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COEXISTENCE CHAOTIC BEHAVIOR ON THE EVOLUTION OF POPULATIONS OF THE BIOLOGICAL SYSTEMS MODELING BY THREE DIMENSIONAL QUADRATIC MAPPINGS

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Abstract. The present paper is devoted to the investigation of three - dimensional case of the quadratic mapping on the hyperplane to itself which the mathematical model of populations of the biological systems having three kinds. In this paper we learnt the properties of the asymptotical behavior of the trajectories of certain mappings. The main results of this paper are existence of the chaotic orbits and Sharkovsky order bifurcations three - dimensional case of the quadratic mapping on the hyperplane $H_1 \subset R^3$.

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1. Introduction

The study of dynamics of the mapping

$$F_c : x \to x^2 + c \tag{1.1}$$

on the real line to itself and its various generalizations are devoted hundreds of papers. Von Neumann and Ulam studied this problem and considered by helping first computers the behaviors of the orbits are chaotic. Mathematical theory of this numerical experiments constructed by Sharkosvky [9], Li-Yorke [7], Feigenbaum [5] and others. In the present time the theory of one dimensional mapping is the most learned part of the general theory of dynamical systems. This paper is devoted to investigate the evolution of populations of the biological systems modeling by three dimensional quadratic mappings which depend on the problem Von Neumann and Ulam [1]. The following definitions are the analogue of the complex case to the real case [4], [6].

Definition 1.1. The filled Julia set $K(F_c)$ of the mapping (1.1) is defined as the set of all points $x \in R$, that have bounded orbits with respect to the mapping (1.1).

Definition 1.2. The Julia set is the common boundary of the filled Julia set

$$J(F_c) = \partial K\left(F_c\right).$$

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Definition 1.3. The Mandelbrot set [8] M_{F_c} of the mapping (1.1) is the set of all points $c \in R$ on the parameter line, which the orbits of the all critical points are bounded.

We know from [1] and [3] the following statements for the mapping (1.1).

Proposition 1.4. *If $c > 0.25$ then $K(F_c) = \emptyset$ and $J(F_c) = \emptyset$.*

Proposition 1.5. *If $-2 \leq c \leq 0.25$ then $K(F_c) = \left[-\frac{1+\sqrt{1-4c}}{2}, \frac{1+\sqrt{1-4c}}{2}\right]$ and $J(F_c) = \{-\frac{1+\sqrt{1-4c}}{2}, \frac{1+\sqrt{1-4c}}{2}\}$.*

Proposition 1.6. *If $c < -2$ then $K(F_c)$ is the Cantor type set.*

Proposition 1.7. *Mandelbrot set of (1.1) is*

$$M_{F_c} = [-2, 0.25].$$

Theorem 1.8. *If $c = -2$ then there is exist many chaotic orbits in $K_{F_c} = [-2, 2]$.*

Let $S^{m-1} = \{x = (x_1, x_2, ..., x_m) : \sum_{k=1}^{m} x_k = 1, x_k \geq 0\} \subset R^m$, $(m-1)-$ dimensional simplex in R^n. Quadratic stochastic operator (q.s.o) $V : S^{m-1} \to S^{m-1}$ is defined by

$$(Vx)_k = x'_k = \sum_{k=1}^{m} P_{ij,k} x_i x_j, \ k = \overline{1, m}$$

where $P_{ij,k} \geq 0, P_{ij,k} = P_{ji,k}, \sum_{k=1}^{m} P_{ij,k} = 1, \ x = (x_1, x_2, ..., x_m) \in S^{m-1}$.

In mathematical genetics, V is called the evolutionary operator of populations. A population is defined as closed relative to the propagation of a community of organisms. Successive generations $F_1, \ F_2, \ ...$ are distinguished in the population. The state of a population is a set of $x = (|x_1|, |x_2|, ..., |x_m|)$ densities of varieties. Inheritance coefficients $P_{ij,k}$ is the probability of the birth of an individual belonging to the $k-th$ variety when the individuals of the $i-th$ and $j-th$ species are crossbreeding. In [2] investigated some properties of the evolutionary operator of populations on S^{m-1}.

Let $H_p = \{x = (x_1, x_2, ..., x_m) : \sum_{k=1}^{m} x_k = p\} \subset R^m, \ p \in R, \ (m-1)$ - dimensional hyperspace in R^m. When $p = 1$ the hyperspace H_1 is the expansion of the $(m-1)$ -dimensional simplex i.e. $S^{m-1} \subset H_1$.

In this paper we study the properties of the quadratic stochastic operator (q.s.o) in the case $m = 3$, $S^2 \subset H_1$, $V : H_1 \to H_1$.

2. Main Part

We investigate population of dynamics of the biological systems which has three kinds creature. Let the populations kinds have a following laws

$$I + I \mapsto I$$

$$II + II \mapsto II$$

$$III + III \mapsto III$$

$$I + II \mapsto II$$

$$II + III \mapsto I$$

$$I + III \mapsto III$$

then the evolution of this biological system defines by the following mapping $V_0 : H_1 \to H_1$

$$V_0 : \begin{cases} x_1' = x_1^2 + 2x_2x_3, \\ x_2' = x_2^2 + 2x_1x_2, \\ x_3' = x_3^2 + 2x_1x_3. \end{cases} \tag{2.1}$$

If the populations kinds have a following laws

$$I + I \mapsto I$$
$$II + II \mapsto II$$
$$III + III \mapsto III$$
$$I + II \mapsto I$$
$$II + III \mapsto II$$
$$I + III \mapsto III$$

then the evolution of this biological system defines by the following mapping $V_1 : H_1 \to H_1$

$$V_1 : \begin{cases} x_1' = x_1^2 + 2x_1x_2, \\ x_2' = x_2^2 + 2x_2x_3, \\ x_3' = x_3^2 + 2x_1x_3. \end{cases} \tag{2.2}$$

Let

$$V = \lambda V_0 + (1 - \lambda)V_1$$

V is the new evolution operator has both of the properties of V_0 and V_1 depends on $0 \leq \lambda \leq 1$.

Theorem 2.1. *Operator V has chaotic orbits and Sharkovsky order bifurcations at the value $\lambda = \frac{1}{2}$ on H_1 .*

Proof. Since $\lambda = \frac{1}{2}$ and $x_3 = 1 - x_1 - x_2$

$$V : \begin{cases} x_1' = x_1^2 + x_2 - x_2^2, \\ x_2' = x_2, \\ x_3' = x_2^2 - 2x_2 - x_1^2 + 1. \end{cases} \tag{2.3}$$

If we consider x_2 as a parameter then interesting dynamics occurs at the values of $-2 \leq x_2 - x_2^2 \leq 0.25$.

We know from [3] if c decrease from 0.25 to -2 then occurs period doubling bifurcation with Sharkovsky ordering [9]. From theorem (1.8) that at $c = -2$ orbits are chaotic. It means that if x_2 increase from 0.5 to 2 and decrease from 0.5 to -1 operator V has Sharkovsky ordering bifurcations. At $x_2 = -1$ and $x_2 = 2$ operator V has chaotic orbits on H_1. $\square$

By the propositions (1.5),(1.7) we get following statements.

Theorem 2.2. *The filled Julia set for V is*

$$V:\begin{cases}-\frac{1}{2}-\left|x_2-\frac{1}{2}\right|\le x_1\le\frac{1}{2}+\left|x_2-\frac{1}{2}\right|,\\ -1\le x_2\le 2,\\ -\frac{3}{2}-\left|x_2-\frac{1}{2}\right|\le x_3\le\frac{5}{2}+\left|x_2-\frac{1}{2}\right|.\end{cases}\tag{2.4}$$

Theorem 2.3. *Mandelbrot set for V is the set of all values λ*

$$M=[0,1].$$

3. Conclusion

In the present paper we investigated mathematical model of populations of the biological systems having three kinds by three - dimensional case of the quadratic mapping on the hyperplane to itself. We used the methods of one dimensional dynamical systems. The main results of this paper are existence of the chaotic orbits and Sharkovsky order bifurcations three - dimensional case of the quadratic mapping on the hyperplane $H_1 \subset R^3$.

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