigma) & G^{(4)}(y-x,\sigma) \end{array} \right\|,$$

$$G^{(i)}(y-x,\sigma) = \left\| G^{(i)}_{kj}(y-x,\sigma) \right\|_{3\times 3}, \quad i = 1,2,3,4,$$

$$G^{(1)}_{kj}(y-x,\sigma) = \sum_{q=1}^{4}\left(\delta_{kj} a_q + b_q \frac{\partial^2}{\partial x_k \partial x_j}\right)\varphi_\sigma(y-x,k_q), \quad k,j = 1,2,3,$$

$$G^{(2)}_{kj}(y-x,\sigma) = G^{(3)}_{kj}(y-x,\sigma) = \frac{2\alpha}{\mu+\alpha}\sum_{q=1}^{4}\sum_{m=1}^{3}\varepsilon_q \varepsilon_{kjm}\frac{\partial}{\partial x_m}\varphi_\sigma(y-x,k_q),$$
$$k,j = 1,2,3,$$

$$G^{(4)}_{kj}(y-x,\sigma) = \sum_{q=1}^{4}\left(\delta_{kj} c_q + d_q \frac{\partial^2}{\partial x_k \partial x_j}\right)\varphi_\sigma(y-x,k_q), \quad k,j = 1,2,3.$$

Direct computations show that the matrix $G(y-x,\sigma)$ in the first variable satisfies system (2.1) everywhere in D_ρ.

Based on (3.12), (3.13), and (3.16), we obtain

$$\int\limits_{\partial D_\rho \setminus S} \left(|\Pi(y,x,\sigma)| + |T(\partial_y, n(y))\Pi(y,x,\sigma)|\right) ds_y \leq \frac{C_1(x)}{1+\sigma^3}, \tag{3.18}$$

where $C_1(x)$ some bounded function inside D_ρ. The lemma is proved.

We put

$$U_\sigma(x) = \int\limits_S [\{T(\partial_y, n(y))\Pi(y,x,\sigma)\}^\top U(y) - \\ -\Pi(y,x,\sigma)\{T(\partial_y, n(y))U(y)\}]ds_y \quad x \in D_\rho. \tag{3.19}$$

The following theorem is true

Theorem 3.6. *Let $U(x)$ be a regular solution to system (2.1) in the domain D_ρ, satisfying the condition*

$$|U(y)| + |T(\partial_y, n(y))U(y)| \le M, \quad y \in \partial D_\rho \backslash S.$$

Then for $\sigma \ge 1$ and $x \in D_\rho$ it is true

$$|U(x) - U_\sigma(x)| \le \frac{MC_2(x)}{1+\sigma^3},$$

where $C_2(x)$ is some bounded function in D_ρ.

The proof of the theorem follows from formulas (3.11), (3.19) and from inequality (3.18).

Corollary 3.7. *Under the condition of the theorem 3.6, the following equivalent extension formulas are valid*

$$U(x) = \lim_{\sigma\to\infty} \int\limits_S [\{T(\partial_y, n)\Pi(y,x,\sigma)\}^\top U(y) - \Pi(y,x,\sigma)\{T(\partial_y, n)U(y)\}]ds_y, \tag{3.20}$$

$$U(x) = \int\limits_S [\{T(\partial_y, n)\Pi(y,x)\}^\top U(y) - \Pi(y,x)\{T(\partial_y, n)U(y)\}]ds_y + \int\limits_0^\infty \mathcal{R}(\sigma, x)d\sigma, \tag{3.21}$$

where

$$\mathcal{R}(\sigma, x) = \int\limits_S [\{T(\partial_y, n)\Omega(y,x,\sigma)\}^\top U(y) - \Omega(y,x,\sigma)\{T(\partial_y, n)U(y)\}]ds_y, \tag{3.22}$$

$$\Omega(y,x,\sigma) = \frac{\partial}{\partial\sigma}\Pi(y,x,\sigma) = \left\| \frac{\partial}{\partial\sigma}\Pi^{(i)}_{kj}(y,x,\sigma) \right\|, \ i = 1,2,3,4,$$

$\Pi(y,x)-$ *matrix constructed by formula (3.12) and (3.13) for*

$$\Phi_\sigma(y-x,\lambda) = \Phi(y-x, i\lambda) = \frac{\exp(i\lambda r)}{4\pi r}, \ r = |x-y|.$$

Equivalence of extension formulas (3.20) and (3.21) follows from the formula

$$\lim_{\sigma\to\infty} U_\sigma(x) = \int\limits_0^\infty \frac{dU_\sigma(x)}{d\sigma}d\sigma + U_0(x).$$

and here from the existence of the limit on the left is equivalent to the existence of an improper integral on the right.

Based on the extension formula (3.20) and (3.21), we give criterion for the solvability of the Cauchy problem (3.10). For this denote by S_0 the interior points of the surface S, that is, the surface without boundary.

Theorem 3.8. *Let $S \in C^2$, $U_0 \in C^1(S) \cap L(S)$, $U_1 \in C(S) \cap L(S)$. Then , for problem (3.10) to be solvable, it is necessary and sufficient that*

$$\left| \int_0^\infty \mathcal{R}(\sigma, x) d\sigma \right| < \infty, \tag{3.23}$$

uniformly on any compact $K \in D_\rho$, $x \in K$.If these conditions are satisfied then the solutions are defined by the equivalent formulas (3.20) and (3.21).

Proof. Necessity. Let there exists a regular solution $U(x)$ in domain D_ρ, of the system (2.1) satisfying the conditions $U(y) = U_0(y)$, $T(\partial_y, n(y))U(y) = U_1(y)$, $y \in S_0$, where $U_0 \in C^1(S) \cap L(S)$, $U_1 \in C(S) \cap L(S)$, K is compact in D_ρ. Choose $\varepsilon > 0$ such that $K \subset \overline{G}_\rho^{2\varepsilon} \subset G_\rho^\varepsilon \subset G_\rho$. Distance K to $\partial G_\rho^\varepsilon$ is not less than $\tau_1 \varepsilon$. Denote by S_ε, the part of S lying in a closed corner $\overline{G}_\rho^\varepsilon$, and through D_ρ^ε to fill in the $\overline{G}_\rho^\varepsilon$ parts of the limited surface S_ε. By the definition of the matrix-valued function $\mathcal{R}(\sigma, x)$, we need to consider the function

$$\frac{\partial}{\partial \sigma} \Phi_\sigma(y - x, \lambda) = \frac{1}{-2\pi^2} \int_0^\infty Im\big[exp(w^2) E_\rho'(\sigma w)\big] \frac{\cos \lambda u du}{\sqrt{u^2 + \alpha^2}}$$

where $w = i\sqrt{u^2 + \alpha^2} + y_1 - x_1$, $\alpha^2 = (y_2 - x_2)^2 + (y_3 - x_3)^2$, $\alpha > 0$, is regular in y and x in the whole space, hence all elements of the matrix $\frac{\partial}{\partial \sigma} \Pi(y, x, \sigma)$ are regular. Then, according to Green's formula applied in the area D_ρ^ε with the boundary $S_\varepsilon \bigcup P_\varepsilon$, where $P_\varepsilon = \partial D_\rho^\varepsilon \backslash S_\varepsilon$ we get

$$\int_{S_\varepsilon} [\{T(\partial_y, n)\Omega(y, x, \sigma)\}^\top U(y) - \Omega(y, x, \sigma)\{T(\partial_y, n)U(y)\}] ds_y =$$

$$= \int_{P_\varepsilon} [\{T(\partial_y, n)\Omega(y, x, \sigma)\}^\top U(y) - \Omega(y, x, \sigma)\{T(\partial_y, n)U(y)\}] ds_y,$$

Using this equality and from formula (3.22), we obtain

$$\big|\mathcal{R}(\sigma, x)\big| \leq \int_{S_\varepsilon} \Big|\{T(\partial_y, n)\Omega(y, x, \sigma)\}^\top U(y) - \Omega(y, x, \sigma)\{T(\partial_y, n)U(y)\}\Big| ds_y +$$

$$+ \int_{S \backslash S_\varepsilon} \Big|\{T(\partial_y, n)\Omega(y, x, \sigma)\}^\top U(y) - \Omega(y, x, \sigma)\{T(\partial_y, n)U(y)\}\Big| ds_y \leq$$

$$\leq \int_{P_\varepsilon} \big[|\Omega(y, x, \sigma)| + |\{T(\partial_y, n)\Omega(y, x, \sigma)\}^\top|\big] \left[|U_0(y)| + |U_1(y)|\right] ds_y +$$

$$+\int\limits_{S\setminus S_\varepsilon}\left[|\Omega(y,x,\sigma)|+|\{T(\partial_y,n)\Omega(y,x,\sigma)\}^\top|\right]\left[|U_0(y)|+|U_1(y)|\right]ds_y$$

Let $x\in K$ $(|x'|<\tau(x_1-2\varepsilon),x_1>2\varepsilon)$, $y\in P_\varepsilon\cup S\setminus S_\varepsilon$, $(|y'|\geq\tau(x_1-\varepsilon),y_1>\varepsilon)$. Then

$$\frac{\tau y_1-\tau x_1}{\sqrt{u^2+\alpha^2}}\leq\frac{|y'|-|x'|-\varepsilon}{|y'-x'|}\leq 1-\varepsilon_1, u\geq 0, y'\neq x'$$

and for the argument $\arg w=\arg\tau w$, $\tau w=i\tau\sqrt{u^2+\alpha^2}+\tau y_1-\tau x_1$. Thus, inequality (3.7) is true, if in this case $y'=x'$ then $Rew<0$ and this inequality is even more true. Therefore, for $\Phi_\sigma(y-x,\lambda),F_\sigma(y-x,\lambda)$ estimates (3.16), (3.17) hold, where $\delta\geq\varepsilon\tau_1$. Now, based on inequalities (3.8), (3.9) and Lemma 3.3, we have

$$|\mathcal{R}(\sigma,x)|\leq\frac{C(\rho,\varepsilon)}{1+\varepsilon^3\tau_1^3\sigma^3},$$

where $C(\rho,\varepsilon)$ is a limited number. From the last inequality we obtain condition (3.23). Necessity is proved.

Sufficiency. Let $S\in C^2$, $U_0\in C^1(S)$, $U_1\in C(S)$ and the inequality is true (3.23). Let us show that there exists a regular solution $U(x)$ of the system (3) such that $U(y)=U_0(y)$, $T(\partial_y,n(y))U(y)=U_1(y)$, $y\in S_0$. Consider the function $U(x)$ given by two equivalent formulas of the form (3.20) and (3.21). The first term on the right-hand side of formula (3.21) defines two functions that are regular solutions of the elliptic system (2.1), respectively, in the domains D_ρ and $R^3_+\setminus\overline{D}_\rho$, such that the difference between their limit values along the normals and their stresses ($x^{(1)},x^{(2)}$ two points on the normal, symmetric with respect to the point $y\in S_0$, when approaching y) on S_0 is equal to the vector functions $U_0(y)$ and $U_1(y)$, respectively, and if one of these functions is continuous in the corresponding domain up to S_0, then the other also has this property. The second term on the right side of (3.21) by virtue of (3.23) is a regular solution of system (2.1) in $\mathbb{R}^3_+$. So, the right side of formula (3.21) defines two regular solutions $U^+(x)$ and $U^-(x)$ in domains D_ρ and $\mathbb{R}^3_+\setminus\overline{\mathrm{D}}_\rho$,respectively such that for any point $y\in S_0$ the equality

$$\begin{cases}U^+(y)-U^-(y)=U_0(y)\\ T(\partial_y,n)U^+(y)-T(\partial_y,n)U^-(y)=U_1(y),\end{cases}\tag{3.24}$$

moreover, the limit relations are fulfilled uniformly with respect to y on each compact part of S_0. If $\max\{y_1 : y\in\overline{D}_\rho\}<x_1$, where $y\in S,x\in G_\rho$, then $Re\,w=y_1-x_1<0$ and for $\Phi_\sigma(y-x,\lambda)$ and its derivatives, inequality (3.16) is valid. Now from formula (3.19), we see that $U^-(x)=0$ and, according to the uniqueness theorem, $U^-(x)\equiv 0,x\in D_\rho$. It is clear that $U^-(x)$ extends smoothly to $D_\rho\cup S_0$. Then $U^+(x)$ also extends smoothly as a function of the class $C^1(D_\rho)\cup S_0$ (see [24], lemma 1.1). Further, note that from formula (3.20) and inequalities (3.18) it follows that $U^-(x)=0$ for $x_1>\max\{y_1 : y\in\overline{D}_\rho\}$. Then, according to the uniqueness theorem (since solution of elliptic systems is analytical [29]) $U^-(x)\equiv 0$, $x\in\mathbb{R}^3_+\setminus\overline{\mathrm{D}}_\rho$. Now from (3.24) we obtain the statement theorems. The theorem is proved. □

Acknowledgment

In conclusion, the author expresses his sincere gratitude to the referee for helpful advice.

References

1. Aizenberg L.A.: *Carleman's formulas in complex analysis*, Nauka, Novosibirsk, 1990.
2. Carleman T.: *Les fonctions quasi analytiques*, Gautier-Villars et Cie., Paris, 1926.
3. Dzharbashyan M.M.: *Integral Transformations and Representations of Functions in Complex Domain*, Nauka, Moscow, 1926.
4. Fock V.A. and Kuni, F.M.: On introduction of damping function into dispersion relations, *Dokl. Akad.Nauk SSSR*, **127** (1959), no. 6, 1195–1198.
5. Goluzin G.M., Krylov V.M.: The generalized Carleman formula and its application to the analytic continuation of functions, *Sbornik: Mathematics*, **40** (1993), no. 2, 144–149.
6. Gonchar A.A.: On analytic continuation from the edge of wedge, *Ann. Acad. Sci. Fenn., Ser. A1: Math.-Phys.*, **10** (1985), 221–225.
7. Kupradze, V.D., Gegelia, T.G., Basheleishvili, M.O., and Burchuladze, T.V.: *Three-Dimensional Problems of Mathematical Theory of Elasticity and Thermoelasticity*, Moscow: Nauka, 1976.
8. Kytmanov A.M.: *The Martinelli Bochner integral and its applications*, Novosibirsk: Nauka, 1991.
9. Lavrent'ev M.M.: *On some ill-posed problems of mathematical physics*, Nauka, Novosibirsk, 1962.
10. Lavrentev, M.M., Romanov, V.G., and Shishatskii, S.P.: *Ill-Posed Problems of Mathematical Physics and Analysis*, NovosibirskMoscow: Nauka, 1980.
11. Juraev D.A.: The Cauchy problem for matrix factorizations of the Helmholtz equation in an unbounded domain, *Siberian Electronic Mathematical Reports*, **14** (2017), 752–764.
12. Juraev D.A.: Cauchy problem for matrix factorizations of the Helmholtz equation, *Ukrainian Mathematical Journal*, **69** (2017), no. 10, 1364-1371.
13. Juraev D.A.: On the Cauchy problem for matrix factorizations of the Helmholtz equation in a bounded domain, *Siberian Electronic Mathematical Reports*, **15** (2018), 11–20.
14. Juraev D.A.: On the Cauchy problem for matrix factorizations of the Helmholtz equation in an unbounded domain in $\mathbb{R}^2$, *Siberian Electronic Mathematical Reports*, **15** (2018), 1865–1877.
15. Juraev D.A., Noeiaghdam S.: Regularization of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation on the plane, *Axioms*, **10** (2021), no. 2, 1–14.
16. Juraev D.A.: Solution of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation on the plane, *Global and Stochastic Analysis*, **8** (2021), no. 3, 1–17.
17. Mergelyan S.N.: Harmonic approximation and approximate solution of the Cauchy problem for Laplace equation, *Usp. Mat. Nauk*, **5** (1956), no. 71, 337–340.
18. Makhmudov, O.I. and Niezov, I.E.: Regularization of a solution to the Cauchy problem for the system of equations of elasticity in displacements, *Sib. Math. J.*, **39** 1998, no. 2, 323–330.
19. Makhmudov, O.I. and Niezov, I.E.,*A Cauchy problem for the system of elasticity equations*, Differ. Equations, **36** (2000), no. 5, 749–754.
20. Makhmudov O., Niyozov I.: Regularization of a solution to the Cauchy Problem for the System of Thermoelasticity, *Contemporary Mathematics.AMS, Primary* **382**, (2005), 285–289.
21. Makhmudov O., Niyozov I.: The Cauchy problem for the Lame system in infinite domains in R^m, *Journal of inverse and Ill-Posed Problems*, **14** (2006), no. 8, 905–924.
22. Makhmudov O. I., Niyozov I. E., Tarkhanov N.: The Cauchy Problem of Couple-Stress Elasticity, *Contemporary Mathematics. AMS*, **455**, (2008), 207–310.
23. Makhmudov, O. and Niyozov, I.: Regularization of solutions to the Cauchy problem for systems of elasticity theory, *in Proceedings of the International 5th ISAAK Congress, More progresses on Analysis, Ed. by H. G. W. Begehr*, Singapore, (2009), 69–84.

24. Shlapunov A.A.: On the Cauchy problem for the Laplace equation, *Siberian Math. J.*, **33** (1992), no. 3, 205–215.
25. Tarkhanov N.N.: *The Cauchy Problem for solutions of Elliptic Equations*, Academie Varlag, Berlin, 1995.
26. Yarmukhamedov Sh.: On the Cauchy problem for the Laplace equation, *Reports of the USSR Academy of Sciences*, **235** (1977), no. 2, 281–283.
27. Yarmukhamedov Sh.: On the continuation of the solution of the equation of the theory of elasticity, *Reports of the USSR Academy of Sciences*, *358* (1998), no. 1, 54–56.
28. Yarmukhamedov Sh.: The Carleman function and the Cauchy problem for the Laplace equation, *Siberian Mathematical Journal*, **45** (2004), no. 3, 702–719.
29. Petrovskii I.G.: *Lectures on Partial Differential Equations*, Moscow, Fizmatgiz, 1961.
30. Zhuraev D.A.: Cauchy problem for matrix factorizations of the Helmholtz equation, *Ukrainian Mathematical Journal*, **69** (2018), no. 10, 1583–1592.

Iqbol Ergashevich Niyozov: Samarkand State University, University Boulevard, 15, 140104, Samarkand, Uzbekistan

E-mail address: `iqboln@mail.ru`